Cohomology of Hopf Algebras

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Motivation

- Homological algebra has a large number of applications in many different fields
- Hopf algebra structure
- The finite generation of the cohomology of Hopf algebra makes it easier to compute and able to apply algebraic geometry/commutative algebra in the study

Bialgebra

Definition

A bialgebra over a field k is a k-vector space B endowed with algebra and coalgebra structures:

an algebra structure:

$$m : B \otimes B \longrightarrow B \qquad u : k \longrightarrow B$$

satisfying:

- $m \circ (m \otimes id) = m \circ (id \otimes m)$
- $m \circ (u \otimes id) = 1_k.id_B$
- $m \circ (id \otimes u) = id_B.1_k$

Bialgebra

Definition

A bialgebra over a field k is a k-vector space B endowed with algebra and coalgebra structures:

a coalgebra structure:

$$\Delta : B \to B \otimes B \qquad \varepsilon : B \to k$$

satisfying:

- $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$
- $(id \otimes \varepsilon) \circ \Delta = id_B \otimes 1_k$
- $(\varepsilon \otimes id) \circ \Delta = 1_k \otimes id_B$

Notation: $\Delta(b) = \sum b_1 \otimes b_2, \forall b \in B$.

Bialgebra

Definition

A bialgebra over a field k is a k-vector space B endowed with algebra and coalgebra structures:
such that either (Δ and ε are algebra morphisms) OR (m and u are coalgebra morphisms).

(Co)commutativity

- An algebra (B, m, u) is said to be commutative if $ab = ba, \forall a, b \in B$.
- A coalgebra (B, Δ, ε) is called cocommutative if $\sum b_1 \otimes b_2 = \sum b_2 \otimes b_1, \forall b_i \in B$.

Hopf Algebra

Definition

A Hopf algebra is a bialgebra H with a linear map $S : H \to H$, such that $\forall h \in H$:

$$\sum S(h_1)\, h_2 \;=\; \varepsilon(h)1_H \;=\; \sum h_1\, S(h_2)$$

The map S is called the antipode of H.

Chain Complex

Definition

Let R be a ring. A chain complex $\{C_\bullet, d\} = \{C_n, d_n : C_n \to C_{n-1}\}$ is a family of R-modules C_n and R-module homomorphisms d_n:

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

such that $d_n \circ d_{n+1} = 0, \forall n \in \mathbb{Z}$.

Definition

The nth homology of the complex $\{C_n, d_n\}$ is the quotient

$$H_n(C_\bullet) = Ker(d_n)/Im(d_{n+1})$$

Cochain Complex

Definition

Let R be a ring. A cochain complex is a family $\{C_\bullet, d\} = \{C^n, d^n : C^n \to C^{n+1}\}$:

$$\cdots \leftarrow C^{n+1} \xleftarrow{d^n} C^n \xleftarrow{d^{n-1}} C^{n-1} \xleftarrow{d^{n-2}} \cdots$$

such that $d^n \circ d^{n-1} = 0, \forall n \in \mathbb{Z}$.

Definition

The nth cohomology of the cochain complex $\{C^n, d^n\}$ is the quotient

$$H^n(C\bullet) = Ker(d^n)/Im(d^{n-1})$$

Projective resolution

Definition

Let $B \in_R \mathcal{M}$, a projective resolution of B, denoted by $P_\bullet = \{P_n, d_n\}$, is an exact sequence of R-projective modules

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} B \longrightarrow 0$$

Recursively, choose P_0 projective and $\varepsilon : P_0 \to B$ surjective map
Choose P_1 projective and $\varepsilon_1 : P_1 \to Ker(\varepsilon)$ surjective map
Choose P_2 projective and $\varepsilon_2 : P_2 \to Ker(\varepsilon_1)$ surjective map
etc.
Let $d_n = \iota_n \circ \varepsilon_n$, where $\iota : Ker(\varepsilon_n) \hookrightarrow P_{n-1}$ is inclusion map.
Since $Im(d_n) = Ker(d_{n-1})$, by this way, for any left R-module B, we can construct a projective resolution of B.

