Universal Deformation Formulas

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Outline

Preliminaries

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Algebra

An algebra is a \( \mathbb{C} \)-vector space \( A \) together with two \( \mathbb{C} \)-linear maps:
- multiplication \( m : A \otimes A \to A \)
- unit \( u : \mathbb{C} \to A \)

s.t.

a) associativity

\[
\begin{align*}
A \otimes A \otimes A & \xrightarrow{m \otimes \text{id}} A \otimes A \\
\text{id} \otimes m & \downarrow \quad \downarrow \quad m \\
A \otimes A & \xrightarrow{m} A
\end{align*}
\]

b) unit

\[
\begin{align*}
\mathbb{C} \otimes A & \xrightarrow{\text{scalar mult.}} A \\
A \otimes A & \xrightarrow{m} A \otimes \mathbb{C} \\
\text{id} \otimes u & \downarrow \quad \downarrow \\
A & \xrightarrow{\text{scalar mult.}} A
\end{align*}
\]

\( \text{id} \otimes u \)
Coalgebra

A coalgebra is a $C$-vector space $C$ together with two $C$-linear maps:
- comultiplication $\Delta : C \to C \otimes C$
- counit $\varepsilon : C \to C$

s.t.

a) coassociativity

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow{\Delta} & & \downarrow{\Delta \otimes \text{id}} \\
C \otimes C & \xrightarrow{\text{id} \otimes \Delta} & C \otimes (C \otimes C)
\end{array}
\]

b) counit

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow{\text{id} \otimes \varepsilon} & & \downarrow{\text{id} \otimes \varepsilon} \\
C \otimes C & \xrightarrow{\varepsilon \otimes \text{id}} & C \otimes C
\end{array}
\]

\[
\begin{array}{ccc}
C & \xrightarrow{1 \otimes \varepsilon} & C \\
\varepsilon \otimes \text{id} & & \text{id} \otimes \varepsilon \\
C \otimes C & \xrightarrow{\Delta} & C \otimes C
\end{array}
\]
Bialgebra

Let $B$ be a $C$-vector space. We say that $(B, m, u, \Delta, \varepsilon)$ is a bialgebra if
- $(B, m, u)$ is an algebra
- $(B, \Delta, \varepsilon)$ is a coalgebra
- $\Delta$ and $\varepsilon$ are algebra maps

**Notation**

The *sigma notation* for $\Delta$ is given by

$$\Delta(b) = \sum b_1 \otimes b_2$$

for all $b \in B$. 
A Hopf algebra is a bialgebra \((H, m, u, \Delta, \varepsilon)\) with a \(\mathbb{C}\)-linear map \(S : H \to H\) such that

\[
\sum S(h_1) h_2 = \varepsilon(h) 1_H = \sum h_1 S(h_2)
\]

for all \(h \in H\).

The map \(S\) is called the antipode of \(H\).
Let $A$ be an algebra and $B$ a bialgebra. Suppose $A$ is a left $B$-module via

$$\rho : B \otimes A \to A$$

$$b \otimes x \mapsto b(x)$$

for $x \in A, b \in B$. Then $A$ is a left $B$-module algebra if

$$b(xy) = \sum b_1(x) \, b_2(y)$$

$$b(1_A) = \varepsilon(b) \, 1_A$$

for all $x, y \in A, b \in B$. 

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**Module Algebra**
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Formal Deformation

Let $t$ be an indeterminate. A formal deformation of an algebra $A$ is an associative algebra $A[[t]]$ over the formal power series $\mathbb{C}[[t]]$ with multiplication

$$a \ast b = ab + \mu_1(a \otimes b) \, t + \mu_2(a \otimes b) \, t^2 + \cdots$$

for all $a, b \in A$, where

- $ab$ is the multiplication in $A$
- $\mu_i : A \otimes A \to A$ are $\mathbb{C}$-linear maps extended to be $\mathbb{C}[[t]]$-linear
Recall: $A$ is a left $B$-module algebra if

\[ b(xy) = \sum b_1(x) \, b_2(y) \]
\[ b(1_A) = \varepsilon(b) \, 1_A \]

for all $x, y \in A, b \in B$.

