Invariant Theory
and
Hochschild Cohomology of Skew Group Algebras

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Outline

1. Hochschild Cohomology and Associative Deformations

2. Hochschild Cohomology of Skew Group Algebras

3. Classical invariant theory tools

Hochschild Cohomology and Associative Deformations

Deformations of algebras

Start with an associative $\mathbb{F}$-algebra: A

Adjoin a central parameter: $A[t]$
(this will be underlying vector space structure)

Define a new multiplication: $a * b = ab + \mu_1(a,b)t + \mu_2(a,b)t^2 + \cdots$
(first define for pairs of elements in A; then extend $\mathbb{F}[t]$-bilinearly)

Specialize to $t \in \mathbb{F}$ to get many algebras
$t = 0 \rightarrow$ original algebra
$t = 1 \rightarrow$???

Hochschild cohomology and associative deformations

$A \overset{\text{deform}}{\rightsquigarrow} A[t]$ with multiplication $a * b = ab + \mu_1(a,b)t + \mu_2(a,b)t^2 + \cdots$

- picking any old μ_k's will not usually yield an **associative** algebra
- to even get started, μ_1 must be a Hochschild 2-cocycle:

$$a\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$$

What is Hochschild cohomology?

Hochschild cohomology of an algebra A

- 1-maps: $A \to A$
- 2-maps: $A \otimes A \to A$
- 3-maps: $A \otimes A \otimes A \to A$

  $\vdots$
- $\mathrm{HH}^k(A)$ =equivalence classes of k-maps satisfying cocycle conditions

$$\mathrm{HH}^0(A) \quad = \quad \text{center of } A$$

$$\mathrm{HH}^1(A) \quad = \quad \frac{\text{derivations}}{\text{inner derivations}} \quad \text{(can use to construct some 2-cocycles!)}$$

$$\mathrm{HH}^2(A) \quad \supset \quad \text{first multiplication maps of deformations}$$

$$\mathrm{HH}^3(A) \quad \supset \quad \text{``obstructions'' to obtaining associative deformations}$$

Hochschild Cohomology of Skew Group Algebras

Skew group algebras

group G $\xrightarrow{\text{acts on}}$ vectorspace V

group algebra $\mathbb{C}G$ $\xrightarrow{\text{acts on}}$ polynomial algebra $S(V) \cong \mathbb{C}[v_1, \ldots, v_n]$

Skew group algebra $S(V)\#G$

Elements: $\mathbb{C}$-linear combos of monomials $v_1^{e_1} \cdots v_n^{e_n}\mathbf{g}$

Relations:

- $vw - wv = 0$
- group elements multiply as in group
- $\mathbf{g}v = \vec{g}(v)\mathbf{g}$

$\text{HH}^0(S(V)\#G)$ - Invariant polynomials

The center of a skew group algebra is the set of G-invariant polynomials:

$$S(V)^G = \{f \in S(V) \,:\, \vec{g}(f) = f \text{ for all } g \in G\}$$

Example - Sym_3 acts on $\mathbb{C}[x, y, z]$ by permuting the variables

- elementary symmetric polynomials

$$\begin{aligned} f_1 &= x + y + z \\ f_2 &= yz + xz + xy \\ f_3 &= xyz \end{aligned}$$

- use these to build more: e.g. $3f_1^5 - 7f_2f_3$ is also invariant
- How many invariant polynomials are there?

$\mathrm{HH}^1(S(V)\#G)$ - Invariant derivations

$$\mathrm{HH}^1(S(V)\#G) \cong \{\text{invariant derivations on } S(V)\}$$

Example - Sym_3 also permutes the "dual vectors" ∂_x, ∂_y, ∂_z

- a few invariant derivations

$$\begin{aligned}\theta_0 &= \partial_x + \partial_y + \partial_z\\ \theta_1 &= x\partial_x + y\partial_y + z\partial_z\\ \theta_2 &= yz\partial_x + xz\partial_y + xy\partial_z\end{aligned}$$

- build more by multiplying by invariant polynomials:

$$f_3\theta_2 = xy^2z^2\partial_x + x^2yz^2\partial_y + x^2y^2z\partial_z$$

- How many invariant derivations are there?

Big invariant theory description of HH($S(V)\#G$)

Theorem (Ginzburg-Kaledin, Farinati)

$$HH^k(S(V)\#G) \cong (S(V) \otimes \bigwedge^k V^*)^G \oplus \textit{more!}$$

The remaining summands are spaces of semi-invariants under centralizer subgroups:

$$\left(S(V^g) \otimes \bigwedge^{k-c} (V^g)^* \otimes \mathbb{C}_\chi \right)^{Z(g)}$$

(one summand per conjugacy class)

Classical invariant theory tools

Graded vector spaces and Poincaré series

Start with a graded vector space:

$$S = \bigoplus_{d \geq 0} S_d \qquad \text{(with } S_d S_e \subset S_{d+e}\text{)}$$

The Poincaré series for S is a power series with the coefficient of t^d recording the dimension of S_d:

$$P_t(S) = \sum_{d \geq 0} (\dim_{\mathbb{C}} S_d) t^d$$

Poincaré series for $\mathbb{C}[f_1, f_2, f_3] \subset \mathbb{C}[x, y, z]^{\mathrm{Sym}_3}$

$$f_1 = x + y + z, \qquad f_2 = yz + xz + xy, \qquad f_3 = xyz$$

Build more polynomials $\longrightarrow$

Note: f_1, f_2, f_3 are algebraically independent, which forces all these new polynomials to be linearly independent.

