THE SIGN REPRESENTATION FOR SHEPHARD GROUPS

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Abstract. Shephard groups are unitary reflection groups arising as the symmetries of regular complex polytopes. For a Shephard group, we identify the representation carried by the principal ideal in the coinvariant algebra generated by the image of the product of all linear forms defining reflecting hyperplanes. This representation turns out to have many equivalent guises making it analogous to the sign representation of a finite Coxeter group. One of these guises is (up to a twist) the cohomology of the Milnor fiber for the isolated singularity at 0 in the hypersurface defined by any homogeneous invariant of minimal degree.

1. Introduction

Let W be a finite reflection group acting in a Euclidean space V, that is, a finite subgroup of $GL(V)$ generated by reflections. An important role in the structure and representation theory of W is played by its *sign character*

$$\epsilon : W \to \mathbb{Z}^{\times} = \{\pm 1\}.$$

This character appears in many different guises:

- $\epsilon(w) = \det(w) = \det^{-1}(w)$.
- ϵ is the character of W acting on the top (reduced) cohomology group of the unit sphere $\mathbb{S}^{\dim(V)-1}$.
- If R is a set of Coxeter generators for W, then ϵ is the virtual character $\sum_{J \subset R} (-1)^{|R-J|} \operatorname{Ind}_{W_J}^{W} 1_{W_J}$, where W_J is the parabolic subgroup generated by J, and 1_{W_J} denotes its trivial character.
- Let $S = \mathbb{C}[V]$ denote the ring of polynomial functions on V, and I the ideal generated by the W-invariant polynomials of positive degree. Then the quotient ring S/I is a graded ring which is finite-dimensional over $\mathbb{C}$, and whose nonvanishing graded component of top degree $(S/I)_t$ carries the representation ϵ.
- This top graded component $(S/I)_t$ can also be described as the principal ideal $Q \cdot (S/I)$ within S/I, where Q is the product of the linear forms defining the reflecting hyperplanes for W.

This paper concerns an analogue of ϵ for the class of unitary reflection groups known as Shephard groups. Let V be a finite-dimensional complex unitary space. Recall that a *unitary reflection group* is a finite subgroup $G \subset GL(V)$ generated by

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unitary reflections, i.e., elements of finite order that fix a hyperplane in V. Such groups include the finite Euclidean reflection groups, called *Coxeter groups*, and were classified by Shephard and Todd [25].

Shephard groups are the symmetry groups of the *regular complex polytopes* defined and classified by Shephard [24] (see also Coxeter [9]). These groups generalize the finite reflection groups which occur as the Euclidean symmetry groups of regular convex polytopes, or equivalently, those whose Coxeter diagrams are unbranched. In particular, each Shephard group can be generated by a distinguished set R of $\ell := \dim V$ generators which yield a particularly nice presentation for the group, in fact, a presentation which can be expressed by a "Coxeter-like" diagram which is unbranched; see Section 3.

We show in this paper that every Shephard group G has a representation, defined over $\mathbb{Z}$, which occurs in many guises, analogous to ϵ. We introduce some notation to make this precise. Most of our notation follows [20].

Let $\mathcal{P}$ be a regular complex polytope in a unitary space V of dimension ℓ having G as its group of unitary automorphisms. Let Δ be the *order complex* of its poset of proper faces, that is, the simplicial complex of totally ordered subsets in this poset. Let R be a distinguished set of generators for G (as defined in Section 3 below). For $J \subset R$, let G_J denote the subgroup generated by J.

As before, let S denote the algebra $\mathbb{C}[V]$ of polynomial functions on V, and I the ideal generated by the G-invariants of positive degree. Let d denote the minimal degree of a G-invariant, and let f_1 denote any homogeneous G-invariant of this degree (this turns out to define f_1 uniquely up to a scalar multiple; see Lemma 7). The *Milnor fiber* of the singularity at 0 on the hypersurface $f_1^{-1}(0)$ is the level set $F := f_1^{-1}(1)$, where we regard f_1 as a map $f_1 : V \to \mathbb{C}$. Let K denote the ideal of S generated by the first partial derivatives $\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_\ell}$. Let Q denote the product $\prod_{H \in \mathcal{A}} \alpha_H$ where $\mathcal{A}$ is the collection of reflecting hyperplanes H, and α_H is any linear form that vanishes on H. Given a graded vector space $U = \oplus_i U_i$ carrying a graded representation of G with character $\chi_{U_i} : G \to \mathbb{C}$ on U_i, define its *graded character*

$$\chi_{U,t}(g) := \sum_i \chi_{U_i}(g) t^i.$$

After establishing notation and reviewing facts about unitary reflection groups in Section 2 and Shephard groups in Section 3, we prove the following result, which is essentially a collection of previously known results.

