DEFORMING GROUP ACTIONS ON KOSZUL ALGEBRAS

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Abstract. We give Braverman-Gaitsgory style conditions for general PBW deformations of skew group algebras formed from finite groups acting on Koszul algebras. When the characteristic divides the order of the group, this includes deformations of the group action as well as of the Koszul relations. Techniques involve a twisted product resolution and the Gerstenhaber bracket on Hochschild cohomology expressed explicitly for cocycles on this resolution.

1. Introduction

Braverman and Gaitsgory [2] gave conditions for an algebra to be a PBW deformation of a Koszul algebra. Etingof and Ginzburg [4] adapted these conditions to the setting of a Koszul ring over a semisimple group ring $\mathbb{C}G$ using results of Beilinson, Ginzburg, and Soergel [1] in order to study symplectic reflection algebras. These are certain kinds of deformations of a skew group algebra $\mathbb{C}[x_1, \dots, x_n] \rtimes G$ that preserve a symplectic group action. Drinfeld [3] considered similar deformations of $\mathbb{C}[x_1, \dots, x_n] \rtimes G$ for an arbitrary finite group G acting linearly.

We showed in [14] how to adapt the techniques of Braverman and Gaitsgory to an algebra defined over a group ring kG that is not necessarily semisimple. This approach aids exploration of deformations of skew group algebras of the form $S \rtimes G$ for any Koszul algebra S and any finite group G. In [14], we examined deformations preserving the action of G on the Koszul algebra S. However, other types of deformations are possible, some arising only in the modular setting, where the characteristic of the underlying field k divides the order of G. Here, we study deformations of $S \rtimes G$ that deform not only the generating relations of the Koszul algebra S but also deform the action of G on S. This construction recollects the graded affine Hecke algebras of Lusztig [11], in which a group action is deformed; in the nonmodular setting, these were shown by Ram and the first author [12] to be isomorphic to Drinfeld's deformations.

We show how PBW deformations of algebras of the form $S \rtimes G$ for S a Koszul algebra and G a finite group are determined by conditions in Hochschild cohomology using a new twisted resolution $X_{\bullet}$ of $S \rtimes G$. We summarize Theorem 5.3:

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Theorem. *Suppose G is a finite group acting on a Koszul algebra S by graded automorphisms. A quadratic algebra over kG is a PBW deformation of $S \rtimes G$ if and only if its relations are parameterized by functions α, β, λ satisfying*

(a) $d^*(\alpha + \lambda) = 0$,
(b) $[\alpha + \lambda, \alpha + \lambda] = 2d^*\beta$, *and*
(c) $[\lambda + \alpha, \beta] = 0$,

for α, β, λ identified with cochains on a twisted product resolution $X_\bullet$ of $S \rtimes G$.

The twisted product resolution in the theorem statement was constructed by Guccione, Guccione, and Valqui [8] and adapted in [14] to $S \rtimes G$. The differentials d^* and the Gerstenhaber bracket $[\ ,\ ]$ in the statement are defined in Section 5, and the parameter functions α, β, γ are defined in Section 2. Theorem 5.3 implies our main Theorem 2.5, giving explicit PBW conditions. In the nonmodular setting, a simpler resolution suffices, one that is induced directly from the Koszul resolution of S itself. The twisted product resolution $X_\bullet$ we use here partitions homological information according to type; cochains corresponding to deformations of the group action and to deformations of the Koszul relations live on two distinct parts of the resolution. Conditions for PBW deformations include interaction among the parts.

The results here apply to many algebras of interest that arise as deformations of algebras of the form $S \rtimes G$. For example, one might take S to be the symmetric algebra (polynomial ring) $S(V)$ on a finite dimensional vector space V, or a skew (quantum) polynomial ring $S_{\mathbf{q}}(V)$ with multiplication skewed by a tuple $\mathbf{q} = (q_{ij})$ of scalars, or a skew exterior algebra, or even the Jordan plane or a Sklyanin algebra.

Every deformation of an algebra defines a Hochschild 2-cocycle of that algebra. A central question in deformation theory asks: *which cocycles may be lifted to deformations?* We use homological techniques to answer this question in our context: We show in Theorem 2.5 that obstructions to lifting cocycles on algebras of the form $S \rtimes G$ correspond to concrete conditions on parameter functions defining potential PBW deformations. Such deformations are filtered algebras with associated graded algebra precisely $S \rtimes G$. Our theorem generalizes [14, Theorem 5.4] to include deformations of the group action.

We obtain explicit conditions in the special case that the Koszul algebra S is a polynomial ring in Theorem 6.1, generalizing [15, Theorem 3.1]. This result may also be proven directly via the Composition-Diamond Lemma, used by Khare [9, Theorem 27] for deformations of the action of a cocommutative algebra on a polynomial ring. An advantage of our approach is that it yields conditions much more generally for all Koszul algebras.

When the characteristic does not divide the group order, we strengthen [15, Theorem 4.1] by showing in Theorem 7.1 that a deformation of the group action and Koszul relations together is isomorphic to one in which *only* the Koszul relations are deformed. We give an example to show that Theorem 7.1 is false in the modular setting.

Let k be any field. We assume the characteristic of k is not 2 throughout to make some results easier to state. All tensor products are over k unless otherwise indicated, that is, $\otimes = \otimes_k$. We assume that in each graded or filtered k-algebra, elements of k have degree 0.

2. PBW Deformations of Koszul algebras twisted by groups

In this section, we recall some definitions and state our main result giving Braverman-Gaitsgory style conditions for PBW deformations. The proof will be given in Section 5 after we recall and develop the needed homological algebra.

PBW deformations. Let $\mathbf{k}$ be a k-algebra (for example, the field k itself or a group ring kG) and let U be a finitely generated $\mathbf{k}$-bimodule. An element of the tensor algebra $T_{\mathbf{k}}(U)$ has *filtered degree* d if it lies in the d-th filtered piece $\oplus_{i\leq d}\,(U)^{\otimes_{\mathbf{k}} i}$ of $T_{\mathbf{k}}(U)$ but not in the $(d+1)$-st.

Consider a finitely generated filtered $\mathbf{k}$-algebra $\mathcal{H}$, so that we may write $\mathcal{H} = T_{\mathbf{k}}(U)/(P)$ for some finitely generated $\mathbf{k}$-bimodule U and ideal (P) generated by a subset P of $T_{\mathbf{k}}(U)$. Note that elements of P may be nonhomogeneous with respect to the grading on the free algebra $T_{\mathbf{k}}(U)$ with U in degree 1.

We associate to any presentation of a filtered algebra a homogenous version,

$$\mathrm{HomogeneousVersion}\big(T_{\mathbf{k}}(U)/(P)\big) = T_{\mathbf{k}}(U)/(R),$$

where $R = \cup_d \{\pi_d(p) : p \in P \text{ of filtered degree } d\}$ and $\pi_d : T_{\mathbf{k}}(U) \to (U)^{\otimes_{\mathbf{k}} d}$ projects onto the homogeneous component of degree d.

We say that a filtered algebra $\mathcal{H}$ with a given presentation is a *PBW deformation* of its homogeneous version if it has the *PBW property*, i.e., the associated graded algebra of $\mathcal{H}$ coincides with the homogeneous version:

$$\mathrm{Gr}(\mathcal{H}) \cong \mathrm{HomogeneousVersion}(\mathcal{H}) \qquad \text{as graded algebras.}$$

Given a fixed presentation in terms of generators and relations, we often merely say that $\mathcal{H}$ is a PBW deformation. This terminology originated from the Poincaré-Birkhoff-Witt Theorem, which states that the associated graded algebra of the universal enveloping algebra of a Lie algebra is its homogeneous version, namely, a polynomial ring.

Remark 2.1. The reader is cautioned that authors use the adjective *PBW* in slightly different ways. For example, in Braverman-Gaitsgory [2] and also in [14], the homogeneous version of a filtered *quadratic* algebra is defined by projecting every generating relation onto its degree 2 part, instead of its highest homogeneous part. This merely means that filtered relations of degree 1 must be considered separately in PBW theorems there.

Group twisted quadratic algebras. Let S be a graded quadratic k-algebra generated by some finite dimensional k-vector space V (in degree 1), that is, S is determined by relations R, where R is some k-subspace of $V \otimes V$:

$$S = T_k(V)/(R)\,.$$

Let G be a finite group acting by graded automorphisms on S. This is equivalent to G acting linearly on V with the relations R preserved set-wise (so R is a kG-module). We denote the action of g in G on v in V by ${}^g v$ in V. The *skew group algebra* (or semidirect product algebra) $S \rtimes G$ (also written $S\#G$) is the k-algebra generated by the group algebra kG and the vector space V subject to the relations given by R together with the relations $gv - {}^g v g$ for g in G and v in V. We identify $S \rtimes G$ with a filtered algebra over the ring $\mathbf{k} = kG$ generated by $U = kG \otimes V \otimes kG$:

$$S \rtimes G \cong T_{kG}(kG \otimes V \otimes kG)/(R \cup R')$$

as graded algebras, where elements of G have degree 0 and elements of V have degree 1, and where

$$R' = \mathrm{Span}_k\{g \otimes v \otimes 1 - 1 \otimes {}^g v \otimes g : v \in V,\ g \in G\} \subset kG \otimes V \otimes kG\,. \tag{2.2}$$

Here we identify $R \subset V \otimes V$ with a subspace of

$$k \otimes V \otimes k \otimes V \otimes k \subset kG \otimes V \otimes kG \otimes V \otimes kG \cong (kG \otimes V \otimes kG) \otimes_{kG} (kG \otimes V \otimes kG).$$

PBW deformations of group twisted quadratic algebras. Now suppose $\mathcal{H}$ is a PBW deformation of $S \rtimes G$. Then $\mathcal{H}$ is generated by kG and V subject to nonhomogeneous relations of degrees 2 and 1 of the form

$$\begin{aligned} P &= \{r - \alpha(r) - \beta(r) : r \in R\} \quad \text{and} \\ P' &= \{r' - \lambda(r') : r' \in R'\} \end{aligned}$$

for some k-linear parameter functions

$$\alpha : R \to V \otimes kG,\ \ \beta : R \to kG,\ \ \lambda : R' \to kG\,.$$

That is, $\mathcal{H}$ can be realized as the quotient

$$\mathcal{H} = T_{kG}(kG \otimes V \otimes kG)/(P \cup P').$$

Note we may assume that α takes values in $V \otimes kG \cong k \otimes V \otimes kG$, rather than more generally in $kG \otimes V \otimes kG$, without changing the k-span of $P \cup P'$, since the relations P' allow us to replace elements in $kG \otimes V \otimes kG$ with those in $k \otimes V \otimes kG$.