Ext

Definition

Ext^*_R: Let $C \in_R \mathcal{M}$, apply $Hom_R(-, C)$ to projective resolution $P_\bullet$ of B and drop the last term $Hom_R(B, C)$, we get:

$0 \xrightarrow{0} Hom(P_0, C) \xrightarrow{d_1^*} Hom(P_1, C) \to \cdots \to Hom(P_n, C) \xrightarrow{d_n^*} Hom(P_{n+1}, C) \to \cdots$

where $d_i^*(f) = f \circ d_i, i \neq 0$.

The nth homology of this complex is isomorphic to

$$Ext^n_R(B, C) \cong H_n(Hom_R(P_\bullet, C)) = Ker(d^*_{n+1})/Im(d^*_n)$$

Augmented Algebra

Definition

An augmented algebra over a commutative ring k is a k-algebra A together with an algebra homomorphism $\varepsilon : A \to k$.

Example

$A = k[x_1, x_2, \ldots, x_n]$, where k is a field

$\varepsilon(x_i) = 0, \forall i \qquad \varepsilon(r) = r, \forall r \in k.$

k is an A-module via ε, i.e. for $a \in A, r \in k$, then $r.a = \varepsilon(a)r$.

Let $M \in_A \mathcal{M}$: $\qquad H^n(A, M) = Ext^n_A(k, M)$

$$H^*(A, M) = \bigoplus_{n \geq 0} H^n(A, M) = \bigoplus_{n \geq 0} Ext^n_A(k, M)$$

Special case: $M = k$, $H^*(A, k)$ turns out to be a (graded) algebra under a cup product.

Example

In particular, let $A = k[x]$. Consider k to be an A-module on which x acts as multiplication by 0.
Let $\varepsilon : A \to k$ be evaluation at 0, i.e. $f(x) \mapsto f(0)$.
Consider projective resolution of k:

$$0 \to A \xrightarrow{.x} A \xrightarrow{\varepsilon} k \to 0$$

Example

Apply $Hom_A(-, k)$ and delete the term $Hom_A(k, k)$, we get:

$$0 \to Hom_A(A, k) \xrightarrow{(.x)^*} Hom_A(A, k) \to 0$$

which is equivalent to

$$0 \to k \xrightarrow{0} k \to 0$$

since $Hom_A(A, k) \cong k$. Thus:

$$Ext_A^n(k, k) = \begin{cases} k & n = 0, 1 \\ 0 & \forall n \geq 2 \end{cases}$$

Example

Let $A = k[x, y]$: Note that $k \cong A/(x, y)$. Consider k to be an A-module via the quotient map (i.e. x, y act as 0 on elements of k. Consider projective resolution of k:

$$0 \longrightarrow A \xrightarrow{\alpha} A \oplus A \xrightarrow{\beta} A \xrightarrow{\varepsilon} k \longrightarrow 0$$

where $\alpha = \begin{pmatrix} y \\ -x \end{pmatrix}$, and $\beta = \begin{pmatrix} x & y \end{pmatrix}$.

Similar computation as before, we get:

$$Ext_A^n(k, k) = \begin{cases} k \oplus k & n = 1 \\ k & n = 0, 2 \\ 0 & \forall n > 2 \end{cases}$$

Example

Note: $A = k[x, y]$ is a Hopf algebra via:

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\Delta(y) = y \otimes 1 + 1 \otimes y$$

$Ext_A^*(k, k)$ is graded commutative, i.e. $ab = (-1)^{deg(a)deg(b)}ba$.
We showed above that $Ext_A^*(k, k) \cong \Lambda(V)$, where V is a vector of dimension 2.

Example

In general, let $A = k[x_1, x_2, \dots, x_n]$, we get:

$$Ext_A^i(k,k) \cong k^{\binom{n}{i}}, \forall i \geq 0$$

so

$$H^*(A,k) = Ext_A^*(k,k) = \bigoplus_{i \geq 0} Ext_A^i(k,k) \cong \bigoplus_{i \geq 0} k^{\binom{n}{i}} \cong \Lambda^*(V)$$

where V is a vector space of dimension n.