Theorem 1. *Let G be a Shephard group with distinguished generators R. Then the following graded (complex) representations of G are equivalent:*

(i) $S/K \otimes \det^{-1}$.

(ii) *the representation U affording the graded character*

$$\chi_{U,t}(g) = \frac{\det(1 - gt^{d-1})}{\det(g - t)}.$$

Furthermore, as ungraded representations, both are equivalent to the following representations defined over $\mathbb{Z}$:

(iii) *the dual of the virtual representation $\sum_{J \subset R} (-1)^{|R-J|} \operatorname{Ind}_{G_J}^{G} 1_{G_J}$.*

(iv) *the representation on the (reduced) cohomology $\tilde{H}^{\ell-1}(F, \mathbb{C})$ of the Milnor fiber.*

(v) *the representation on the (reduced) cohomology $\tilde{H}^{\ell-1}(\Delta, \mathbb{C})$.*

Our main result, proven in Section 4, describes another natural occurrence of this representation. Let $\phi : S \to S/I$ be the composite map

$$S \xrightarrow{Q} S \to S/I$$

where the first map is multiplication by Q and the second is the canonical surjection.

Theorem 2. *For any Shephard group G, the kernel of ϕ is the ideal K generated by the first partial derivatives of f_1. Therefore, S/K maps isomorphically onto the principal ideal $Q \cdot (S/I)$ within S/I.*

It is known (see Lemma 5 below) that

$$g(Q) = det^{-1}(g)Q \text{ for all } g \text{ in } G.$$

Consequently, Theorem 2 shows that the graded representation carried by $Q \cdot (S/I)$ is equivalent (up to a shift in grading) with those in (i), (ii) of Theorem 1, and equivalent as an ungraded representation with those in (iii), (iv), and (v) of Theorem 1.

Note that in a suitable coordinate system, the polynomial $f_1 := x_1^2 + \ldots + x_\ell^2$ is a minimal degree invariant for any Coxeter group. Every Coxeter group W acting on $\mathbb{R}^\ell$ can be considered as a unitary reflection group acting on $\mathbb{C}^\ell$. The sphere $\mathbb{S}^{\ell-1} := \{v \in \mathbb{R}^\ell : f_1(v) = 1\}$ is a W-equivariant strong deformation retract of the Milnor fiber $F := \{v \in \mathbb{C}^\ell : f_1(v) = 1\}$; see e.g. [13]. Thus, when G is simultaneously a Coxeter group and a Shephard group, the sign character described in the introduction and the sign representation described in Theorems 1 and 2 coincide.

Section 5 contains remarks and open questions.

2. Notation and review of unitary reflection groups

Let V be an ℓ-dimensional *unitary space*, that is, a $\mathbb{C}$-vector space of dimension ℓ with a positive definite Hermitian form. A *unitary reflection* (or *pseudo-reflection*) is a non-identity element g of $GL(V)$ of finite order which fixes some hyperplane H in V, called the *reflecting hyperplane* for g. A finite subgroup $G \subset GL(V)$ is called a *unitary group generated by reflections* (or *u.g.g.r.*) if it is generated by unitary reflections. A u.g.g.r. is *irreducible* if V contains no G-invariant subspaces. Irreducible u.g.g.r.'s were classified by Shephard and Todd [25]. They proved that u.g.g.r's are distinguished by a rich invariant theory which we discuss next. Good references for most of this material are [11, 20, 27].

A subgroup $G \subset GL(V)$ acts on the dual space V^* in the usual (contragradient) way: for g in G, f in V^*, and $v \in V$ we have

$$g(f)(v) = f(g^{-1}(v)).$$

This extends to an action on the symmetric algebra $S := Sym(V^*)$, which we can view as the algebra of polynomial functions $f : V \to \mathbb{C}$. When G is finite, it is well-known [27, §1] that the subalgebra of invariant polynomials S^G is a finitely generated $\mathbb{C}$-algebra, and S is a finitely generated module over S^G. Shephard and Todd [25] and Chevalley [8] proved the following.

Theorem 3. *Let $G \subset GL(V)$ be a finite subgroup. Then S^G is isomorphic to a polynomial algebra generated by ℓ algebraically independent homogeneous elements $f_1, \ldots, f_\ell$ if and only if G is a u.g.g.r.* $\square$

We call $f_1, \dots, f_\ell$ a set of *basic invariants* of G. Let I be the ideal in S generated by $f_1, \dots, f_\ell$. Note that although the invariants $f_1, \dots, f_\ell$ themselves are not unique, their degrees and the ideal I are uniquely determined by G. Let d be the minimum non-zero degree in I, and note that there can be more than one G-invariant of degree d up to scaling. On the other hand, this does not happen when G is a Shephard group; see Lemma 7.

