Global $\widetilde{SL(2,\mathbb{R})}$ representations of the Schrödinger equation with time-dependent potentials

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Outline

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Problem Setup

We consider the one-dimensional Schrödinger equation

$$\Box u = 2V(t,x)u, \tag{1}$$

where

$$\Box = 2i\partial_t + \partial_x^2$$

is the potential free Schrödinger operator. We look at potentials of the form

$$V(t,x) = g_2(t)x^2 + g_1(t)x + g_0(t) + \lambda x^{-2}$$

with $\lambda \cdot g_1 = 0$. It has been shown that this is the general form of all the potentials whose symmetry algebra contains $\mathfrak{sl}(2,\mathbb{R})$ as a subalgebra.

Problem Setup

Using Lie's original prolongation algorithm, it can be shown that potentials of the form

$$V_1(x) = \lambda \tag{2a}$$

$$V_2(x) = \lambda x \tag{2b}$$

$$V_3(x) = \lambda x^2, \tag{2c}$$

where $\lambda \in \mathbb{R}$ or the more general potential $V(t,x) = g_2(t)x^2 + g_1(t)x + g_0(t)$ have symmetry Lie algebra isomorphic to

$$\mathfrak{g} := \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{h}_3(\mathbb{R}).$$

We will show that the study of the representation theory associated to these potentials reduces to the study of the representation theory of the potential free case, which has already been studied.

Problem Setup

Potentials,

$$V_4(x) = \lambda x^{-2} \tag{3a}$$

$$V_5(x) = \lambda_1 x^2 + \lambda_2 x^{-2}, \tag{3b}$$

where $\lambda, \lambda_i \in \mathbb{R}$ are arbitrary constants, or the more general potential $V(t,x) = \lambda x^{-2} + g_2(t)x^2 + g_0(t)$ have symmetry Lie algebras isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}$.
We will show that the study of the potential (3b) and the general potential, $V(t,x) = \lambda x^{-2} + g_2(t)x^2 + g_0(t)$, reduces at least locally to the study of potential (3a) and we will study the solution space for (3a).

Goal

- Our goal is to use representation theory to study the solution space of the Schrödinger equation with potential (3a).
- The problem is that the resulting actions are not necessarily global, that is they do not always extend to an action of the group. Then, the techniques of representation theory do not always apply.
- What saves the day? It is sometimes possible to look at special subspaces of solution that carry the structure of a Lie group representation.

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3. Compact Picture and K-types
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The Group

In order to study the representation theory of the solution space associated with (3a), we need to introduce the appropriate notation. We start with the Lie group that will act on the solution space. Let $G_0 = SL(2,\mathbb{R})$ and let H_3 denote the three dimensional Heisenberg group. Write $\widetilde{G_0}$ for the 2-fold cover of $SL(2,\mathbb{R})$ and let

$$G := \widetilde{G_0} \ltimes H_3.$$

Here $\widetilde{G_0}$ projects to G_0, which acts on H_3 by the standard action on the first two coordinates and leaves the third fixed.

Parabolic Subgroup

We consider the parabolic subalgebra of lower triangular matrices $\overline{\mathfrak{q}} \subset \mathfrak{sl}(2,\mathbb{R})$ with Langlands decomposition $\mathfrak{m} \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}$. Let K be the maximal compact subgroup of G and write M for the centralizer of $A := \exp(\mathfrak{a})$ in K. Define $W \subset H_3$ by

$$W = \{(0, v, w) | v, w \in \mathbb{R}\} \cong \mathbb{R}^2$$

and

$$X := \{(x, 0, 0) | x \in \mathbb{R}\}.$$

Write $\mathfrak{w}$ for the Lie algebra of W. Then

$$\overline{P} = MA\overline{N} \ltimes W$$

is the analogue of a parabolic subgroup in G corresponding to $\overline{\mathfrak{p}} := \overline{\mathfrak{q}} \ltimes \mathfrak{w}$.

Induced Representations

We construct three isomorphic representations. In the first, the representation theory takes place. This is a the space of smooth sections of a particular line bundle. The second version, is where the physiscs live. The third picture, which is attained by partially compactifying $\mathbb{R}^2$, will be particularly useful to study the K-finite vectors of the space of solutions.

A character on $\overline{P}$ that is trivial on N is parametrized by a triplet (q, r, s) where $s, r \in \mathbb{C}$ and $q \in \mathbb{Z}_4$ and denoted by $\chi_{q,r,s}$.