In our main theorem below, we determine which such quotients define PBW deformations of $S \rtimes G$ in case S is a Koszul algebra. (We recall a definition of Koszul algebra in Section 4.) For the statement of the theorem, we first need some notation for decomposing any functions α, β, λ as above. We identify $\lambda : R' \to kG$

with the function (of the same name) $\lambda : kG \otimes V \to kG$ mapping $g \otimes v$ to $\lambda(g \otimes v \otimes 1 - 1 \otimes {}^g v \otimes g)$ for all g in G and v in V. We write

$$\alpha(r) = \sum_{g \in G} \alpha_g(r) g, \quad \beta(r) = \sum_{g \in G} \beta_g(r) g, \quad \lambda(h \otimes v) = \sum_{g \in G} \lambda_g(h \otimes v) g$$

for functions $\alpha_g : R \to V$, $\beta_g : R \to k$, $\lambda_g : kG \otimes V \to k$ (identifying V with $V \otimes k$ in $V \otimes kG$). Write $\lambda(g \otimes -) : V \to kG$ for the function induced from λ by fixing g in G. Let $m : kG \otimes kG \to kG$ be multiplicaton on kG and let $\sigma : kG \otimes V \to V \otimes kG$ be the twist map given by

$$\sigma(g \otimes v) = {}^g v \otimes g \quad \text{for } g \text{ in } G,\ v \text{ in } V. \tag{2.3}$$

For the statement of the theorem, we set

(2.4)
$$\mathcal{H}_{\lambda,\alpha,\beta} = T_{kG}(kG \otimes V \otimes kG) \ / \ \big(r - \alpha(r) - \beta(r),\ r' - \lambda(r') : r \in R,\ r' \in R'\big)$$

for linear parameter functions $\alpha : R \to V \otimes kG$, $\beta : R \to kG$, $\lambda : R' \to kG$ and for R the space of quadratic relations and R' the space of group action relations (2.2). The functions α and β are extended uniquely to right kG-module homomorphisms from $R \otimes kG$ to $V \otimes kG$ and kG, respectively.

Theorem 2.5. *Let G be a finite group and let V be a kG-module. Let $S = T_k(V)/(R)$ be a Koszul algebra with R a kG-submodule of $V \otimes V$. Then a filtered algebra $\mathcal{H}$ is a PBW deformation of $S \rtimes G$ if and only if*

$$\mathcal{H} \cong \mathcal{H}_{\lambda,\alpha,\beta}$$

for linear parameter functions $\alpha : R \to V \otimes kG,\ \beta : R \to kG,\ \lambda : kG \otimes V \to kG$ satisfying

(1) $1 \otimes \lambda - \lambda(m \otimes 1) + (\lambda \otimes 1)(1 \otimes \sigma) = 0,$
(2) $\lambda(\lambda \otimes 1) - \lambda(1 \otimes \alpha) = (1 \otimes \beta) - (\beta \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1),$
(3) $(1 \otimes \alpha) - (\alpha \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1) = \lambda \otimes 1 + (1 \otimes \lambda)(\sigma \otimes 1),$
(4) $\alpha((1 \otimes \sigma)(\alpha \otimes 1) - 1 \otimes \alpha) + \sum_{g \in G} \alpha_g \otimes \lambda(g \otimes -) = 1 \otimes \beta - \beta \otimes 1,$
(5) $\beta((1 \otimes \sigma)(\alpha \otimes 1) - 1 \otimes \alpha) = -\lambda(\beta \otimes 1),$
(6) $\alpha \otimes 1 - 1 \otimes \alpha = 0,$

upon projection of images of the maps to $S \rtimes G$. Here, the map in (1) *is defined on $kG \otimes kG \otimes V$, the maps in* (2) *and* (3) *are defined on $kG \otimes R$, the map in* (6) *is defined on $(V \otimes R) \cap (R \otimes V) \subset V \otimes V \otimes V$, and* (6) *implies that the maps in* (4) *and* (5) *are also defined on $(V \otimes R) \cap (R \otimes V)$.*

We will prove the theorem in Section 5 as a corollary of Theorem 5.3, after first developing some homological algebra in Sections 3 and 4.

The theorem above includes the case of filtered quadratic algebras defined over the ring kG instead of the field k. Such algebras preserve the action of kG and correspond to the case $\lambda = 0$ in the theorem above. We recover a result from [14] which we rephrase below to highlight the role of the twisting map σ. The theorem was developed to provide tools particularly in the case that kG is not semsimple.

Note that the action of G on itself by conjugation induces an action on the parameter functions α and β (with $({}^g\alpha)(r) = {}^g(\alpha({}^{g^{-1}}r))$ as usual and ${}^g(v \otimes h) = {}^g v \otimes ghg^{-1}$ for r in R, g in G, and v in V).

Theorem 2.6. [14, Theorem 5.4] *Let G be a finite group and let V be a kG-module. Let $S = T_k(V)/(R)$ be a Koszul algebra for which R is a kG-submodule of $V \otimes V$. Then a filtered quadratic algebra $\mathcal{H}$ is a PBW deformation of $S \rtimes G$ preserving the action of G if and only if*

$$\mathcal{H} \cong \mathcal{H}_{0,\alpha,\beta}$$

for some G-invariant linear parameter functions $\alpha : R \to V \otimes kG$, $\beta : R \to kG$ satisfying, upon projection to $S \rtimes G$,

(i) $\alpha \otimes 1 - 1 \otimes \alpha = 0$,
(ii) $\alpha((1 \otimes \sigma)(\alpha \otimes 1) - 1 \otimes \alpha) = 1 \otimes \beta - \beta \otimes 1,$
(iii) $\beta((1 \otimes \sigma)(\alpha \otimes 1) - 1 \otimes \alpha) = 0.$

Here, the map in (i) is defined on $(V \otimes R) \cap (R \otimes V)$, and (i) implies that the maps in (ii) and (iii) are also defined on $(V \otimes R) \cap (R \otimes V)$.

Proof. The additional hypothesis, that the action of G is preserved in the deformation, is equivalent to setting $\lambda = 0$ in Theorem 2.5. In this case, Condition (1) of Theorem 2.5 is vacuous, and Conditions (2) and (3) are equivalent to G-invariance of α and β. Conditions (4), (5), (6) become Conditions (ii), (iii), (i) here, respectively. □

Remark 2.7. The conditions of the above theorems generalize those of Braverman and Gaitsgory [2, Lemma 3.3] from Koszul algebras S to skew group algebras $S \rtimes G$. Their Condition (I) corresponds to our Conditions (1), (2), and (3) in Theorem 2.5; these conditions limit the possible relations of filtered degree 1. The nonmodular case can be proven using the theory of Koszul rings over the semisimple ring kG, as in [4]. In the modular case, when $\mathrm{char}(k)$ divides $|G|$, we found in [14] that more complicated homological information is required to obtain PBW conditions using this approach.

3. Deformations

In this section, we recall the general theory of deformations and Hochschild cohomology that we will need and show how it applies to the algebras $\mathcal{H}_{\lambda,\alpha,\beta}$ of Theorem 2.5.

Recall that for any k-algebra A, the Hochschild cohomology of an A-bimodule M in degree n is

$$\mathrm{HH}^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M),$$

where $A^e = A \otimes A^{op}$ is the enveloping algebra of A, and the bimodule structure of M defines it as an A^e-module. In the case that $M = A$, we abbreviate $\mathrm{HH}^n(A) = \mathrm{HH}^n(A, A)$.

Bar and reduced bar resolutions. Hochschild cohomology can be defined using the *bar resolution*, that is, the free resolution of the A^e-module A given as:

$$\cdots \xrightarrow{\delta_3} A \otimes A \otimes A \otimes A \xrightarrow{\delta_2} A \otimes A \otimes A \xrightarrow{\delta_1} A \otimes A \xrightarrow{\delta_0} A \to 0,$$

where

$$\delta_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \tag{3.1}$$

for all $n \geq 0$ and $a_0, \ldots, a_{n+1} \in A$. If A is an $\mathbb{N}$-graded algebra, then each tensor power of A is canonically a graded A-bimodule. The Hochschild cohomology of A inherits this grading from the bar resolution and thus is bigraded: $\mathrm{HH}^i(A) = \bigoplus_j \mathrm{HH}^{i,j}(A)$ with $\mathrm{HH}^{i,j}(A)$ the subspace consisting of homogeneous elements of graded degree j as maps. For our arguments, we will need to use the *reduced bar resolution*, which replaces the A^e-module $A^{\otimes(n+2)}$, for each n, by its vector space quotient $A \otimes (\overline{A})^{\otimes n} \otimes A$, where $\overline{A} = A/k$ (the vector space quotient by all scalar multiples of the multiplicative identity 1_A in A). The differentials on the bar resolution factor through these quotients to define differentials for the reduced bar resolution, and we will use the same notation δ_n for these.

Deformations. A *deformation of A over $k[t]$* is an associative $k[t]$-algebra A_t with underlying vector space $A[t]$ such that $A_t|_{t=0} \cong A$ as algebras. The product $*$ on a deformation A_t of A is determined by its values on pairs of elements of A,

$$a_1 * a_2 = a_1 a_2 + \mu_1(a_1 \otimes a_2)t + \mu_2(a_1 \otimes a_2)t^2 + \cdots \tag{3.2}$$

where $a_1 a_2$ is the product of a_1 and a_2 in A and each $\mu_j : A \otimes A \to A$ is some k-linear map (called the *j-th multiplication map*) extended to be linear over $k[t]$. (We require that only finitely many terms in the above expansion for each pair a_1, a_2 are nonzero.) We may (and do) assume that 1_A is the multiplicative identity with respect to the multiplication $*$ of A_t. (Each deformation is equivalent to one with 1_A serving as the multiplicative identity; see [7, p. 43].)