The fact that S is finite over S^G means that $f_1, \dots, f_\ell$ form a *homogeneous system of parameters* (*h.s.o.p.*) for S, and consequently also an S-*regular sequence*, since S is *Cohen-Macaulay*; see e.g. [27, §3] and [28]. This says that S/I is a graded *complete intersection*, and rings of this form satisfy a version of Poincaré duality:

Lemma 4. *Let S/L be a graded complete intersection, that is, L is an ideal in S generated by an h.s.o.p. $h_1, \dots, h_\ell$ which is also an S-regular sequence. Let $t_i := \deg(h_i)$. Then*

(i) [27, §8, §9] [28, p. 12] *S is a Gorenstein ring of Krull dimension 0, with top non-zero degree*

$$\tau := \sum_{i=1}^{\ell} (t_i - 1).$$

Consequently, the bilinear pairing

$$(S/L)_j \times (S/L)_{\tau-j} \to (S/L)_\tau \cong \mathbb{C}$$

is non-degenerate.

(ii) [21, p. 187] *$(S/L)_\tau$ is spanned by the image of the Jacobian determinant*

$$Jac(h_1, \dots, h_\ell) := \det \left(\frac{\partial h_i}{\partial x_j} \right)_{i,j=1,\dots,\ell}. \quad \square$$

For a u.g.g.r. G, the algebra S/I is called the *coinvariant algebra*. A theorem of Chevalley [8] asserts that S/I is equivalent to the regular representation as an ungraded G-representation. We wish to be explicit about the occurrences of certain degree one characters in this representation. Let $\mathcal{A}$ denote the collection of reflecting hyperplanes of the unitary reflections in G, and for each such unitary reflection, let α_H be a linear form that vanishes on its reflecting hyperplane H. Let e_H denote the order of the cyclic subgroup of G which fixes H. Given any degree one character $\chi : G \to \mathbb{C}^\times$, for each hyperplane $H \in \mathcal{A}$ there is a unique integer $e_{H,\chi}$ with $0 \leq e_{H,\chi} < e_H$ defined by $\chi(g) = \det(g)^{-e_{H,\chi}}$ for all unitary reflections g fixing H. One can then define a (minimal) χ-*relative invariant*

$$Q_\chi := \prod_{H \in \mathcal{A}} (\alpha_H)^{e_{H,\chi}},$$

i.e. $g(Q_\chi) = \chi(g) Q_\chi$ for all g in G.

Lemma 5. [27, Proposition 4.12] *The set of χ-relative invariants $S^{G,\chi}$ is a free S^G-module of rank 1 with Q_χ as generator.* $\square$

The following particular cases of Q_χ are important for what follows:

$$\begin{aligned} J(G) &:= Q_{\det} = \prod_{H \in \mathcal{A}} (\alpha_H)^{e_H - 1}, \\ H(G) &:= Q_{\det^2} = \prod_{H \in \mathcal{A}} (\alpha_H)^{e_H - 2}, \\ Q(G) &:= Q_{\det^{-1}} \prod_{H \in \mathcal{A}} \alpha_H. \end{aligned}$$

When no confusion will result, we will use J, H, Q to refer to $J(G), H(G), Q(G)$, respectively. Note that one has by definition

$$QH = J. \tag{1}$$

Let t denote the top degree of S/I (see Lemma 4).

Lemma 6. [30] [25, p. 283] *For any u.g.g.r. G, J is equal to the Jacobian determinant $Jac(f_1, \ldots, f_\ell)$ up to a scalar multiple. Consequently, the image of J spans $(S/I)_t$, and in particular, J does not lie in I.* $\square$

3. Shephard groups

We now turn to the special case of Shephard groups, which enjoy special properties not shared by all u.g.g.r.'s. For a more detailed treatment of Shephard groups, see Coxeter's wonderful book [9].

A *regular complex polytope* $\mathcal{P}$ in V is a collection of complex affine subspaces of V, called *faces* of $\mathcal{P}$, satisfying certain conditions [9, p. 115]. One of these conditions is that the group $G \subset GL(V)$ of unitary automorphisms of $\mathcal{P}$ acts transitively on the maximal flags of faces in $\mathcal{P}$. Such a group G is called a *Shephard group*, and will always be an irreducible u.g.g.r. One obtains a *distinguished* set of generators $R := \{r_0, \ldots, r_{\ell-1}\}$ for a Shephard group G as follows: let

$$\mathcal{F}_0 := (F_0 \subset F_1 \subset \cdots \subset F_{\ell-2} \subset F_{\ell-1}) \tag{2}$$

be a fixed maximal flag of (proper) faces in $\mathcal{P}$, which we will call the *base flag*. For each i, choose r_i to be a generator for the (cyclic) group that stabilizes (not necessarily pointwise) each F_j with $j \neq i$. Let p_i denote the order of r_i; then there exist positive integers $q_i \geq 3$ such that G has the following very simple presentation with respect to these generators:

$$\begin{aligned} r_i^{p_i} &= 1, \\ r_i r_j &= r_j r_i \text{ if } |i - j| > 1, \\ \underbrace{r_i r_{i+1} r_i r_{i+1} \cdots}_{q_i \text{ letters}} &= \underbrace{r_{i+1} r_i r_{i+1} r_i \cdots}_{q_i \text{ letters}}. \end{aligned}$$