We may extend this $\mathbb{C}$-linear action of $B$ to a $\mathbb{C}[[t]]$-linear action of $B[[t]]$. 
A universal deformation formula based on a bialgebra $B$ is an element $F \in (B \otimes B)[[t]]$ of the form

$$F = 1_B \otimes 1_B + tF_1 + t^2F_2 + \ldots$$

with $F_i \in B \otimes B$, satisfying

$$(\varepsilon \otimes \text{id})(F) = 1 \otimes 1_B \quad (\text{id} \otimes \varepsilon)(F) = 1_B \otimes 1$$

and

$$[ (\Delta \otimes \text{id})(F) ] (F \otimes 1_B) = [ (\text{id} \otimes \Delta)(F) ] (1_B \otimes F)$$
Giaquinto and Zhang (1998)

Let $A$ be an algebra and $B$ a bialgebra. Let $m : A \otimes A \to A$ be the multiplication of $A$, extended to be $\mathbb{C}[[t]]$-linear.

**Proposition**

If $A$ is a left $B$-module algebra and $F$ a universal deformation formula based on $B$, then there is a formal deformation of $A$ given by

$$a \ast b = (m \circ F)(a \otimes b)$$

for all $a, b \in A$.

$F$ is universal in the sense that it applies to any $B$-module algebra to yield a formal deformation.
The Hopf Algebra $H_q$

Let $q \in \mathbb{C}^\times$ and let $H$ be the algebra generated by

$$D_1, \ D_2, \ \sigma, \ \sigma^{-1}$$

subject to the relations

$$D_1 \ D_2 = D_2 \ D_1$$
$$\sigma \ D_1 = q^{-1} \ D_1 \ \sigma$$
$$\sigma \ D_2 = q^{-1} \ D_2 \ \sigma$$
$$\sigma \ \sigma^{-1} = \sigma^{-1} \ \sigma = 1_H$$
The Hopf Algebra $H_q$

Then $H$ is a Hopf algebra with

$$
\Delta(D_1) = D_1 \otimes \sigma + 1_H \otimes D_1 \\
\Delta(D_2) = D_2 \otimes 1_H + \sigma \otimes D_2 \\
\Delta(\sigma) = \sigma \otimes \sigma
$$

$$
\varepsilon(D_1) = 0 \quad S(D_1) = -D_1 \sigma^{-1} \\
\varepsilon(D_2) = 0 \quad S(D_2) = -\sigma^{-1} D_2 \\
\varepsilon(\sigma) = 1 \quad S(\sigma) = \sigma^{-1}
$$
The Hopf Algebra $H_q$

If $q$ is a primitive $n$th root of unity ($n \geq 2$), then the ideal $I$ generated by $D_1^n$ and $D_2^n$ is a Hopf ideal, that is

\[
\Delta(I) \subseteq I \otimes H + H \otimes I \\
\varepsilon(I) = 0 \\
S(I) \subseteq I
\]

Thus, the quotient $H/I$ is also a Hopf algebra. Define

\[
H_q = \begin{cases} 
H/I, & \text{if } q \text{ is a primitive } n\text{th root of unity } (n \geq 2) \\
H, & \text{if } q = 1 \text{ or is not a root of unity}
\end{cases}
\]
The $q$-exponential function

Let $A$ be an algebra.
If $q = 1$ or is not a root of unity, the $q$-exponential function is given by

$$\exp_q(y) = \sum_{i=0}^{\infty} \frac{1}{(i)_q!} y^i \quad \text{for } y \in A.$$ 

If $q$ is a primitive $n$th root of unity ($n \geq 2$), the $q$-exponential function is given by

$$\exp_q(y) = \sum_{i=0}^{n-1} \frac{1}{(i)_q!} y^i \quad \text{for } y \in A.$$ 

Notation

$(i)_q = 1 + q + q^2 + \cdots + q^{i-1}$ with $(0)_q = 0$

$(i)_q! = (i)_q (i - 1)_q \cdots (1)_q$ with $(0)_q! = 1$
**Theorem**
Let $q \in \mathbb{C}^\times$. Then
\[
\exp_q(t D_1 \otimes D_2)
\]
is a universal deformation formula based on $H_q$.