We identify the maps μ_i with 2-cochains on the reduced bar resolution using the canonical isomorphism $\mathrm{Hom}_k(\overline{A} \otimes \overline{A}, A) \cong \mathrm{Hom}_{A^e}(A \otimes \overline{A} \otimes \overline{A} \otimes A, A)$. (Our assumptions imply that the value of μ_i is 0 if either argument is the multiplicative identity of A.) We will use the same notation for elements of A and $\overline{A}$ when no confusion will arise.

Associativity of the multiplication $*$ implies certain conditions on the maps μ_i which are elegantly phrased in [6] in terms of the differential δ and the Gerstenhaber bracket [,], as we explain next. The *Gerstenhaber bracket* for 2-cochains

ξ, ν on the (reduced) bar resolution is the 3-cochain defined by

$$\begin{aligned}[\xi,\nu](a_1\otimes a_2\otimes a_3) &= \xi(\nu(a_1\otimes a_2)\otimes a_3)-\xi(a_1\otimes\nu(a_2\otimes a_3))\\ &\quad +\nu(\xi(a_1\otimes a_2)\otimes a_3)-\nu(a_1\otimes\xi(a_2\otimes a_3))\end{aligned} \tag{3.3}$$

for all $a_1, a_2, a_3 \in A$. See [5] for the definition of Gerstenhaber bracket in other degrees.

Obstructions. If A_t is a deformation of a k-algebra A over $k[t]$, associativity of multiplication $*$ implies in particular (see [6]) that

$$\delta_3^*(\mu_1) = 0 \qquad (\mu_1 \text{ is a Hochschild 2-cocycle}), \tag{3.4}$$

$$\delta_3^*(\mu_2) = \tfrac{1}{2}[\mu_1, \mu_1] \qquad (\text{the first obstruction vanishes}), \text{ and} \tag{3.5}$$

$$\delta_3^*(\mu_3) = [\mu_1, \mu_2] \qquad (\text{the second obstruction vanishes}). \tag{3.6}$$

Here, δ_3^* denotes the map from $\operatorname{Hom}_{A^e}(A\otimes\overline{A}^{\otimes 2}\otimes A, A)$ to $\operatorname{Hom}_{A^e}(A\otimes\overline{A}^{\otimes 3}\otimes A, A)$ induced by δ_3, and we have identified μ_1, μ_2, μ_3 with functions on $A\otimes\overline{A}^{\otimes 2}\otimes A$, as described above. Associativity of the multiplication $*$ also implies that higher degree "obstructions" vanish, i.e., it forces necessary conditions on all the μ_j. We will only need to look closely at the above beginning obstructions: Higher degree obstructions relevant to our setting will automatically vanish because of the special nature of Koszul algebras (see the proof of Theorem 5.3).

Graded deformations. Assume that the k-algebra A is $\mathbb{N}$-graded. Extend the grading on A to $A[t]$ by setting $\deg(t) = 1$. A *graded deformation of A over $k[t]$* is a deformation of A over $k[t]$ that is graded, i.e., each map $\mu_j : A \otimes A \to A$ is homogeneous of degree $-j$. An *i-th level graded deformation of A* is a deformation over $k[t]/(t^{i+1})$, i.e., an algebra A_i with underlying vector space $A[t]/(t^{i+1})$ and multiplication as in (3.2) in which terms involving powers of t greater than i are 0. An i-th level graded deformation A_i of A *lifts* (or *extends*) to an $(i+1)$-st level graded deformation A_{i+1} if the j-th multiplication maps of A_i and A_{i+1} coincide for all $j \le i$.

We next point out that the algebra $\mathcal{H}_{\lambda,\alpha,\beta}$ defined in (2.4) gives rise to a graded deformation of $S \rtimes G$ in case it has the PBW property.

Proposition 3.7. *If $\mathcal{H}_{\lambda,\alpha,\beta}$ is a PBW deformation of $S \rtimes G$, then $\mathcal{H}_{\lambda,\alpha,\beta}$ is the fiber of a deformation of $A = S \rtimes G$:*

$$\mathcal{H}_{\lambda,\alpha,\beta} \cong A_t\big|_{t=1}$$

as filtered algebras, for A_t a graded deformation of $S \rtimes G$ over $k[t]$.

Proof. We define the algebra A_t by

$$A_t = T_{kG}(kG\otimes V\otimes kG)[t]/(r-\alpha(r)t-\beta(r)t^2,\ r'-\lambda(r')t : r\in R, r'\in R').$$

We claim that A_t and $(S \rtimes G)[t]$ are isomorphic as $k[t]$-modules. To see this, consider the natural quotient map

$$T_{kG}(kG \otimes V \otimes kG) \to T_{kG}(kG \otimes V \otimes kG)/(R \cup R') \cong S \rtimes G$$

and let ι from $S \rtimes G$ to $T_{kG}(kG \otimes V \otimes kG)$ be its natural k-linear section. Composing ι with the quotient map onto $\mathcal{H}_{\lambda,\alpha,\beta}$ yields an isomorphism of filtered vector spaces since $\mathcal{H}_{\lambda,\alpha,\beta}$ has the PBW property. Now extend ι to a $k[t]$-module homomorphism from $S \rtimes G[t]$ to $T_{kG}(kG \otimes V \otimes kG)[t]$. The composition of this map with the quotient map onto A_t can be seen to be an isomorphism of vector spaces by the above discussion and a degree argument. The rest of the proof is a straightforward generalization of the proof of [15, Proposition 6.5], which is the case $S = S(V)$ and $\alpha = 0$. Here, r replaces $v \otimes w - w \otimes v$ and the first and second multiplication maps μ_1 and μ_2 satisfy

$$\begin{aligned}
&\lambda(g \otimes v \otimes 1 - 1 \otimes {}^g v \otimes g) = \mu_1(g \otimes v) - \mu_1({}^g v \otimes g),\\
&\alpha(r) = \mu_1(r), \quad \text{and} \quad \beta(r) = \mu_2(r)
\end{aligned}$$

for all g in G, v in V, and r in R. □

4. Hochschild cohomology of group twisted Koszul algebras

We will look more closely at the Hochschild 2-cocycle condition (3.4) and the obstructions (3.5) and (3.6) in the case that A is a group twisted quadratic algebra $S \rtimes G$. A convenient resolution for this purpose was introduced by Guccione, Guccione, and Valqui [8]. We now recall a definition of a Koszul algebra and from [14] a modified version of this construction for Koszul algebras.

Twisted product resolution. Let S be a quadratic algebra with finite dimensional generating k-vector space V and subspace of relations $R \subset V \otimes V$:

$$S = T_k(V)/(R)\,.$$

Recall that S is a *Koszul algebra* if the complex

$$\cdots \to K_3 \xrightarrow{d_3} K_2 \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0 \xrightarrow{d_0} S \to 0$$

is a free S^e-resolution of S, where $K_n = S \otimes \tilde{K}_n \otimes S$ with $\tilde{K}_0 = k$, $\tilde{K}_1 = V$, and

$$\tilde{K}_n = \bigcap_{j=0}^{n-2} (V^{\otimes j} \otimes R \otimes V^{\otimes(n-2-j)}), \quad n \geq 2,$$

and the differential is restricted from that of the (reduced) bar resolution of S, defined in (3.1), so that $d_n = \delta_n|_{K_n}$. (We have identified K_0 with $S \otimes S$. This definition of a Koszul algebra is equivalent to several more standard definitions; see, e.g. [10].)

Let G be finite group acting by graded automorphisms on S and set $A = S \rtimes G$. The *twisted product resolution* $X_\bullet$ of A as an A^e-module is the total complex of the double complex $X_{\bullet,\bullet}$, where

$$X_{i,j} = A \otimes (\overline{kG})^{\otimes i} \otimes \tilde{K}_j \otimes A, \tag{4.1}$$

and A^e acts by left and right multiplication on the outermost tensor factors A:

$$\begin{tikzcd}
X_{0,3} \arrow[d, "d^v_{0,3}"] & \vdots \arrow[d] & \vdots \arrow[d] & \\
X_{0,2} \arrow[d, "d^v_{0,2}"] & X_{1,2} \arrow[l, "d^h_{1,2}"'] \arrow[d, "d^v_{1,2}"] & X_{2,2} \arrow[l, "d^h_{2,2}"'] \arrow[d, "d^v_{2,2}"] & \cdots \arrow[l] \\
X_{0,1} \arrow[d, "d^v_{0,1}"] & X_{1,1} \arrow[l, "d^h_{1,1}"'] \arrow[d, "d^v_{1,1}"] & X_{2,1} \arrow[l, "d^h_{2,1}"'] \arrow[d, "d^v_{2,1}"] & \cdots \arrow[l] \\
X_{0,0} & X_{1,0} \arrow[l, "d^h_{1,0}"'] & X_{2,0} \arrow[l, "d^h_{2,0}"'] & \cdots \arrow[l]
\end{tikzcd}$$

To define the differentials, we first identify each $X_{i,j}$ with a tensor product over A (see [14, Section 4]),

$$X_{i,j} \cong (A \otimes (\overline{kG})^{\otimes i} \otimes kG) \otimes_A (S \otimes \tilde{K}_j \otimes A), \tag{4.2}$$

where the right action of A on $A \otimes (\overline{kG})^{\otimes i} \otimes kG$ is given by

$$(a \otimes g_1 \otimes \cdots \otimes g_i \otimes g_{i+1})sh = a({}^{g_1 \cdots g_{i+1}}s) \otimes g_1 \otimes \cdots \otimes g_i \otimes g_{i+1}h$$

and the left action of A on $S \otimes \tilde{K}_j \otimes A$ is given by

$$sh(s' \otimes x \otimes a) = s({}^h s') \otimes {}^h x \otimes ha$$

for all $g_1, \ldots, g_{i+1}, h$ in G, s, s' in S, and a in A. (We have suppressed tensor symbols in writing elements of A to avoid confusion with tensor products defining the resolution.) The horizontal and vertical differentials on the bicomplex $X_{\bullet,\bullet}$, given as a tensor product over A via (4.2), are then defined by $d^h_{i,j} = d_i \otimes 1$ and $d^v_{i,j} = (-1)^i \otimes d_j$, respectively, where the notation d is used for both the differential on the reduced bar resolution of kG (induced to an $A \otimes (kG)^{op}$-resolution) and on the Koszul resolution of S (induced to an $S \otimes A^{op}$-resolution). Setting $X_n = \oplus_{i+j=n} X_{i,j}$ for each $n \geq 0$ yields the total complex $X_\bullet$:

$$\cdots \to X_2 \to X_1 \to X_0 \to A \to 0, \tag{4.3}$$

with differential in positive degrees n given by $d_n = \sum_{i+j=n}(d^h_{i,j} + d^v_{i,j})$, and in degree 0 by the multiplication map. By [14, Theorem 4.3], $X_\bullet$ is a free resolution of the A^e-module $A = S \rtimes G$.