The Shephard group G with the above presentation is denoted by the shorthand *symbol*

$$p_0[q_0]p_1[q_1]p_2 \cdots p_{\ell-2}[q_{\ell-2}]p_{\ell-1}. \tag{3}$$

It may also be represented by a "Coxeter-like" linear diagram with vertices labeled by the p_i and edges labeled by the q_i. The classification of Shephard groups is relatively short. There is one infinite family $r[4]2[3]2[3] \cdots 2[3]2$ isomorphic to the wreath product $C_r \wr S_\ell$ of a cyclic group with a symmetric group (corresponding to $G(r,1,\ell)$ in the notation of Shephard and Todd [25]). There is a finite list of exceptional Shephard groups:

- symmetry groups of real regular polytopes (Coxeter groups with unbranched diagrams)
- $p_0[q]p_1$ where $p_0, p_1 \geq 2$ and $q \geq 3$ satisfy $p_0 = p_1$ if q is odd,
$$\frac{1}{p_0} + \frac{1}{p_1} + \frac{2}{q} > 1,$$
and at least one of p_0, p_1 is > 2. There are twelve such groups.
- 2[4]3[3]3
- 3[3]3[3]3
- 3[3]3[3]3[3]3.

From the invariant-theoretic point of view, Shephard groups have the following extra properties, which were verified using the above classification in [17, Corollaries 5.4 and 5.8, Theorem 5.10]:

Lemma 7. *Let G be a Shephard group, and d the minimal degree of a G-invariant.*

(i) *Up to scalar multiples, there is a unique G-invariant f_1 of degree d.*
(ii) *The hypersurface $f_1^{-1}(0)$ has an isolated critical point at 0 in V, i.e., f_1 defines a smooth hypersurface in the projective space $\mathbb{P}(V)$.*
(iii) *The Hessian determinant*
$$\begin{aligned} Hess(f_1) &:= Jac\left(\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_\ell}\right) \\ &= \det\left(\frac{\partial^2 f_1}{\partial x_i x_j}\right)_{i,j=1,\ldots,r} \end{aligned}$$
is equal (up to scalar multiple) to $H(G) = Q_{\det^2} = \prod_{H\in\mathcal{A}}(\alpha_H)^{e_H-2}$. □

We remark that property (iii) above actually *characterizes* the union of the Coxeter and the Shephard groups among all u.g.g.r.'s, see [20, Theorem 6.121].

We next discuss some consequences of Lemma 7. As in the introduction, for a Shephard group G, let K denote the ideal in S generated by the first partial derivatives $\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_\ell}$.

Corollary 8. *For a Shephard group G with notation as above, S/K is a graded complete intersection with top degree $\ell(d-2)$ and with $(S/K)_{\ell(d-2)}$ spanned by the image of H. Consequently, for any f in $S - K$, there exists f' in S with $\overline{ff'} = \overline{H}$ in S/K.*

Proof. The fact that 0 is an isolated singular point of $f_1^{-1}(0)$ implies that the first partial derivatives of f_1 form an S-regular sequence by [4, Chapter V, p. 137, Exercise 5]. The rest follows from Lemma 4 and Lemma 5. □

Part (ii) of Lemma 7 also has the following consequences for the topology of the *Milnor fiber* $F := f_1^{-1}(1)$ (see [14, 15] and the references therein).

Theorem 9. *Let G be a Shephard group with minimal degree invariant f_1 of degree d as above. Then*

(i) *The Milnor fiber F is homotopy equivalent to a wedge of $(d-1)^\ell$ spheres of dimension $\ell - 1$.*
(ii) *There is a G-equivariant isomorphism $S/K \to \tilde{H}^{\ell-1}(F;\mathbb{C})$.* □

Let Δ denote the *order complex* of the poset of proper faces of the regular complex polytope $\mathcal{P}$. In other words, Δ is the simplicial complex having vertex set indexed by the proper faces of $\mathcal{P}$ and simplices corresponding to flags of nested faces. Note that the choice of a base flag $\mathcal{F}_0$ as in definition (2) then corresponds to the choice of maximal face in Δ which we call the *base chamber*.

The following is proven in [13, Thms. 4.1 and 5.1] somewhat nonconstructively; see [19, 31] for more explicit case-by-case constructions that use the classification of Shephard groups.