**Corollary**
For every $H_q$-module algebra $A$,
\[
m \circ \exp_q(t D_1 \otimes D_2)
\]
gives a formal deformation of $A$. 
Example (Taft Algebra)

Let $A$ be the algebra generated by $a, b, x, y$ subject to the relations

\[
\begin{align*}
    a^2 &= a & b^2 &= b \\
    ab &= 0 & ba &= 0 \\
    x^2 &= 0 & y^2 &= 0 \\
    xy &= 0 & yx &= 0 \\
    ax &= 0 & by &= 0 \\
    ay &= y & bx &= x \\
    xa &= x & yb &= y \\
    xb &= 0 & ya &= 0 \\
    a + b &= 1_A
\end{align*}
\]
Example (Taft Algebra)

Let $q = -1$. Then $H_{-1}$ is generated by

$$D_1, D_2, \sigma, \sigma^{-1}$$

subject to the relations

$$D_1D_2 = D_2D_1$$
$$-\sigma D_1 = D_1\sigma$$
$$-\sigma D_2 = D_2\sigma$$
$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1_H$$
$$D_1^2 = 0$$
$$D_2^2 = 0$$
Example (Taft Algebra)

Define an action of $H_{-1}$ on the generators of $A$ by

\[
\begin{align*}
D_1(a) &= 0 & D_1(b) &= 0 \\
D_2(a) &= 0 & D_2(b) &= 0 \\
\sigma(a) &= b & \sigma(b) &= a \\
D_1(x) &= b & D_1(y) &= a \\
D_2(x) &= a & D_2(y) &= b \\
\sigma(x) &= -y & \sigma(y) &= -x
\end{align*}
\]

Extend this action to all of $A$ under the conditions

\[
\begin{align*}
D_1(fg) &= D_1(f) \sigma(g) + f D_1(g) \\
D_2(fg) &= D_2(f) g + \sigma(f) D_2(g) \\
\sigma(fg) &= \sigma(f) \sigma(g)
\end{align*}
\]

for all $f, g \in A$. 
Example (Taft Algebra)

A is an $H_{-1}$-module algebra:

- the relations of $H_{-1}$ are preserved by the generators of $A$:

Example

Check that $D_1 D_2 = D_2 D_1$ is preserved by $x$:

$$D_1 D_2(x) = D_1(a) = 0 = D_2(b) = D_2 D_1(x).$$

- the relations of $A$ are preserved by the generators of $H_{-1}$:

Example

Check that $xy = 0$ is preserved by $D_1$:

$$D_1(xy) = D_1(x) \sigma(y) + x D_1(y) = -bx + xa = -x + x = 0.$$
Example (Taft Algebra)

Since $A$ is an $H_{-1}$-module algebra, by Corollary, we have that

$$m \circ \exp_q(t \, D_1 \otimes D_2) = m \circ \left( \sum_{i=0}^{n-1} \frac{1}{(i)!} (t \, D_1 \otimes D_2)^i \right)$$

$$= m \circ (1 + t \, D_1 \otimes D_2)$$

yields a formal deformation of $A$.
Recall: a formal deformation of $A$ has multiplication given by

$$a \ast b = ab + \mu_1(a \otimes b) \, t + \mu_2(a \otimes b) \, t^2 + \cdots$$

for all $a, b \in A$.
In this case, $\mu_1 = m \circ (D_1 \otimes D_2)$ and $\mu_j = 0$ for all $j \geq 2$. 
Example (Taft Algebra)

To find the new relations in the deformed algebra $A[[t]]$, consider

$$x * y = (m \circ (1 + t D_1 \otimes D_2)) (x \otimes y)$$

$$= m (x \otimes y) + m ((t D_1 \otimes D_2) (x \otimes y))$$

$$= xy + m (t D_1 (x) \otimes D_2 (y))$$

$$= xy + m (t b \otimes b)$$

$$= xy + tb^2$$

$$= tb$$

Similarly, $y * x = yx + ta^2 = ta.$
Example (Taft Algebra)

The deformation of $A$ is generated by $a, b, x, y$ subject to the new relations

\[
\begin{align*}
    a^2 &= a \\
    ab &= 0 \\
    x^2 &= 0 \\
    xy &= tb \\
    ax &= 0 \\
    ay &= y \\
    xa &= x \\
    xb &= 0 \\
    a + b &= 1_A
\end{align*}
\]
Thank you!!!