Chain maps between reduced bar and twisted product resolutions. We found in [14] useful chain maps converting between the bar resolution and the (nonreduced) twisted product resolution $X_\bullet$ of $A = S \rtimes G$. We next extend [14, Lemma 4.7], adding more details and adapting it for use with the reduced bar resolution. See also [15, Lemma 7.3] for the special case $S = S(V)$. We consider elements of $\tilde{K}_j \subset V^{\otimes j}$ to have graded degree j and elements of $(\overline{kG})^{\otimes i}$ to have graded degree 0.

Lemma 4.4. *For $A = S \rtimes G$, there exist morphisms of complexes of graded A-bimodules,*

$$\phi_n : X_n \to A \otimes \overline{A}^{\otimes n} \otimes A \quad \textit{and} \quad \psi_n : A \otimes \overline{A}^{\otimes n} \otimes A \to X_n\,,$$

i.e., A-bimodule homomorphisms ϕ_n and ψ_n such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & X_2 \arrow[r, "d_2"] \arrow[d, shift right, "\phi_2"'] & X_1 \arrow[r, "d_1"] \arrow[d, shift right, "\phi_1"'] & X_0 \arrow[r, "d_0"] \arrow[d, shift right, "\phi_0"'] & A \arrow[r] \arrow[d, shift right] & 0 \\
\cdots \arrow[r] & A \otimes \overline{A} \otimes \overline{A} \otimes A \arrow[r, "\delta_2"] \arrow[u, shift right, "\psi_2"'] & A \otimes \overline{A} \otimes A \arrow[r, "\delta_1"] \arrow[u, shift right, "\psi_1"'] & A \otimes A \arrow[r, "\delta_0"] \arrow[u, shift right, "\psi_0"'] & A \arrow[r] \arrow[u, shift right, "="'] & 0
\end{tikzcd}$$

commutes, the maps ϕ_n, ψ_n are of each of graded degree 0, and $\psi_n\phi_n$ is the identity map on X_n for all $n \geq 0$.

Proof. We will not need in fact the general statement of the lemma in this paper, but rather some of the explicit values of the maps in low degrees. These we give in this proof and in an additional lemma; here we show explicitly that $\psi_n\phi_n$ is the identity map for $n = 0, 1, 2$ and indicate a technique for proving the general statement. Alternatively, the general statement may be proven by taking advantage of properties of twisted product resolutions: Chain maps between the Koszul resolution and the bar resolution exist for which the composition is the identity map on the Koszul resolution; we compose these chain maps with an isomorphism between the reduced bar resolution of $S \rtimes G$ and the twisted product resolution of two other reduced bar resolutions, one for S and one for kG, to obtain the desired morphisms in the statement of the lemma.

We again suppress tensor symbols in writing elements of A to avoid confusion with tensor products defining the resolution. In degree 0, ψ_0 and ϕ_0 may be chosen to be identity maps on $A \otimes A$. As in [14, Lemma 4.7], we may set

$$\phi_1(1 \otimes g \otimes 1) = 1 \otimes g \otimes 1 \ \ (\text{on } X_{1,0}), \qquad \phi_1(1 \otimes v \otimes 1) = 1 \otimes v \otimes 1 \ \ (\text{on } X_{0,1}),$$

for all nonidentity g in G and v in V, and these values determine ϕ_1 as an A-bimodule map. Moreover, we set

$$\begin{aligned}\psi_1(1\otimes g\otimes 1) &= 1\otimes g\otimes 1 \quad (\text{in } X_{1,0}),\\ \psi_1(1\otimes vg\otimes 1) &= 1\otimes v\otimes g + v\otimes g\otimes 1 \quad (\text{in } X_{0,1}\oplus X_{1,0})\end{aligned}$$

for nonidentity g in G and v in V and use the identification (4.2) for evaluating the differential to check that $d_1\psi_1=\psi_0\delta_1$ on these arguments. In order to extend ψ_1 to an A-bimodule map on $A\otimes\overline{A}\otimes A$, we first choose a homogeneous vector space basis of $\overline{A}$ consisting of the elements $g\neq 1_G$, vg, and sg as g ranges over the elements of G, v ranges over a k-basis of V, and s ranges over a k-basis of homogeneous elements of S of degree ≥ 2. Tensoring each of these elements on the left and right by 1 then gives a free A-bimodule basis of $A\otimes\overline{A}\otimes A$. The function ψ_1 is already defined on elements of the form $1\otimes g\otimes 1$ and $1\otimes vg\otimes 1$; we may define ψ_1 on elements of the form $1\otimes s\otimes 1$ so that $d_1\psi_1=\psi_0\delta_1$ on these elements and then define

$$\psi_1(1\otimes sg\otimes 1)=\psi_1(1\otimes s\otimes g)+s\otimes g\otimes 1$$

so that $d_1\psi_1=\psi_0\delta_1$ on these elements as well. Then $\psi_1\phi_1$ is the identity map on X_1, by construction.

Define ϕ_2 by setting

$$\begin{aligned}\phi_2(1\otimes g\otimes h\otimes 1) &= 1\otimes g\otimes h\otimes 1 && (\text{on } X_{2,0}),\\ \phi_2(1\otimes g\otimes v\otimes 1) &= 1\otimes g\otimes v\otimes 1 - 1\otimes {}^g v\otimes g\otimes 1 && (\text{on } X_{1,1}),\\ \phi_2(1\otimes r\otimes 1) &= 1\otimes r\otimes 1 && (\text{on } X_{0,2})\end{aligned}$$

for all nonidentity g,h in G, v in V, and r in R. One may check that $\delta_2\phi_2=\phi_1 d_2$. Now set

$$\begin{aligned}\psi_2(1\otimes g\otimes h\otimes 1) &= 1\otimes g\otimes h\otimes 1 && (\text{in } X_{2,0}),\\ \psi_2(1\otimes vh\otimes g\otimes 1) &= v\otimes h\otimes g\otimes 1 && (\text{in } X_{2,0}),\\ \psi_2(1\otimes g\otimes vh\otimes 1) &= 1\otimes g\otimes v\otimes h + {}^g v\otimes g\otimes h\otimes 1 && (\text{in } X_{1,1}\oplus X_{2,0}),\\ \psi_2(1\otimes r\otimes 1) &= 1\otimes r\otimes 1 && (\text{in } X_{0,2}),\end{aligned}$$

for all g,h in G, v in V, and r in R. A calculation shows that $d_2\psi_2=\psi_1\delta_2$ on these elements. Letting g,h range over the elements of G, v over a k-basis of V, and r over a k-basis of R, we obtain a linearly independent set consisting of elements of the form $1\otimes g\otimes h\otimes 1$, $1\otimes vh\otimes g\otimes 1$, $1\otimes g\otimes vh\otimes 1$, and $1\otimes r\otimes 1$ on which ψ_2 has already been defined. Extend the k-basis of R to a k-basis of $V\otimes V$ by including additional elements of the form $v\otimes w$ for v,w in V. Now define $\psi_2(1\otimes v\otimes w\otimes 1)$ arbitrarily subject to the condition that $d_2\psi_2=\psi_1\delta_2$ on these elements. Let

$$\begin{aligned}\psi_2(1\otimes vg\otimes w\otimes 1) &= \psi_2(1\otimes v\otimes {}^g w\otimes g)+v\otimes g\otimes w\otimes 1, \quad \text{and}\\ \psi_2(1\otimes v\otimes wg\otimes 1) &= \psi_2(1\otimes v\otimes w\otimes g)\end{aligned} \tag{4.5}$$

for all g in G and v, w in V. One checks that $d_2\psi_2 = \psi_1\delta_2$ on these elements as well. Extend these elements to a free A-bimodule basis of $A \otimes \overline{A} \otimes \overline{A} \otimes A$. We may define ψ_2 on the remaining free basis elements so that $d_2\psi_2 = \psi_1\delta_2$. By construction, $\psi_2\phi_2$ is the identity map on X_2. The rest of the proof may be completed by induction, or by properties of twisted tensor products as discussed at the start of the proof. $\square$

We will need some further values of ϕ in homological degree 3, which we set in the next lemma. The lemma is proven by directly checking the chain map condition. Other values of ϕ_3 may be defined by extending to a free A-bimodule basis of $A \otimes (\overline{A})^{\otimes 3} \otimes A$. Recall the map σ is defined by equation (2.3).

Lemma 4.6. *We may choose the map ϕ_3 in Lemma 4.4 so that*

$$\begin{aligned}
\phi_3(1 \otimes x \otimes 1) &= 1 \otimes x \otimes 1 \quad (\textit{on } X_{0,3}),\\
\phi_3(1 \otimes g \otimes r \otimes 1) &= 1 \otimes g \otimes r \otimes 1 - (1 \otimes \sigma \otimes 1 \otimes 1)(1 \otimes g \otimes r \otimes 1)\\
&\quad + (1 \otimes 1 \otimes \sigma \otimes 1)(1 \otimes \sigma \otimes 1 \otimes 1)(1 \otimes g \otimes r \otimes 1) \quad (\textit{on } X_{1,2})
\end{aligned}$$

for all nonidentity g in G, r in R, and x in $(V \otimes R) \cap (R \otimes V)$.

5. Homological PBW conditions

We now give homological conditions for a filtered algebra to be a PBW deformation of a Koszul algebra twisted by a group. These conditions are a translation of the necessary homological Conditions (3.4), (3.5), and (3.6) into conditions on the parameter functions α, β, λ defining a potential deformation; we prove these conditions are in fact sufficient.

Again, let S be a Koszul algebra generated by a finite dimensional vector space V with defining relations R and an action of a finite group G by graded automorphisms. Let R' be the space of group action relations defined in (2.2). Let $A = S \rtimes G$. We use the resolution $X_\bullet$ of (4.1) to express the Hochschild cohomology $\mathrm{HH}^\bullet(A)$.