The standard induced representation space of $\chi_{q,r,s}$ will be denoted by $I(q, r, s)$ and defined by

$$I(q,r,s) := \{\phi : G \to \mathbb{C} | \phi \in C^\infty \text{ and } \phi(g\overline{p}) = \chi^{-1}_{q,r,s}(\overline{p})\phi(g) \text{ for } g \in G, \overline{p} \in \overline{P}\}. \quad (4)$$

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Non-compact Picture

The second picture, that in the semisimple category would be named the non-compact picture, uses the isomorphism between $\mathbb{R}^2$ and N. This isomorphism is given by

$$(t,x) \to N_{t,x} := \left[\left(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix} \right), z \mapsto 1 \right), (x,0,0) \right].$$

The restriction of $\phi \in I(q,r,s)$ to N defines an isomorphism of vector spaces from $I(q,r,s)$ to

$$I'(q,r,s) = \{ f \in C^\infty(\mathbb{R}^2) | f(t,x) = \phi(N_{t,x}) \\ \text{for some } \phi \in I(q,r,s) \}.$$

This space is endowed with the action that makes the map, $\phi \mapsto f$ where $f(t,x) = \phi(N_{t,x})$, intertwining. Thus $I(q,r,s) \cong I'(q,r,s)$ as G-modules.

Casimir elements

Let

$$\Omega = 1/2h^2 - h + 2e^+e^-$$

be the Casimir element in the enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$ and define $\Omega' = 2\Omega - r(r+2)$.

Corollary

On $I'(q,r,s)$, Ω acts by

$$\Omega = \frac{1}{2}\left(4sx^2\partial_t + x^2\partial_x^2 - (1+2r)x\partial_x + r(r+2)\right).$$

In particular, for $r = -1/2$ and $s = i/2$, Ω' acts by

$$\Omega' - 2\lambda = x^2(\Box - 2\lambda/x^2)$$

As a consequence of this corollary, we are interested in the study of $\ker(\Omega' - 2\lambda)$.

Change of coordinates (Pause for a second)

In order to define the appropriate multiplier representation space we start by defining a change of variables $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\gamma(t,x) := \left(\int_0^t \frac{1}{\chi_2^2} \, , \, \frac{1}{\chi_2(t)} x + \int_0^t \frac{\mathcal{C}_2}{\chi_2^2} \right) .$$

Let $f \in I'(q,r,s)$ and define the map $f \mapsto \tilde{f}$ by

$$\tilde{f}(t,x) = e^{\int_0^t \frac{\mathcal{B}_2(u)}{\chi_2^2(u)} + \left(\frac{1}{\chi_2^2(u)} \left(\frac{1}{2} \varphi_2'(u) x + \mathcal{A}_2(u) \right) \right)^2 du} f(\gamma(t,x)). \quad (5)$$

The space $I'(q,r,s)_\mu$ is defined as the image of $I'(q,r,s)$ under this map, and is given the structure of a G-module that makes the map intertwining.

Action of the Casimir Elements

Corollary

For the parameters $r = -1/2$ and $s = i/2$, the Casimir element acts on $I(q,r,s)_\mu$ by

$$\Omega = \frac{1}{2}\Big[(x - \chi_1\mathcal{C}_2 + \chi_2\mathcal{C}_1)^2 \cdot(\Box - 2(g_2(t)x^2 + g_1(t)x + g_0(t))) - 3/4\Big].$$

In particular,

$$\ker\Omega' = \ker\left(\Box - 2(g_2(t)x^2 + g_1(t)x + g_0(t))\right)$$

in $I'(q,r,s)_\mu$.

Action of the Casimir Elements (continued)

Corollary

If $\lambda \neq 0$ *then* $g_1(t) \equiv 0$ *and* Ω' *acts by*

$$\Omega' = x^2(\Box - 2(g_2(t)x^2 + g_0(t))).$$

Thus

$$\ker(\Omega' - 2\lambda) = \ker\left(\Box - 2(g_2(t)x^2 + g_0(t) + \lambda/x^2)\right)$$

in $I'(q,r,s)_\mu$.

This shows that when $\lambda = 0$, at least locally, the time-dependent cases analyzed reduce to the potential free case. It also shows that when $\lambda \neq 0$ the study reduces to the study of the eigenvalue problem of Ω'.