Group algebra

Let G be a (multiplicative) group and k is a field, then $kG = \{\sum_{g \in G} a_g g\colon\ a_g \in k\}$ is the associated group algebra.

kG is a Hopf algebra:

- $\left(\sum_{g \in G} a_g g\right) \cdot \left(\sum_{h \in G} b_h h\right) = \sum_{g,h \in G} (a_g b_h)(gh)$
  $\forall a_g, b_h \in k; \forall g, h \in G.$
- $\Delta(g) = g \otimes g$, and $\varepsilon(g) = 1, \forall g \in G.$
- since every $g \in G$ is invertible, define the antipode $S \in Hom_k(kG, kG)$ by $S(g) = g^{-1}$,

Cohomology of finite groups

Notation: the group cohomology

$$H^*(kG, M) = H^*(G, M) = \bigoplus_{n\geq 0} Ext^n_{kG}(k, M)$$

Common example: $M = k$, look at $H^*(G, k)$.

Example

Let $G = \langle g \rangle$, cyclic group of order m. kG-projective resolution:

$$\ldots \xrightarrow{.T} kG \xrightarrow{.(g-1)} kG \xrightarrow{.T} kG \xrightarrow{.(g-1)} kG \xrightarrow{\varepsilon} k \to 0$$

where $T = 1 + g + g^2 + g^3 + ... + g^{m-1}$
$\varepsilon(g^i) = 1, \forall g^i \in G$
$\varepsilon(1 - g^i) = 0, \forall g^i \in G$

Example

Case 1: k is a field, $char(k) \mid m$

$$H^n(G,k) \cong k, \forall n \geq 0$$

Case 2: k is a field, $char(k) \nmid m$

$$H^n(G,k) = \begin{cases} k & n = 0 \\ 0 & \forall n > 0 \end{cases}$$

Case 3: $k = \mathbb{Z}$

$$H^n(G,\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/(m\mathbb{Z}) & n > 0, n \text{ is even} \\ 0 & n > 0, n \text{ is odd} \end{cases}$$

Theorem (Golod '59, Venkov '59, Evens '61)

If G is a finite group and k is a field of positive characteristic , then $H^*(G, k)$ is finitely generated as k-algebra.

And it's also graded commutative since kG is a Hopf algebra.

Theorem (Friedlander-Suslin '97)

If G is a finite group scheme and k is a field of positive characteristic, then $H^*(G, k)$ is finitely generated.

Equivalently,

Theorem

If R is a finite dimensional cocommutative Hopf algebra, then $H^*(R, k)$ is finitely generated.

Theorem (Ginzburg-Kumar '93, Bendel-Nakano-Parshall-Pillen '07)

The cohomology ring of finite dimensional (Lusztig's) small quantum group $u_q(\mathfrak{g})$ over $\mathbb{C}$ is finitely generated.

Theorem (Mastnak-Pevtsova-Schauenburg-Witherspoon '10)

More generally, if H is finite dimensional "pointed" Hopf algebra (under some assumptions), then $H^*(H, k)$ is finitely generated.

Conjecture: *The cohomology ring of a finite dimensional Hopf algebra, $H^*(H, k)$, is finitely generated.*

OPEN QUESTION!

Reference

E 1991 Evens, Leonard, *The cohomology of groups*, Oxford University Press, New York, 1991.

M 1993 Montgomery, Susan, *Hopf algebras and their actions on rings*, Vol. 82 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

MPSW10 M. Mastnak, J. Pevtsova, P. Schauenburg and S. Witherspoon. *Cohomology of finite-dimensional pointed Hopf algebras*, Proc. London Math. Soc., 100, pp. 377-404, 2010.

O 2000 Osborne, M. Scott, *Basic homological algebra*, (Graduate Text in Mathematics), Springer-Verlag, New York, 2000.

Thank You!