Remark 5.1. Just as in [15, Lemma 8.2], we may identify the k-linear functions

$$\alpha : R \to V \otimes kG, \quad \beta : R \to kG, \quad \text{and } \lambda : R' \to kG$$

with 2-cochains on the resolution $X_\bullet$, i.e., A-bimodule homomorphisms from X_2 to A. Indeed, both α and β extend uniquely to cochains on $X_{0,2} = A \otimes R \otimes A$ since a cochain is an A-bimodule homomorphism and thus determined there by its values on R. Similarly, λ corresponds to a unique cochain on $X_{1,1}$ taking the value $\lambda(g \otimes v \otimes 1 - 1 \otimes {}^g v \otimes g)$ on elements of the form $1 \otimes g \otimes v \otimes 1$. Here we identify the target spaces of α, β, λ with subspaces of A. We extend these cochains defined by α, β, λ to all of $X_\bullet$ by setting them to be 0 on the components of $X_\bullet$ on which we did not already define them.

We fix choices of chain maps ϕ, ψ satisfying Lemmas 4.4 and 4.6. We define the Gerstenhaber bracket of cochains on $X_\bullet$ by transferring the Gerstenhaber bracket (3.3) on the (reduced) bar resolution to $X_\bullet$ using these chain maps: If ξ,ν are Hochschild cochains on $X_\bullet$, we define

$$[\xi,\nu]=\phi^*([\psi^*(\xi),\psi^*(\nu)]), \tag{5.2}$$

another cochain on $X_\bullet$. At the chain level, this bracket depends on the choice of chain maps ϕ,ψ, although at the level of cohomology, it does not. The choices we have made in Lemmas 4.4 and 4.6 provide valuable information, as we see next.

Theorem 5.3. *Let S be a Koszul algebra over the field k generated by a finite dimensional vector space V. Let G be a finite group acting on S by graded automorphisms. The algebra $\mathcal{H}_{\lambda,\alpha,\beta}$ defined in (2.4) is a PBW deformation of $S\rtimes G$ if and only if*

(a) $d^*(\alpha+\lambda)=0$,
(b) $[\alpha+\lambda,\alpha+\lambda]=2d^*\beta$, *and*
(c) $[\lambda+\alpha,\beta]=0$,

where α,β,λ are identified with cochains on the twisted product resolution $X_\bullet$ as in Remark 5.1.

Proof. We adapt ideas of [2, Theorem 4.1], first translating the above Conditions (a), (b), and (c) to conditions on the reduced bar resolution itself. The proof is similar to that of [14, Theorem 5.4], but certain arguments must be altered to allow for the additional parameter function λ.

If $\mathcal{H}_{\lambda,\alpha,\beta}$ is a PBW deformation of $S\rtimes G$, then by Proposition 3.7, there are Hochschild 2-cochains μ_1 and μ_2 on the (reduced) bar resolution such that the Conditions (3.4), (3.5), and (3.6) hold, that is, μ_1 is a Hochschild 2-cocycle, $[\mu_1,\mu_1]=2\delta^*(\mu_2)$, and $[\mu_1,\mu_2]$ is a coboundary. By the proofs of Proposition 3.7 and Lemma 4.4,

$$\alpha+\lambda=\phi_2^*(\mu_1)\quad\text{and}\quad\beta=\phi_2^*(\mu_2).$$

Since μ_1 is a cocycle, it follows that $d^*(\alpha+\lambda)=0$, that is, Condition (a) holds. For Condition (b), note that each side of the equation is automatically 0 on $X_{3,0}$ and on $X_{2,1}$, by a degree argument. We will evaluate each side of the equation on $X_{1,2}$ and on $X_{0,3}$. By definition,

$$\begin{aligned}[\alpha+\lambda,\alpha+\lambda] &= \phi^*[\psi^*(\alpha+\lambda),\psi^*(\alpha+\lambda)]\\ &= \phi^*([\psi^*\phi^*(\mu_1),\psi^*\phi^*(\mu_1)])\\ &= 2\phi^*(\psi^*\phi^*(\mu_1)(\psi^*\phi^*(\mu_1)\otimes 1-1\otimes\psi^*\phi^*(\mu_1))).\end{aligned}$$

We evaluate on $X_{1,2}$. By Lemma 4.6, the image of ϕ_3 on $X_{1,2}$ is contained in

$$(kG\otimes \operatorname{Im}(\phi_2|_{X_{0,2}}))\oplus(V\otimes\operatorname{Im}(\phi_2|_{X_{1,1}}))\cap(\operatorname{Im}(\phi_2|_{X_{0,2}})\otimes kG)\oplus(\operatorname{Im}(\phi_2|_{X_{1,1}})\otimes V).$$

Therefore, since $\psi_2\phi_2$ is the identity map, applying $\psi^*\phi^*(\mu_1)\otimes 1 - 1\otimes\psi^*\phi^*(\mu_1)$ to an element in the image of ϕ_3 is the same as applying $\mu_1\otimes 1 - 1\otimes\mu_1$. Since μ_1 is a Hochschild 2-cocycle, the image of $\mu_1\otimes 1-1\otimes\mu_1$ is 0 upon projection to $S\rtimes G$, which implies that the image of $\mu_1\otimes 1 - 1\otimes\mu_1$ on $\phi_3(X_{1,2})$ is contained in the subspace of $\overline{A}\otimes\overline{A}$ spanned by all $g\otimes v - {}^g v\otimes g$ for nonidentity g in G and v in V. This is in the image of ϕ_1, and so again, applying $\psi^*\phi^*(\mu_1)$ is the same as applying μ_1. Hence $[\alpha+\lambda,\alpha+\lambda] = \phi^*([\mu_1,\mu_1])$ on $X_{1,2}$. Condition (3.5) then implies that Condition (b) holds on $X_{1,2}$. A similar argument verifies Condition (b) on $X_{0,3}$. Condition (c) holds by a degree argument: The bracket $[\lambda+\alpha,\beta]$ is cohomologous to $[\mu_1,\mu_2]$, which by (3.6) is a coboundary. So $[\alpha+\lambda,\beta]$ is itself a coboundary: $[\lambda+\alpha,\beta] = d^*(\xi)$ for some 2-cochain ξ. Now $[\lambda+\alpha,\beta]$ is of graded degree -3, and the only 2-cochain on $X_\bullet$ of graded degree -3 is 0.

For the converse, assume Conditions (a), (b), and (c) hold. We may now set $\mu_1 = \psi^*(\alpha+\lambda)$ and $\mu_2 = \psi^*(\beta)$. Set

$$\gamma = \delta_3^*(\mu_2) - \tfrac{1}{2}[\mu_1,\mu_1].$$

Condition (a) of the theorem implies that $\alpha+\lambda$ is a 2-cocycle and thus μ_1 is a 2-cocycle on the reduced bar resolution of A. The 2-cocycle μ_1 then is a first multiplication map on A and defines a first level deformation A_1 of $A = S\rtimes G$.

Next we will see that Condition (b) implies this first level deformation can be extended to a second level deformation. By Lemma 4.4,

$$\begin{aligned}
\phi_3^*(\gamma) &= \phi_3^*(\delta_3^*\,(\psi_2^*(\beta))) - \tfrac{1}{2}\phi_3^*[\psi_2^*(\alpha+\lambda),\psi_2^*(\alpha+\lambda)]\\
&= d^*(\phi_2^*\psi_2^*(\beta)) - \tfrac{1}{2}[\alpha+\lambda,\alpha+\lambda]\\
&= d_3^*(\beta) - \tfrac{1}{2}[\alpha+\lambda,\alpha+\lambda].
\end{aligned}$$

Hence $\phi_3^*(\gamma)=0$ by Condition (b). This forces γ to be a coboundary, say $\gamma=\delta^*(\mu)$ for some 2-cochain μ on the reduced bar resolution, necessarily of graded degree -2. Now,

$$d^*(\phi^*(\mu)) = \phi^*(\delta^*(\mu)) = \phi^*(\gamma) = 0\,,$$

so $\phi^*(\mu)$ is a 2-cocycle. Then there must be a 2-cocycle μ' on the reduced bar resolution with $\phi^*\mu' = \phi^*\mu$. We replace μ_2 by $\tilde{\mu}_2 = \mu_2 - \mu + \mu'$ so that $\phi^*(\tilde{\mu}_2) = \beta$ but

$$2\delta^*(\tilde{\mu}_2) = 2\delta^*(\mu_2+\mu') - 2\gamma = [\mu_1,\mu_1]$$

by the definition of γ, since μ' is a cocycle. Thus the obstruction to lifting A_1 to a second level deformation using the multiplication map $\tilde{\mu}_2$ vanishes, and μ_1 and $\tilde{\mu}_2$ together define a second level deformation A_2 of A.

We now argue that Condition (c) implies A_2 lifts to a third level deformation of A. Adding the coboundary $\mu'-\mu$ to μ_2 adds a coboundary to $[\mu_2,\mu_1]$, and hence $[\tilde{\mu}_2,\mu_1] = \delta_3^*(\mu_3)$ for some cochain μ_3 on the reduced bar resolution of graded degree -3. Thus the obstruction to lifting A_2 to a third level deformation vanishes and the multiplication maps $\mu_1,\tilde{\mu}_2,\mu_3$ define a third level deformation A_3 of A.

The obstruction to lifting A_3 to a fourth level deformation of A lies in $\mathrm{HH}^{3,-4}(A)$ by [2, Proposition 1.5]. Applying the map ϕ^* to this obstruction gives a cochain of graded degree -4 on X_3, as ϕ is of graded degree 0 as a chain map by Lemma 4.4. But X_3 is generated, as an A-bimodule, by elements of graded degree 3 or less, and thus ϕ^* applied to the obstruction is 0, implying that the obstruction is a coboundary. Thus the deformation A_3 lifts to a fourth level deformation A_4 of A. Similarly, the obstruction to lifting an i-th level deformation A_i of A lies in $\mathrm{HH}^{3,-(i+1)}$, and again since S is Koszul, the obstruction is a coboundary. So the deformation A_i lifts to A_{i+1}, an $(i+1)$-st level deformation of A, for all $i \geq 1$.

The corresponding graded deformation A_t of A is the vector space $A[t]$ with multiplication determined by

$$a * a' = aa' + \mu_1(a,a')t + \mu_2(a,a')t^2 + \mu_3(a,a')t^3 + \dots$$

for all $a, a' \in A$.