Theorem 10. *The geometric realization of Δ is G-equivariantly isomorphic to a (G-equivariant) strong deformation retraction of the Milnor fiber $F := f_1^{-1}(1)$.* □

The following corollary is [13, Corollary 5.3]; see [1, 2] for more on Cohen-Macaulay complexes.

Corollary 11. *Δ is a Cohen-Macaulay complex.*

Proof. Theorems 9 and 10 imply that Δ has only top-dimensional cohomology. The same follows for all links of faces in Δ, since these are always joins of order complexes of regular complex polytopes which are *medial polytopes* of $\mathcal{P}$ (see [9, p. 116] or the proof of Lemma 12 below).

□

We wish to describe explicitly the permutation action of G on faces of Δ. Most of the following lemma seems implicit in the discussion of medial polytopes from [9, p. 116], but we include a proof for the sake of completeness.

Lemma 12. *Let $\mathcal{P}$ be a regular complex polytope, and G its Shephard group. Then G acts simply transitively on maximal flags of faces in $\mathcal{P}$, and hence on maximal faces (chambers) of Δ.*

More generally, consider a partial flag

$$\mathcal{F} = (F_{a_0} \subset F_{a_1} \subset \cdots \subset F_{a_k})$$

contained in the base flag $\mathcal{F}_0$ of definition (2), or equivalently, a face contained in the base chamber of Δ. Then the stabilizer subgroup within G of $\mathcal{F}$ is the subgroup G_J generated by the subset of distinguished generators $J := R - \{r_{a_0}, r_{a_1}, \ldots, r_{a_k}\}$.

Proof. The first assertion is [9, p. 116, lines 1-2].

For the second assertion, note that G_J is a subset of the stabilizer of $\mathcal{F}$, and hence it suffices to show that they have the same cardinality. By the first assertion, the order of a Shephard group G is the number of maximal flags in the corresponding polytope $\mathcal{P}$, and a group element g may be identified with the image $g\mathcal{F}_0$ of the base flag $\mathcal{F}_0$. In particular, the stabilizer of $\mathcal{F}$ has the same cardinality as the set of maximal flags in $\mathcal{P}$ which pass through the partial flag $\mathcal{F}$. This cardinality is clearly the product of the numbers of maximal flags in each interval

$$[F_{a_{i-1}}, F_{a_i}] := \{\text{faces } F \text{ in } \mathcal{P} \text{ with } F_{a_{i-1}} \subset F \subset F_{a_i}\}$$

for $i = 0, \ldots, k+1$ (where we adopt the convention that $a_{-1} := -1, a_{k+1} := \ell$, $F_{-1} := \emptyset$, and $F_\ell := V$). However, each such interval is again the poset of faces in a regular complex polytope $\mathcal{P}_i$, the *medial polytope* [9, p. 116] associated with $F_{a_{i-1}} \subset F_{a_i}$. Since the Shephard group associated to $\mathcal{P}_i$ may be identified with the subgroup G_{J_i} where

$$J_i := \{r_{a_{i-1}+1}, r_{a_{i-1}+2}, \ldots, r_{a_i-2}, r_{a_i-1}\},$$

we conclude that the stabilizer of $\mathcal{F}$ has cardinality $\prod_{i=0}^{k+1}|G_{J_i}|$. On the other hand, since $J = \cup_{i=0}^{k+1} J_i$, and G_{J_r}, G_{J_s} commute for $r \neq s$ by the presentation of G discussed in Section 3, we conclude that $|G_J| = \prod_{i=0}^{k+1}|G_{J_i}|$, as desired. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. The equivalence of (i) and (ii) is, up to a twist by $\det^{-1}$, exactly [15, Theorem 1.3]. The equivalence of (i) and (iv) is the main theorem of [14]. The equivalence of (iv) and (v) follows from Theorems 9 and 10.

The equivalence of (iii) and (v) will follow from the equivalent statement that the virtual representation $\sum_{J\subset R}(-1)^{|R-J|}\operatorname{Ind}_{G_J}^{G} 1_{G_J}$ is equivalent to the reduced homology representation $\tilde{H}_{\ell-1}(\Delta,\mathbb{C})$. We use the Hopf trace formula:

$$\sum_{i\geq -1}(-1)^i Trace\left(g|_{\tilde{C}_i(\Delta,\mathbb{C})}\right) = \sum_{i\geq -1}(-1)^i Trace\left(g|_{\tilde{H}_i(\Delta,\mathbb{C})}\right), \tag{4}$$