Compact Picture

Using the maximal compact subgroup $K \subset \widetilde{G_0}$ and the fact that there exists an isomorphism $K \times X \cong S^1 \times \mathbb{R}$ that induces a 4π-periodic isomorphism into $\mathbb{R}^2$, we realize a G-isomorphism between $I'(q,r,s)$ and

$$I''(q,r,s) = \{F \in C^\infty(\mathbb{R}^2) | F(\theta + j\pi, (-1)^j y) = i^{-jq} F(\theta, y)\}.$$

The map is given by restriction, that is $\phi \mapsto F$ iff

$$\phi([(g_\theta, \epsilon_\theta), (y, 0, 0)]) = F(\theta, y).$$

This realization is particularly useful to write the K-types of $I'(q,r,s)$ explicitly.

Action of the Casimir element on I”(q,r,s)

The action of the algebra $\mathfrak{g}$ on $I'(q,r,s)$ can be translated into an action on $I''(q,r,s)$. We define a standard basis of $\mathfrak{sl}_2(\mathbb{C})$ given by

$$\kappa = i(e^- - e^+)$$

and

$$\eta^\pm = 1/2(h \pm i(e^+ + e^-)).$$

Theorem

If Ω'' denotes the differential operator by which the central element Ω' acts on $I''(q,r,s)$ then

$$\Omega'' = y^2\left(4s\partial_\theta + 4s^2y^2 + \partial_y^2 + \frac{1+2r}{y}\partial_y\right).$$

K-Finite Vectors

A K-finite vector in $I''(q,r,s)$ is shown to be of the form

$$F_m(\theta, y) = e^{-im\theta/2}\tilde{F}_m(y)$$

for $m \in \mathbb{Z}$. Moreover, it is in $\ker(\Omega'' - 2\lambda)$ if and only if $\tilde{F}_m(y)$ is annihilated by the differential operator $\mathcal{D} = y^2\partial_y^2 - (2\lambda - my^2 + y^4)$. This imposes a condition on the eigenvalue λ.

Theorem

There exist a K-finite vector of weight m in $\ker(\Omega'' - 2\lambda) \subset I''(q,r,s)$ iff

$$l = \frac{1}{2}(1 + \sqrt{1+8\lambda}) \tag{6}$$

is a positive integer (equivalently, $\lambda = l(l-1)/2$ for $l \in \mathbb{Z}^{>0}$) and $m \equiv 2l + q \mod 4$.

K-Finite Vectors (continued)

Theorem

If a K-finite vector of weight m exists in $\ker(\Omega'' - 2\lambda) \subset I''(q, r, s)$ and $\lambda \neq 0$, then it is unique (up to scalar multiples) and it is given by

$$\Psi_{m,l}(\theta, y) = e^{-im\theta/2} e^{-y^2/2} y^l \ {}_1F_1\left(\frac{1 + 2l - m}{4}, l + \frac{1}{2}, y^2\right) \tag{7}$$

where ${}_1F_1$ is a confluent hypergeometric function of the first kind.

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$\mathfrak{sl}_2$-action on $\ker(\Omega'' - 2\lambda)$

Theorem

The $\mathfrak{sl}_2$-triple $\{\kappa, \eta^{\pm}\}$ acts on $\Psi_{m,l}$ by

$$\kappa.\Psi_{m,l} = \frac{m}{2}\Psi_{m,l} \tag{8}$$

$$\eta^{\pm}.\Psi_{m,l} = -\frac{2l+1 \pm m}{4}\Psi_{m\pm 4,l} \tag{9}$$

Lowest (resp. highest) weight vectors occur if $m \equiv 2l + 1 \mod 4$ (resp. $m \equiv -2l - 1 \mod 4$) and the weight vector is of the form

$$e^{-\frac{1}{2}(2l+1)i\theta}e^{-\frac{y^2}{2}}y^l.$$

$$(\textit{resp. } e^{\frac{1}{2}(2l+1)i\theta}e^{\frac{y^2}{2}}y^l).$$

A definition to slow us down

Definition

Let $H_l = \ker(\Omega'' - 2\lambda)_K$ denote the K-finite vectors in $\ker(\Omega'' - 2\lambda) \subset I''(q, r, s)$. For $k \in \mathbb{Z}^{\geq 0}$ define

$$H_k = \operatorname{span}_{\mathbb{C}}\{\Psi_{m,k} : m \equiv 2k + q \mod 4 \text{ for } m \in \mathbb{Z}\}.$$