We next explain that $\mathcal{H}_{\lambda,\alpha,\beta}$ is isomorphic, as a filtered algebra, to the fiber $A_t|_{t=1}$. First note that $A_t|_{t=1}$ is generated by V and G (since the associated graded algebra of A_t is A). Thus we may define an algebra homomorphism

$$T_{kG}(kG \otimes V \otimes kG) \longrightarrow A_t\big|_{t=1}$$

and then use Lemma 4.4 to verify that the elements

$$\begin{aligned} &r - \alpha(r) - \beta(r) && \text{for } r \in R, \text{ and} \\ &g \otimes v \otimes 1 - 1 \otimes {}^g v \otimes g - \lambda(g \otimes v) && \text{for } g \in G, v \in V \end{aligned}$$

lie in the kernel. We obtain a surjective homomorphism of filtered algebras,

$$\mathcal{H}_{\lambda,\alpha,\beta} \longrightarrow A_t\big|_{t=1}.$$

We consider the dimension over k of each of the filtered components in the domain and range: Each filtered component of $\mathcal{H}_{\lambda,\alpha,\beta}$ has dimension at most that of the corresponding filtered component of $S \rtimes G$ since its associated graded algebra is necessarily a quotient of $S \rtimes G$. But the associated graded algebra of $A_t|_{t=1}$ is precisely $S \rtimes G$, and so

$$\dim_k(F^d(S \rtimes G)) \geq \dim_k(F^d(\mathcal{H}_{\lambda,\alpha,\beta})) \geq \dim_k(F^d(A_t\big|_{t=1})) = \dim_k(F^d(S \rtimes G)),$$

where F^d indicates the summand of filtered degree d in $\mathbb{N}$. Thus these dimensions are all equal. It follows that $\mathcal{H}_{\lambda,\alpha,\beta} \cong A_t\big|_{t=1}$, and $\mathcal{H}_{\lambda,\alpha,\beta}$ is a PBW deformation. $\square$

We now prove Theorem 2.5 as a consequence of Theorem 5.3, translating the homological conditions into Braverman-Gaitsgory style conditions.

Proof of Theorem 2.5. We explained in Section 2 that each PBW deformation of $S \rtimes G$ has the form $\mathcal{H}_{\lambda,\alpha,\beta}$ as defined in (2.4) for some parameter functions α, β, λ. Theorem 5.3 gives necessary and sufficient conditions for such an algebra $\mathcal{H}_{\lambda,\alpha,\beta}$

to be a PBW deformation of $S \rtimes G$. We will show that the Conditions (a), (b), and (c) of Theorem 5.3 are equivalent to those of Theorem 2.5.

When convenient, we identify

$$\mathrm{Hom}_{A^e}(A \otimes \overline{A}^n \otimes A, A) \cong \mathrm{Hom}_k(\overline{A}^n, A)\,.$$

Condition (a): $d^*(\alpha+\lambda)=0$. The cochain $d^*(\alpha+\lambda)$ has homological degree 3 and is the zero function if and only if it is 0 on each of $X_{3,0}$, $X_{2,1}$, $X_{1,2}$, and $X_{0,3}$. It is automatically 0 on $X_{3,0}$ since $d(X_{3,0})$ trivially intersects $X_{1,1} \oplus X_{0,2}$ on which $\alpha+\lambda$ is defined.

On $X_{2,1}$, $d^*(\alpha)=0$ automatically, as α is 0 on $X_{2,0} \oplus X_{1,1}$. We evaluate $d^*(\lambda)$ on the elements of a free A^e-basis of $X_{2,1}$, using the identification (4.2) for evaluating the differential:

$$\begin{aligned}
&d^*(\lambda)(1 \otimes g \otimes h \otimes v \otimes 1)\\
&\quad= \lambda(g \otimes h \otimes v \otimes 1 - 1 \otimes gh \otimes v \otimes 1 + 1 \otimes g \otimes {}^{h}v \otimes h\\
&\qquad + {}^{gh}v \otimes g \otimes h \otimes 1 - 1 \otimes g \otimes h \otimes v)\\
&\quad= g\lambda(h \otimes v) - \lambda(gh \otimes v) + \lambda(g \otimes {}^{h}v)h
\end{aligned}$$

in A for all g, h in G and v in V, which can be rewritten as Theorem 2.5(1). Therefore $d^*(\alpha+\lambda)|_{X_{2,1}}=0$ if and only if Theorem 2.5(1) holds. (If g or h is the identity group element 1_G, then in the evaluation above, some of the terms are 0 as we are working with the reduced bar resolution. The condition remains the same in these cases, and merely corresponds to the condition $\lambda(1_G \otimes v)=0$ for all v in V.)

On $X_{1,2}$, $d^*(\alpha+\lambda)=0$ if and only if

$$\begin{aligned}
&d^*(\alpha+\lambda)(1 \otimes g \otimes r \otimes 1)\\
&\quad= (\alpha+\lambda)(g \otimes r \otimes 1 - 1 \otimes {}^{g}r \otimes g - (\sigma \otimes 1 \otimes 1)(g \otimes r \otimes 1) - 1 \otimes g \otimes r)\\
&\quad= g\alpha(r) - \alpha({}^{g}r)g - (1 \otimes \lambda)(\sigma \otimes 1)(g \otimes r) - (\lambda \otimes 1)(g \otimes r)
\end{aligned}$$

vanishes for all g in G and r in R. (Note that the multiplication map takes r to 0 in A.) This is equivalent to the equality

$$1 \otimes \alpha - (\alpha \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1) = (1 \otimes \lambda)(\sigma \otimes 1) + \lambda \otimes 1$$

as functions on $kG \otimes R$ with values in A. Thus $d^*(\alpha+\lambda)|_{X_{1,2}}=0$ if and only if Theorem 2.5(3) holds.

On $X_{0,3}$, $d^*(\lambda)$ is automatically 0 since λ is 0 on $X_{0,2}$. So we compute $d^*(\alpha)|_{X_{0,3}}$. Consider x in $(R \otimes V) \cap (V \otimes R)$. Then

$$d^*(\alpha)(1 \otimes x \otimes 1) = \alpha(x \otimes 1 - 1 \otimes x) = (1 \otimes \alpha - \alpha \otimes 1)(x). \tag{5.4}$$

So $d^*(\alpha+\lambda)|_{X_{0,3}}=0$ if and only if $1 \otimes \alpha - \alpha \otimes 1$ has image 0 in A, i.e., Theorem 2.5(6) holds.

Condition (b): $[\alpha+\lambda,\alpha+\lambda]=2d^*(\beta)$**.** On $X_{3,0}$ and on $X_{2,1}$, both sides of this equation are automatically 0, as their graded degree is -2. We will compute their values on $X_{1,2}$ and on $X_{0,3}$. First note that since λ and α each have homological degree 2, by the definition (3.3) of bracket, $[\alpha,\lambda]=[\lambda,\alpha]$ and so

$$[\alpha+\lambda,\alpha+\lambda]=[\alpha,\alpha]+2[\alpha,\lambda]+[\lambda,\lambda].$$

We will compute $[\lambda,\lambda]$, $[\alpha,\lambda]$, and $[\alpha,\alpha]$.

Note that $[\lambda,\lambda]$ can take nonzero values only on $X_{1,2}$. We will compute its values on elements of the form $1\otimes g\otimes r\otimes 1$ for g in G and r in R. By (3.3), (5.2), and Lemmas 4.4 and 4.6, $[\lambda,\lambda](1\otimes g\otimes r\otimes 1)=2\lambda(\lambda\otimes 1)(g\otimes r)$. Similarly,

$$[\alpha,\lambda](1\otimes g\otimes r\otimes 1)=-\lambda(1\otimes\alpha)(g\otimes r).$$

Finally, note that $[\alpha,\alpha]|_{X_{1,2}}=0$ automatically for degree reasons. Just as in our earlier calculation, we find that

$$d^*(\beta)(1\otimes g\otimes r\otimes 1)=(1\otimes\beta-(\beta\otimes 1)(1\otimes\sigma)(\sigma\otimes 1))(g\otimes r).$$

Therefore, $[\alpha+\lambda,\alpha+\lambda]=2d^*(\beta)$ on $X_{1,2}$ if and only if

$$2\lambda(\lambda\otimes 1)-2\lambda(1\otimes\alpha)=2\otimes\beta-2(\beta\otimes 1)(1\otimes\sigma)(\sigma\otimes 1)$$

on $kG\otimes R$. This is equivalent to Theorem 2.5(2).

On $X_{0,3}$, the bracket $[\lambda,\lambda]$ vanishes. We compute $[\alpha,\lambda]$ and $[\alpha,\alpha]$ on an element $1\otimes x\otimes 1$ of $X_{0,3}$ with x in $(V\otimes R)\cap(R\otimes V)$. Note that $\psi^*(\alpha)(r)=\alpha\psi(r)=\alpha(\psi\phi)r=\alpha(r)$ for all r in R. Thus

$$(\psi^*(\alpha)\otimes 1-1\otimes\psi^*(\alpha))(x)=(\alpha\otimes 1-1\otimes\alpha)(x)$$

and therefore

$$\begin{aligned}[\alpha,\alpha](1\otimes x\otimes 1) &= 2\psi^*(\alpha)(\alpha\otimes 1-1\otimes\alpha)(x) \quad\text{and}\\ [\alpha,\lambda](1\otimes x\otimes 1) &= \psi^*(\lambda)(\alpha\otimes 1-1\otimes\alpha)(x).\end{aligned}$$

We apply ψ to $(\alpha\otimes 1-1\otimes\alpha)(x)$ using Lemma 4.4. Since $(\alpha\otimes 1)(x)$ lies in $(V\otimes kG)\otimes V\subset A\otimes A$ and $(1\otimes\alpha)(x)$ lies in $V\otimes(V\otimes kG)\subset A\otimes A$, we use (4.5) to apply ψ:

$$\psi(\alpha\otimes 1-1\otimes\alpha)(x)=(\psi(1\otimes\sigma)(\alpha\otimes 1)-\psi(1\otimes\alpha))(x)\ +\ y$$

for some element y in $X_{1,1}$. However, α is zero on $X_{1,1}$, so

$$\psi^*(\alpha)(\alpha\otimes 1-1\otimes\alpha)(x)=\alpha\psi((1\otimes\sigma)(\alpha\otimes 1)-1\otimes\alpha)(x)\,.$$