where $\tilde{C}_i(\Delta,\mathbb{C})$ denotes the i-dimensional chain group in the augmented simplicial chain complex that computes the reduced homology $\tilde{H}_{\cdot}(\Delta,\mathbb{C})$. Lemma 12 implies that the permutation action of G on $\tilde{C}_i(\Delta)$ is the direct sum of the coset actions G/G_J as J ranges over subsets of R with $|R-J| = i+1$, so that its character is the sum of induced characters $\operatorname{Ind}_{G_J}^{G} 1_{G_J}$ over the same set of J's. Thus the left-hand-side of (4) is $(-1)^{|R|-1}$ times the character $\sum_{J\subset R}(-1)^{|R-J|}\operatorname{Ind}_{G_J}^{G} 1_{G_J}$. Meanwhile, Corollary 11 implies that all $\tilde{H}_i(\Delta,\mathbb{C})$ vanish except when $i = \ell-1$, so that the right-hand side of (4) is $(-1)^{\ell-1}$ times the character of the homology representation $\tilde{H}_{\ell-1}(\Delta,\mathbb{C})$. □

4. Proof of Theorem 2

Before proving Theorem 2, we review some facts about anti-invariant forms. The action of a group $G \subset GL(V)$ on S induces an action on the set of derivations (or vector fields) on V, $\mathrm{Der}_S \simeq S\otimes V$. This in turn induces an action on the set of differential 1-forms on V, $\Omega^1 := \mathrm{Hom}(\mathrm{Der}_S, S) \simeq S\otimes V^*$. The set Der_S is a free S-module with basis $\{\frac{\partial}{\partial x_i}\}$, and Ω^1 is a free S-module with basis $\{dx_i\}$. The modules Der_S and Ω^1 inherit gradings from S: we say that a derivation or form has degree p if the coefficient of each $\frac{\partial}{\partial x_i}$ or dx_i is homogeneous of degree p.

When G is a u.g.g.r., the set of invariant derivations is a free S^G-module of dimension ℓ, and we call a set of homogeneous generators *basic derivations.* Basic derivations are not determined uniquely by G, but their degrees, called the *coexponents* of G, are. The coexponents are intimately connected with the invariant theory of G. If G acts irreducibly in V and hence in V^*, then the representation V^* occurs in S/I with multiplicity ℓ and in homogeneous components given by the coexponents. See [20, Chapter 6] for more on invariant theory and coexponents, especially for Shephard groups.

A differential 1-form ω is called *anti-invariant* if it is relatively invariant with respect to the $\det^{-1}$ character of G, i.e.,

$$g(\omega) = \det{}^{-1}(g)\ \omega,$$

for any g in G. Let $(\Omega^1)^{\det^{-1}}$ be the space of anti-invariant 1-forms. This space is a free S^G-module of rank ℓ.

We construct generators for $(\Omega^1)^{\det^{-1}}$ from a set of basic derivations. Let $n_1, \ldots, n_\ell$ be the coexponents for G and let $\theta_1, \ldots, \theta_\ell$ be a set of basic derivations with $\deg(\theta_i) = n_i$. We follow [23]. Let $\Omega^1(\mathcal{A})$ be the S-module of *logarithmic 1-forms* with poles along $\mathcal{A}$ (see [20]):

$$\Omega^1(\mathcal{A}) := \left\{ \frac{\eta}{Q} : \ \eta \in \Omega^1, \ d\left(\frac{\eta}{Q}\right) \in \frac{1}{Q}\Omega^2 \right\}$$

where d is the exterior differentiation and $\Omega^2 := \Omega^1 \bigwedge_S \Omega^1$ is the S-module of differential 2-forms. By [20, Theorem 6.59], $\theta_1, \ldots, \theta_\ell$ is an S-basis for the module $D(\mathcal{A})$ of $\mathcal{A}$-derivations,

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der_S} \mid \theta(Q) \in QS\}.$$

By the contraction $\langle\ ,\ \rangle$ of a 1-form and a derivation, the S-modules $D(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ are S-dual to each other [20, Theorem 4.75] . Let $\{\omega_1, \ldots, \omega_\ell\} \subset \Omega^1(\mathcal{A})$ be the basis of $\Omega^1(\mathcal{A})$ dual to $\{\theta_1, \ldots, \theta_\ell\}$: ω_i is the unique element of $\Omega^1(\mathcal{A})$ satisfying $\langle\theta_i, \omega_j\rangle = \delta_{ij}$ (Kronecker's delta). The group G acts naturally on $\Omega^1(\mathcal{A})$, and each ω_i is invariant since the $\theta_1, \ldots, \theta_\ell$ are invariant. Let $\mu_i := Q\omega_i$ for each i. Then since Q is anti-invariant, each μ_i is an anti-invariant 1-form of degree $(\deg Q - n_i)$. The following lemma is an application of [23, Thm. 1] or [22, Prop. 1].