For $q \equiv 1 \mod 4$ and $k \in \mathbb{Z}^{\geq 0}$ define

$$H_k^+ = \operatorname{span}_{\mathbb{C}}\{\Psi_{m,k} : m \geq 2k + 1 \text{ and } m \equiv 2k + 1 \mod 4 \text{ for } m \in \mathbb{Z}\}.$$

For $q \equiv -1 \mod 4$ and $k \in \mathbb{Z}^{\geq 0}$ define

$$H_k^- = \operatorname{span}\{\Psi_{m,k} : m \leq -(2k + 1) \text{ and } m \equiv -(2k + 1) \mod 4 \text{ for } m \in \mathbb{Z}\}.$$

Structure of $\ker(\Omega'' - 2\lambda)$ in $I''(q, r, s)$ as an $\mathfrak{sl}_2$-module

Theorem

Given $q \in \mathbb{Z}_4$ and $l = \frac{1}{2}(1 + \sqrt{1 + 8\lambda})$, then as $\mathfrak{sl}_2$-modules:

1. *If $q \equiv 0 \mod 4$ or $q \equiv 2 \mod 4$ then $H_l = \ker(\Omega'' - 2\lambda)_K$ is irreducible as an $\mathfrak{sl}_2$-module.*
2. *If $q \equiv \pm 1 \mod 4$, then $H_l^{\pm}$ is he only irreducible submodule and the composition series for $\ker(\Omega'' - 2\lambda)_K$ is given by*

$$0 \subset H_l^{\pm} \subset H_l$$

Outline

Heisenberg Action

Let
$$E_{\mp} := (1, \pm i, 0) \in \mathfrak{h}_3(\mathbb{C}).$$

Theorem

Let $m \in \mathbb{Z}$ and $k \in \mathbb{Z}^{\geq 0}$. Then,

$$E^{-}.\Psi_{m,k} = \frac{(1+2k-m)(k-1)}{(2k-1)(2k+1)}\Psi_{m-2,k+1} - k\Psi_{m-2,k-1}$$

and

$$E^{+}.\Psi_{m,k} = \frac{(1+2k+m)(k-1)}{2(2k-1)}\Psi_{m+2,k+1} - k\Psi_{m+2,k-1}.$$

Yet Another Definition

Definition

Let
$$H = \bigoplus_{l \in \mathbb{Z}^{\geq 0}} H_l. \tag{10}$$
Whenever the spaces are defined, let
$$H^{\pm} = \bigoplus_{l \in \mathbb{Z}^{\geq 2}} H_l^{\pm}. \tag{11}$$

H as a $\mathfrak{g}$-module

Theorem

Let $q \in \mathbb{Z}_4$ and $k \in \mathbb{Z}^{\geq 0}$. With respect to the action of $\mathfrak{g}$:

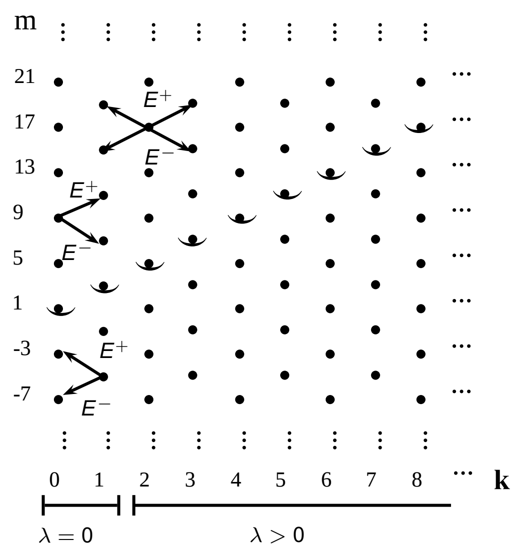
1. *If $q = 0$ or $q = 2$, the composition series of H is*

$$0 \subset H_0 \oplus H_1 \subset H.$$

2. *If $q \equiv \pm 1 \mod 4$, then the composition series of $\mathfrak{g}$-submodules of H is as follows*

$$0 \subset H_0^{\pm} \oplus H_1^{\pm} \subset H_0 \oplus H_1 \subset H_0 \oplus H_1 \oplus H^{\pm} \subset H.$$

Pictorially



Thank you!

Questions?