We assume Condition (a) which we have shown implies Condition (6) of Theorem 2.5, i.e., $((1\otimes\sigma)(\alpha\otimes 1)-1\otimes\alpha)(x)$ lies in $R\otimes kG$ since it is zero upon projection to A. By the proof of Lemma 4.4,

$$\phi((1\otimes\sigma)(\alpha\otimes 1)-1\otimes\alpha)(x)=((1\otimes\sigma)(\alpha\otimes 1)-1\otimes\alpha)(x),$$

and applying $\alpha\psi$ gives $\alpha((1\otimes\sigma)(\alpha\otimes 1)-1\otimes\alpha)(x)$ since $\psi\phi=1$. Hence

$$[\alpha,\alpha](1\otimes x\otimes 1)=2\alpha((1\otimes\sigma)(\alpha\otimes 1)-1\otimes\alpha)(x)\,.$$

Similarly, we apply $\psi^*(\lambda)$ to $(\alpha\otimes 1-1\otimes\alpha)(x)$ again using (4.5). Recall that λ is only nonzero on $X_{1,1}$, and $\psi(1\otimes\alpha)$ intersects $X_{1,1}$ at 0; hence $\psi^*(\lambda)(\alpha\otimes 1-1\otimes\alpha)(x) = \psi^*(\lambda)(\alpha\otimes 1)(x)$ and

$$[\alpha,\lambda](1\otimes x\otimes 1) = \psi^*(\lambda)(\alpha\otimes 1)(x) = (\sum_{g\in G}\alpha_g\otimes\lambda(g\otimes -))(x)\,.$$

Therefore $[\alpha+\lambda,\alpha+\lambda] = 2d^*(\beta)$ on $X_{0,3}$ if and only if Theorem 2.5(4) holds.

Condition (c): $[\alpha+\lambda,\beta]=0$. On $X_{3,0}$ $X_{2,1}$, and $X_{1,2}$, the left side of this equation is automatically 0 for degree reasons. We will compute values on $X_{0,3}$. Similar to our previous calculation, we find

$$[\lambda,\beta] = \lambda(\beta\otimes 1) \quad\text{ and }\quad [\alpha,\beta] = \beta((1\otimes\sigma)(\alpha\otimes 1)-1\otimes\alpha)$$

on $(V\otimes R)\cap(R\otimes V)$. So $[\alpha+\lambda,\beta]=0$ if and only if $\beta((1\otimes\sigma)(\alpha\otimes 1)-1\otimes\alpha) = -\lambda(\beta\otimes 1)$. This is precisely Theorem 2.5(5). □

6. Application: Group actions on polynomial rings

We now consider the special case when S is the symmetric algebra $S(V)$ of a finite dimensional k-vector space V. Let G be a finite group acting on $S(V)$ by graded automorphisms. Let $\mathcal{H}_{\lambda,\kappa}$ be the k-algebra generated by the group ring kG together with the vector space V and subject to the relations

- $gv - {}^g vg - \lambda(g,v)$, for g in G, v in V
- $vw - wv - \kappa(v,w)$, for v,w in V,

where

$$\lambda : kG\times V\to kG,\quad \kappa : V\times V\to kG\oplus(V\otimes kG)$$

are bilinear functions. Letting κ^C and κ^L be the projections of κ onto kG and $V\otimes kG$, respectively, $\mathcal{H}_{\lambda,\kappa}$ is the algebra $\mathcal{H}_{\lambda,\alpha,\beta}$ from earlier sections with $\alpha=\kappa^L$ and $\beta=\kappa^C$. Its homogeneous version is the algebra

$$\text{HomogeneousVersion}(\mathcal{H}_{\lambda,\kappa}) = S(V)\rtimes G = \mathcal{H}_{0,0}\ .$$

We say that $\mathcal{H}_{\lambda,\kappa}$ is a *Drinfeld orbifold algebra* if it has the PBW property:

$$\operatorname{Gr}\mathcal{H}_{\lambda,\kappa}\cong S(V)\rtimes G$$

as graded algebras. Thus Drinfeld orbifold algebras are PBW deformations of $S(V)\rtimes G$.

In characteristic zero, our definition of Drinfeld orbifold algebra coincides with that in [13], up to isomorphism, even though no parameter λ appears there. This is a consequence of Theorem 7.1 in the next section: In this nonmodular case, $\mathcal{H}_{\lambda,\kappa}$ is isomorphic to $\mathcal{H}_{0,\kappa'}$ for some κ'.

The algebras $\mathcal{H}_{\lambda,\kappa}$ include as special cases many algebras of interest in the literature, and our Theorem 6.1 below unifies results giving necessary and sufficient conditions on parameter functions for $\mathcal{H}_{\lambda,\kappa}$ to have the PBW property. When $\lambda=$

0 and $\kappa^L = 0$, Drinfeld orbifold algebras $\mathcal{H}_{0,\kappa}$ include Drinfeld's Hecke algebras [3] and Etingof and Ginzburg's symplectic reflection algebras [4]. When $\lambda = 0$ and $\kappa^C = 0$, Drinfeld orbifold algebras $\mathcal{H}_{0,\kappa}$ exhibit a Lie type structure: Many of the conditions of Theorem 6.1 below are vacuous in this case, while Condition (3) states that κ^L is G-invariant and Conditions (4) and (6) are analogs of the Jacobi identity twisted by the group action. When $\kappa = 0$, Drinfeld orbifold algebras $\mathcal{H}_{\lambda,0}$ include Lusztig's graded affine Hecke algebras [11].

The following theorem simultaneously generalizes [13, Theorem 3.1] and [15, Theorem 3.1].

Theorem 6.1. *Let G be a finite group acting linearly on V, a finite dimensional k-vector space. Then $\mathcal{H}_{\lambda,\kappa}$ is a PBW deformation of $S(V) \rtimes G$ if and only if*

(1) $\lambda(gh, v) = \lambda(g, {}^h v)h + g\lambda(h, v)$,

(2) $\kappa^C({}^g u, {}^g v)g - g\kappa^C(u, v) = \lambda\big(\lambda(g, v), u\big) - \lambda\big(\lambda(g, u), v\big) + \sum_{a\in G} \lambda\big(g, \kappa^L_a(u, v)\big)a\,,$

(3) ${}^g\big(\kappa^L_{g^{-1}h}(u, v)\big) - \kappa^L_{hg^{-1}}({}^g u, {}^g v) = ({}^h v - {}^g v)\lambda_h(g, u) - ({}^h u - {}^g u)\lambda_h(g, v)$,

(4) $$\begin{aligned} 0 \;=\; & 2 \sum_{\sigma\in \mathrm{Alt}_3} \kappa^C_g(v_{\sigma(1)}, v_{\sigma(2)})(v_{\sigma(3)} - {}^g v_{\sigma(3)}) \\ & + \sum_{\substack{a\in G\\ \sigma\in \mathrm{Alt}_3}} \kappa^L_{ga^{-1}}\big(v_{\sigma(1)} + {}^a v_{\sigma(1)}, \kappa^L_a(v_{\sigma(2)}, v_{\sigma(3)})\big) \\ & - 2 \sum_{\substack{a\in G\\ \sigma\in \mathrm{Alt}_3}} \kappa^L_a(v_{\sigma(1)}, v_{\sigma(2)})\lambda_g(a, v_{\sigma(3)})\,, \end{aligned}$$

(5) $$\begin{aligned} & 2 \sum_{\sigma\in \mathrm{Alt}_3} \lambda\big(\kappa^C(v_{\sigma(1)}, v_{\sigma(2)}), v_{\sigma(3)}\big) \\ & \quad = - \sum_{\substack{a\in G\\ \sigma\in \mathrm{Alt}_3}} \kappa^C_{ga^{-1}}\big(v_{\sigma(1)} + {}^a v_{\sigma(1)}, \kappa^L_a(v_{\sigma(2)}, v_{\sigma(3)})\big), \end{aligned}$$

(6) $0 = \kappa^L_g(u, v)(w - {}^g w) + \kappa^L_g(v, w)(u - {}^g u) + \kappa^L_g(w, u)(v - {}^g v)$,

in $S(V) \rtimes G$, for all g, h in G and all u, v, w, v_1, v_2, v_3 in V.

Proof. The theorem follows from Theorem 2.5 by rewriting the conditions explicitly on elements. □

Alternatively, the conditions of the theorem follow from strategic and tedious application of the Composition-Diamond Lemma (such as in the proof of [13, Theorem 3.1]). Condition (1) follows from consideration of overlaps of the form ghv for g, h in G, v in V. For Conditions (2) and (3), we consider overlaps of the form gwv for w in V; terms of degree 1 give rise to Condition (3) while those of degree 0 give rise to Condition (2). Overlaps of the form uvw for u in V give the other conditions: Terms of degree 0 give rise to Condition (5), terms of degree 1 give rise to Condition (4), and terms of degree 2 give rise Condition (6). Note that we assume Condition (6) to deduce Conditions (4) and (5).

In the theorem above, we may set $\kappa^L = 0$ to obtain the conditions of [15, Theorem 3.1] or instead set $\lambda = 0$ to obtain the conditions of [13, Theorem 3.1]. Note that in Theorem 6.1, Condition (3) measures the extent to which κ^L is G-invariant. Indeed, the failure of κ^L to be G-invariant is recorded by $d^*(\lambda)$, and so λ is a cocycle if and only if κ^L is invariant. Condition (3) in particular implies that $\kappa^L_{1_G}$ is G-invariant.

The conditions in the theorem also generalize a special case of Theorem 2.7 in [9] by Khare: He more generally considered actions of cocommutative algebras, while we restrict to actions of group algebras kG. Khare more specifically restricted κ^L to take values in the subspace $V \cong V \otimes k$ of $V \otimes kG$.

We next give some examples of Drinfeld orbifold algebras. The first example exhibits parameters κ^C, κ^L, and λ all nonzero. The second example shows that a new class of deformations is possible in the modular setting; see Remark 7.3.