Lemma 13. *Let G be a u.g.g.r. Then*

$$Q\ \Omega^1(\mathcal{A}) = (\Omega^1)^{\det^{-1}} \otimes_{S^G} S,$$

and hence

$$(\Omega^1)^{det^{-1}} = S^G\mu_1 \oplus \cdots \oplus S^G\mu_\ell.$$

The differential forms ω_i, and hence the μ_i, may be constructed explicitly as follows. Let M be the coefficient matrix of $\{\theta_1, \ldots, \theta_\ell\}$, i.e., the matrix whose (i, j) entry is the polynomial coefficient of $\frac{\partial}{\partial x_j}$ in θ_i. Using Saito's criterion, Terao showed that $\det(M) = Q$; see [20, Chapter 6]. Let w_{ij} be the (i, j) entry of M^{-1}. Then each w_{ij} is a rational function with denominator Q. For each i, ω_i is the rational differential form

$$\omega_i := \sum_{j=1}^{\ell} w_{ji}\ dx_j.$$

Lemma 14. *Let G be any u.g.g.r. and $f \in S$ a G-invariant of positive degree. Then $Q\frac{\partial f}{\partial x_i}$ lies in I for $i = 1, 2, \ldots, l$. In particular, if f_1 is a G-invariant of minimal positive degree, and K is the ideal generated by its first partial derivatives, then*

$$QK \subset I$$

and hence $K \subset \ker\phi$.

Proof. Let df be the exterior derivative of f. Since f is invariant, df is invariant, and hence Qdf is an anti-invariant 1-form. By Lemma 13, Qdf can thus be written as a combination of the μ_i with coefficients from S^G:

$$Qdf = h_1\mu_1 + \ldots + h_\ell\mu_\ell. \tag{5}$$

Since $\deg(Qdf) = \deg Q + \deg f - 1 > \deg Q - n_i = \deg \mu_i$ for each i, each h_i must have positive degree and thus lie in I. By comparing the coefficient of dx_i on each side of equation (5) above, we see that each $Q\frac{\partial f}{\partial x_i}$ is in I. $\square$

Proof of Theorem 2. By Lemma 14, we only need to show that $\ker\phi \subset K$. Assume for the sake of contradiction that f is in $\ker\phi$, i.e. Qf lies in I, but f is not in K. By Lemma 8, there exists some $f' \in S$ with $\overline{ff'} = \overline{H}$ in S/K. Consequently we have

$$\begin{aligned} H &= ff' + k \text{ for some } k \text{ in } K \\ QH &= Qff' + Qk. \end{aligned}$$

The left-hand side is J by equation (1). The right-hand side lies in I, because $QK \subset I$ by Lemma 14 and $Qf \in I$ by assumption. We conclude that J lies in I, contradicting Lemma 6. $\square$.

5. Remarks and questions

We conclude with some remarks and open questions.

5.1. **Coxeter complexes for u.g.g.r.'s.** Lemma 12 shows that for any Shephard group G and a distinguished set of generators R, the simplicial complex Δ has an alternate construction that parallels the construction of *Coxeter complexes* for Coxeter groups [11, §1.15]: it is the unique simplicial complex whose poset of faces is isomorphic to the poset of cosets of "standard parabolic" subgroups

$$P(G,R) := \{gG_J\}_{\substack{g\in G,\\ J\subset R}}$$

ordered by reverse inclusion.

This construction may be carried out more generally. Given any pair (G,R) of a group G and a finite set of generators R which is minimal with respect to inclusion, one can form the poset of cosets $P(G,R)$ as above. It is not always true that this is the face poset of an abstract simplicial complex. However, it *is* always the face poset of a *regular cell complex* $\Delta(G,R)$, in which all faces are isomorphic to simplices, but the intersection of a pair of faces need not be a face of each; see [2, 29] for more on such cell complexes.

The cell complex $\Delta(G,R)$ shares many of the pleasant properties of Coxeter complexes, and its homology carries representations of the group G. We always have the Hopf trace formula relating the virtual character from Theorem 1 (iii) to the alternating sum of the G-representations on the homology groups of $\Delta(G,R)$. Unlike the Coxeter or Shephard group cases, this homology need not be concentrated in a single dimension, so that this virtual character need not be a genuine character (even up to sign).

Question 15. *For u.g.g.r.'s G other than Coxeter or Shephard groups, do there exist minimal generating sets R for which the "Coxeter complex" $\Delta(G,R)$ carries homology representations related to the homology representations of Milnor fibers $f^{-1}(1)$ for some interesting G-invariant or relative-invariant f?*

We have investigated this a tiny bit for the presentations of u.g.g.r.'s given in [5] with inconclusive results.

5.2. **Shellability.** Corollary 11 suggests the following question.

Question 16. *Is Δ shellable for any regular complex polytope $\mathcal{P}$? More strongly, is the poset of faces of $\mathcal{P}$ lexicographically shellable?*

See [2] for the definitions of shellable and lexicographically shellable, and the relation to being Cohen-Macaulay.