degree	polynomials
0	1
1	f_1
2	f_1^2, f_2
3	f_1^3, $f_1 f_2$, f_3
$\vdots$	$\vdots$

$$\begin{aligned} P_t(\mathbb{C}[f_1, f_2, f_3]) &= 1 + t + 2t^2 + 3t^3 + \cdots \\ &= (1 + t + t^2 + \cdots)(1 + t^2 + t^4 + \cdots)(1 + t^3 + t^6 + \cdots) \\ &= \frac{1}{(1-t)(1-t^2)(1-t^3)} \end{aligned}$$

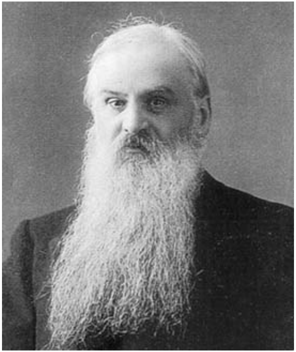
Molien's theorem (1897)

Poincaré series for ring of invariant polynomials

Poincaré series for S^G

$$P_t(S^G) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-gt)}$$

$\det(1-gt)$ is the characteristic polynomial for the matrix of g.



Poincaré series for $\mathbb{C}[x, y, z]^{\mathrm{Sym}_3}$

elements	eigenvalues	characteristic polynomial
1	1, 1, 1	$(1-t)^3$
(12), (13), (23)	1, 1, -1	$(1-t)^2(1+t)$
(123), (321)	1, ω, ω^2	$(1-t)(1-\omega t)(1-\omega^2 t)$

$$P_t(S^G) = \frac{1}{6}\left[\frac{1}{(1-t)^3} + \frac{3}{(1-t)^2(1+t)} + \frac{2}{(1-t^3)}\right]$$
$$= \frac{1}{(1-t)(1-t^2)(1-t^3)}$$

By comparing Poincaré series: $\mathbb{C}[f_1, f_2, f_3] = \mathbb{C}[x, y, z]^{\mathrm{Sym}_3}$

Big invariant theory description of $\mathrm{HH}(S(V)\#G)$

Theorem (Ginzburg-Kaledin, Farinati)

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The remaining summands are spaces of semi-invariants under centralizer subgroups:

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(one summand per conjugacy class)

Generalizations of Molien's theorem

Tensoring with an exterior algebra

Poincaré series for $(S(V) \otimes \bigwedge V^*)^G$

$$P_{X,Y} = \frac{1}{|G|} \sum_{g \in G} \frac{\det(1 + g^* Y)}{\det(1 - gX)}$$

The coefficient of $X^i Y^j$ tells you the dimension of $(S_i(V) \otimes \bigwedge^j V^*)^G$.

More generally, the Poincaré series for $(S(V) \otimes \bigwedge V^* \otimes \mathbb{C}_\chi)^G$ is

$$P_{X,Y} = \frac{1}{|G|} \sum_{g \in G} \frac{\chi^*(g) \det(1 + g^* Y)}{\det(1 - gX)}$$

Predictive power of Poincaré

Module structure for identity component of $\mathrm{HH}(S(V)\#\mathrm{Alt}_4)$

Let $G = \mathrm{Alt}_4$ act irreducibly on $V \cong \mathbb{C}^3$.

Poincaré series for identity component of $\mathrm{HH}(S(V)\#G)$:

$$\frac{1+x^6}{(1-x^2)(1-x^3)(1-x^4)} + \frac{x+x^2+2x^3+x^4+x^5}{(1-x^2)(1-x^3)(1-x^4)}y + \frac{x+x^2+2x^3+x^4+x^5}{(1-x^2)(1-x^3)(1-x^4)}y^2 + \frac{1+x^6}{(1-x^2)(1-x^3)(1-x^4)}y^3$$

invariant polynomials | invariant derivations | invariant 2-forms | invariant 3-forms

Read off finitely-generated free module structure:

- denominators $\rightsquigarrow$ free module over $R = \mathbb{C}[f_2, f_3, f_4]$

  where f_2, f_3, f_4 are algebraically independent invariant polynomials of degrees 2, 3, 4

- numerators $\rightsquigarrow$ how many generators? polynomial degree?

Predictive power of Poincaré

Module structure for identity component of $HH(S(V)\#\mathrm{Alt}_4)$

Let $G = \mathrm{Alt}_4$ act irreducibly on $V \cong \mathbb{C}^3$.

Poincaré series for identity component of $HH(S(V)\#G)$:

$$\underbrace{\frac{1+x^6}{(1-x^2)(1-x^3)(1-x^4)}}_{\text{invariant polynomials}} + \underbrace{\frac{x+x^2+2x^3+x^4+x^5}{(1-x^2)(1-x^3)(1-x^4)}y}_{\text{invariant derivations}} + \underbrace{\frac{x+x^2+2x^3+x^4+x^5}{(1-x^2)(1-x^3)(1-x^4)}y^2}_{\text{invariant 2-forms}} + \underbrace{\frac{1+x^6}{(1-x^2)(1-x^3)(1-x^4)}y^3}_{\text{invariant 3-forms}}$$

Invariant derivations:

- the invariant derivations are the free R-span of six derivations of polynomial degrees 1, 2, 3, 3, 4, 5
- it is now a *finite linear algebra problem* to find all invariant derivations

Thanks!