Example 6.2. Let k have prime characteristic $p > 2$, and $V = kv_1 \oplus kv_2 \oplus kv_3$. Let $G \leq \mathrm{GL}_3(k)$ be the cyclic group of order p generated by the transvection g in $\mathrm{GL}(V)$ fixing v_1, v_2 and mapping v_3 to $v_1 + v_3$:

$$G = \left\langle g = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \right\rangle .$$

Define

$$\lambda(g^i, v_3) = ig^{i-1},\ \kappa^C(v_1, v_3) = g = -\kappa^C(v_3, v_1),\ \kappa^L(v_1, v_3) = v_2 = -\kappa^L(v_3, v_1),$$

and set $\lambda, \kappa^C, \kappa^L$ to be zero on all other pairs of basis vectors. Then

$$\begin{aligned} \mathcal{H}_{\lambda,\kappa} = T_{kG}(kG \otimes V \otimes kG)/(&gv_1 - v_1 g,\ gv_2 - v_2 g,\ gv_3 - v_1 g - v_3 g - 1, \\ &v_1 v_3 - v_3 v_1 - v_2 - g,\ v_1 v_2 - v_2 v_1,\ v_2 v_3 - v_3 v_2) \end{aligned}$$

is a PBW deformation of $S(V) \rtimes G$ by Theorem 6.1.

Example 6.3. Let k have prime characteristic $p > 2$ and $V = kv \oplus kw$. Suppose $G \leq \mathrm{GL}_2(k)$ is the cyclic group of order p generated by $g = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ so that ${}^g v = v$ and ${}^g w = v + w$. Define

$$\lambda(g^i, v) = ig^i,\ \lambda(1, w) = \lambda(g, w) = 0,\ \lambda(g^i, w) = \binom{i}{2} g^i \text{ for } i > 2,$$

and $\kappa = 0$. Then one may check the conditions of Theorem 6.1 to conclude that

$$\mathcal{H}_{\lambda,0} = T_{kG}(kG \otimes V \otimes kG)/(gv - vg - g,\ gw - vg - wg,\ vw - wv)$$

is a PBW deformation of $S(V) \rtimes G$.

7. Comparison of modular and nonmodular settings

We now turn to the nonmodular setting, when the characteristic of the underlying field k does not divide the order of the acting group G. We compare algebras modelled on Lusztig's graded affine Hecke algebra [11] to algebras modelled on Drinfeld's Hecke algebra [3] (such as the symplectic reflection algebras of Etingof and Ginzburg [4]). The following theorem strengthens Theorem 4.1 of [15] while simultaneously generalizing it to the setting of Drinfeld orbifold algebras (see [13]) in the nonmodular setting. The theorem was originally shown for Coxeter groups and Lusztig's graded affine Hecke algebras in [12].

Theorem 7.1. *Suppose G acts linearly on a finite dimensional vector space V over a field k whose characteristic is coprime to $|G|$. If the algebra $\mathcal{H}_{\lambda,\kappa}$ defined in Section 6 is a PBW deformation of $S(V)\rtimes G$ for some parameter functions*

$$\lambda : kG \times V \to kG \quad \textit{and} \quad \kappa : V \times V \to kG \oplus (V \otimes kG),$$

then there exists a parameter function

$$\kappa' : V \times V \to kG \oplus (V \otimes kG)$$

such that

$$\mathcal{H}_{\lambda,\kappa} \cong \mathcal{H}_{0,\kappa'}$$

as filtered algebras and thus $\mathcal{H}_{0,\kappa'}$ also exhibits the PBW property.

Proof. As in [15], define $\gamma : V \otimes kG \to kG$ by

$$\gamma(v \otimes g) = \sum_{a\in G} \gamma_a(v)\, ag$$

for $\gamma_a : V \to k$ the linear map defined by $\gamma_a(v) = \frac{1}{|G|}\sum_{b\in G} \lambda_{ab}(b, {}^{b^{-1}}v)$ for all v in V. (As before, for each h in G, $\lambda_h : kG \times V \to k$ is defined by $\lambda(b,v) = \sum_{h\in G} \lambda_h(b,v)h$ for b in G and v in V.) We abbreviate $\gamma(u)$ for $\gamma(u\otimes 1)$ for u in V in what follows for simplicity of notation. Define a parameter function $\kappa' : V \times V \to kG \oplus (V\otimes kG)$ by

$$\begin{aligned}\kappa'(u,v) \;=\;& \gamma(u)\gamma(v) - \gamma(v)\gamma(u) + \lambda(\gamma(u),v) - \lambda(\gamma(v),u)\\ &+ \kappa(u,v) - \kappa^L(u,v)\\ &+ \tfrac{1}{|G|}\sum_{g\in G}(1-\gamma)\big(({}^{g}\kappa^L)(u,v)\big)\end{aligned}$$

for u,v in V. Here, $\kappa^L(u,v)$ is again the degree 1 part of κ, i.e., the projection of $\kappa(u,v)$ to $V\otimes kG$, and we take the G-action on κ^L induced from the action of G on itself by conjugation, i.e., $({}^{g}\kappa^L)(u,v) = {}^{g}(\kappa^L({}^{g^{-1}}u, {}^{g^{-1}}v))$ with ${}^{g}(v\otimes h) = {}^{g}v \otimes ghg^{-1}$ for g,h in G.

Let

$$F = T_{kG}(kG\otimes V\otimes kG)$$

and identify v in V with $1 \otimes v \otimes 1$ in F. Define an algebra homomorphism

$$f : F \to \mathcal{H}_{\lambda,\kappa} \quad \text{by} \quad v \mapsto v + \gamma(v) \text{ and } g \mapsto g \quad \text{for all } g \in G, v \in V,$$

after identifying $\mathcal{H}_{\lambda,\kappa}$ with a quotient of F. We will use Theorem 6.1 to verify that the relations defining $\mathcal{H}_{0,\kappa'}$ as a quotient of F lie in the kernel of f. It will follow that f extends to a filtered algebra homomorphism

$$f : \mathcal{H}_{0,\kappa'} \to \mathcal{H}_{\lambda,\kappa} .$$

We first check that elements $uv - vu - \kappa'(u,v)$ in F for u, v in V are mapped to zero under f. On one hand, $\kappa'(u,v)$ in F is mapped under f to

$$\kappa'(u,v) + \tfrac{1}{|G|} \sum_{g \in G} \gamma\big(({}^{g}\kappa^L)(u,v)\big).$$

On the other hand, the commutator $[u,v] = uv - vu$ in F maps to the commutator

$$[u + \gamma(u), v + \gamma(v)] = [u,v] + [\gamma(u), \gamma(v)] + \big(u\gamma(v) - \gamma(v)u - v\gamma(u) + \gamma(u)v\big)$$

in $\mathcal{H}_{\lambda,\kappa}$. But $[u,v]$ is $\kappa(u,v)$ in $\mathcal{H}_{\lambda,\kappa}$, and $\kappa'(u,v)$ by definition expresses the commutator $[\gamma(u), \gamma(v)]$ in terms of $\kappa(u,v)$ and other terms. Hence $[u,v] + [\gamma(u), \gamma(v)]$ simplifies to

$$\kappa'(u,v) - \lambda(\gamma(u), v) + \lambda(\gamma(v), u) + \kappa^L(u,v) - \tfrac{1}{|G|} \sum_{g \in G} (1 - \gamma)\big(({}^{g}\kappa^L)(u,v)\big)$$

in $\mathcal{H}_{\lambda,\kappa}$. We may also rewrite

$$u\gamma(v) - \gamma(v)u - v\gamma(u) + \gamma(u)v$$

as

$$\lambda(\gamma(u), v) - \lambda(\gamma(v), u) - \sum_{g \in G} \big(\gamma_g(v)({}^{g}u - u)g - \gamma_g(u)({}^{g}v - v)g\big).$$

Hence, the relation $uv - vu - \kappa'(u,v)$ in F maps under f to

$$\kappa^L(u,v) - \tfrac{1}{|G|} \sum_{g \in G} ({}^{g}\kappa^L)(u,v) - \sum_{g \in G} \big(\gamma_g(v)({}^{g}u - u)g - \gamma_g(u)({}^{g}v - v)g\big). \tag{7.2}$$

We may then argue as in the proof of Theorem 4.1 of [15] to show that Condition (3) of Theorem 6.1 implies that

$$\begin{aligned}\sum_{g \in G} \big(\gamma_g(v)({}^{g}u - u)g - \gamma_g(u)({}^{g}v - v)g\big) &= \kappa^L(u,v) - \tfrac{1}{|G|} \sum_{g,a \in G} {}^{g}\big(\kappa^L_a({}^{g^{-1}}u, {}^{g^{-1}}v)\big) g a g^{-1} \\ &= \kappa^L(u,v) - \tfrac{1}{|G|} \sum_{g \in G} ({}^{g}\kappa^L)(u,v) .\end{aligned}$$

Thus expression (7.2) above is zero and $uv - vu - \kappa'(u,v)$ lies in the kernel of f for all u, v in V.

We may follow the rest of the proof of Theorem 4.1 of [15] to see that $gv - {}^{g}vg$ lies in the kernel of f for all g in G and v in V and that f is an isomorphism. $\square$

Remark 7.3. Theorem 7.1 above is false in the modular setting, i.e., when char (k) divides $|G|$. Indeed, Example 6.3 gives an algebra $\mathcal{H}_{\lambda,0}$ exhibiting the PBW property for some parameter function λ, but we claim that there is no parameter $\kappa' : V \times V \to kG \oplus (V \otimes kG)$ for which $\mathcal{H}_{\lambda,0} \cong \mathcal{H}_{0,\kappa'}$ as filtered algebras.

If there were, then $\mathcal{H}_{0,\kappa'}$ would exhibit the PBW property and any isomorphism $f : \mathcal{H}_{\lambda,0} \to \mathcal{H}_{0,\kappa'}$ would map the relation

$$gv - vg - g = gv - {}^g vg - \lambda(g, v) = 0$$

in $\mathcal{H}_{\lambda,0}$ to 0 in $\mathcal{H}_{0,\kappa'}$. But f is an algebra homomorphism and takes the filtered degree 1 component of $\mathcal{H}_{\lambda,0}$ to that of $\mathcal{H}_{0,\kappa'}$, giving a relation

$$f(g)f(v) - f(v)f(g) - f(g) = 0$$

in $\mathcal{H}_{0,\kappa'}$ with first two terms of the left hand side of filtered degree 1. In particular, the sum of the terms of degree 0 vanish. But this implies that $f(g) = 0$ since the degree 0 terms of $f(g)f(v)-f(v)f(g)$ cancel with each other as kG is commutative. This contradicts the assumption that f is an isomorphism.

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