Shellability of Δ is well-known for the Shephard groups which are also Coxeter groups, so one might try to resolve this question by appealing to the classification of the remaining Shephard groups.

For the infinite family associated with the Shephard group $G(r, 1, \ell)$, one of the two associated regular complex polytopes is Shephard's *generalized cross-polytope* β_ℓ^r [24, 9]. Its face poset (with top element removed) is easily seen to be the ℓ-fold Cartesian product of a poset having $r+1$ elements with one bottom element and the rest atoms. This makes it very easy to produce a lexicographic shelling, answering both questions affirmatively.

For Shephard groups with $\ell = 2$, Δ is a connected graph, hence trivially shellable. We have not checked whether the face poset of $\mathcal{P}$ is lexicographically shellable.

For the remaining three exceptional cases, we have checked using the computer algebra package GAP that the method used by Solomon and Tits to shell Tits buildings [7, Chapter IV, § 6] (ordering the maximal faces by any linear ordering that respects distance from the base chamber) seems not to give a shelling. We also have no candidate for a lexicographic shelling of the face poset of $\mathcal{P}$.

5.3. **Retractions.** Orlik and Solomon [17, 18] observed interesting and mysterious connections between the invariant theory for a Shephard group G having symbol $p_0[q_0]p_1[q_1]p_2 \cdots p_{\ell-2}[q_{\ell-2}]p_{\ell-1}$ and the "associated" Coxeter group W having symbol $2[q_0]2[q_1]2 \cdots 2[q_{\ell-2}]2$. We hypothesize further connections between the Coxeter complexes of G and W.

Let Δ_G be the simplicial complex Δ which was associated to G in Section 2, and let Δ_W be the corresponding complex associated to W (that is, the *Coxeter complex* of W). In the special case of the infinite family of Shephard groups $G = G(r, 1, \ell)$, it is not hard to see that there are many well-defined simplicial inclusions and retractions

$$\iota : \Delta_W \hookrightarrow \Delta_G$$
$$\rho : \Delta_G \twoheadrightarrow \Delta_W$$

satisfying $\rho \circ \iota = \mathrm{id}_{\Delta_W}$ which preserve the "coloring" of vertices by the distinguished generators of each group (this coloring assigns the color i to vertices of Δ which correspond to i-dimensional faces in the regular polytope, or equivalently, to those which correspond to cosets of the form $gG_{R-\{r_i\}}$).

We give an example of such an inclusion-retraction pair using Shephard's generalized cross-polytope β_ℓ^r and the usual cross-polytope β_ℓ^2 as the regular complex polytopes associated to G and W, respectively. In this case, the maps can be determined from their restriction to 0-faces in the associated polytopes. A typical 0-face in the generalized cross-polytope is $\omega^k e_i$ with $\omega = e^{\frac{2\pi i}{r}}$, $0 \le k \le r-1$, $1 \le i \le \ell$. Define

$$\iota(+e_i) = e_i = \omega^0 e_i,$$
$$\iota(-e_i) = \omega^1 e_i,$$

$$\rho(\omega^k e_i) = \begin{cases} +e_i & \text{if } k = 0 \\ -e_i & \text{if } k > 0. \end{cases}$$

Question 17. *Can similar inclusions and retractions be defined for any Shephard group G and associated Coxeter group W?*

One motivation for this question comes from the Weyl group W of a *finite reductive group* G' (see [6, Introduction]). Such groups G' have a BN-pair structure which gives rise to a simplicial complex $\Delta_{G',B,N}$ known as a *Tits building* [7, 10]. The Tits building has many subcomplexes isomorphic to Δ_W (called *apartments*) and there is a *canonical retraction* $\Delta_{G,B,N} \twoheadrightarrow \Delta_W$ onto any apartment that preserves the natural coloring of vertices by the Coxeter generators of W. A positive answer to Question 17 would provide further support for the following analogy:

Weyl group W	Shephard group G	finite reductive group G'
Coxeter system (W, R)	"Shephard system" (G, R)	BN-pair (G', B, N)
sign character	Theorem 1's representation	Steinberg representation
Coxeter complex Δ_W	"Coxeter complex" Δ_G	Tits building $\Delta_{G',B,N}$

Furthermore, whenever a retraction as in Question 17 exists, one can fill in the question mark in the following diagram of simplicial retractions

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & \Delta_G \arrow[d] \\ \Delta_{G',B,N} \arrow[r] & \Delta_W \end{tikzcd}$$

with a simplicial complex defined using the usual pullback construction. We hypothesize that this pullback complex plays the role of the Tits building for the yet-to-be-defined *spetses* investigated by Broué, Malle, and Michel [6]. By analogy to groups with BN-pair, perhaps one can define the spetses to be the group of vertex-color-preserving simplicial automorphisms of this complex?

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