Universal Deformation Formulas

Jeanette Shakalli

Department of Mathematics
Texas A&M University

April 23, 2011

Outline

Preliminaries

Universal Deformation Formulas

Outline

Preliminaries

Universal Deformation Formulas

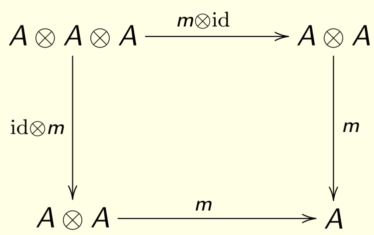
Algebra

An algebra is a $\mathbb{C}$-vector space A together with two $\mathbb{C}$-linear maps:

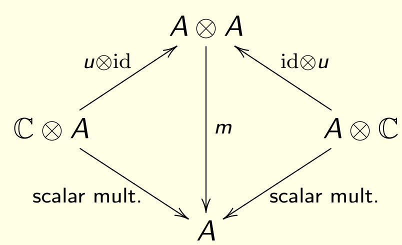
- multiplication $m : A \otimes A \to A$
- unit $u : \mathbb{C} \to A$

s.t.

a) associativity



b) unit



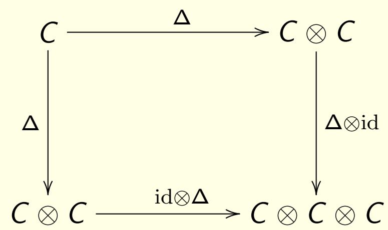
Coalgebra

A coalgebra is a $\mathbb{C}$-vector space C together with two $\mathbb{C}$-linear maps:

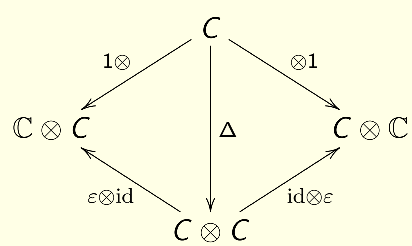
- comultiplication $\Delta : C \to C \otimes C$
- counit $\varepsilon : C \to \mathbb{C}$

s.t.

a) coassociativity



b) counit



Bialgebra

Let B be a $\mathbb{C}$-vector space. We say that $(B, m, u, \Delta, \varepsilon)$ is a bialgebra if

- (B, m, u) is an algebra
- (B, Δ, ε) is a coalgebra
- Δ and ε are algebra maps

Notation

The sigma notation for Δ is given by

$$\Delta(b) = \sum b_1 \otimes b_2$$

for all $b \in B$.

Hopf Algebra

A Hopf algebra is a bialgebra $(H, m, u, \Delta, \varepsilon)$ with a $\mathbb{C}$-linear map

$$S : H \to H$$

such that

$$\sum S(h_1)\, h_2 \;=\; \varepsilon(h)\, 1_H \;=\; \sum h_1\, S(h_2)$$

for all $h \in H$.
The map S is called the antipode of H.

Module Algebra

Let A be an algebra and B a bialgebra. Suppose A is a left B-module via

$$\begin{aligned} \rho : B \otimes A &\longrightarrow A \\ b \otimes x &\mapsto b(x) \end{aligned}$$

for $x \in A, b \in B$. Then A is a left B-module algebra if

$$\begin{aligned} b(xy) &= \sum b_1(x)\, b_2(y) \\ b(1_A) &= \varepsilon(b)\, 1_A \end{aligned}$$

for all $x, y \in A, b \in B$.

Outline

Preliminaries

Universal Deformation Formulas

Formal Deformation

Let t be an indeterminate. A formal deformation of an algebra A is an associative algebra $A[[t]]$ over the formal power series $\mathbb{C}[[t]]$ with multiplication

$$a * b = ab + \mu_1(a \otimes b)\ t + \mu_2(a \otimes b)\ t^2 + \cdots$$

for all $a, b \in A$, where

- ab is the multiplication in A
- $\mu_i : A \otimes A \to A$ are $\mathbb{C}$-linear maps extended to be $\mathbb{C}[[t]]$-linear

Module Algebra

Recall: A is a left B-module algebra if

$$\begin{aligned} b(xy) &= \sum b_1(x)\, b_2(y) \\ b(1_A) &= \varepsilon(b)\, 1_A \end{aligned}$$

for all $x, y \in A, b \in B$.
We may extend this $\mathbb{C}$-linear action of B to a $\mathbb{C}[[t]]$-linear action of $B[[t]]$.

Universal Deformation Formula

A universal deformation formula based on a bialgebra B is an element $F \in (B \otimes B)[[t]]$ of the form

$$F = 1_B \otimes 1_B + t\, F_1 + t^2\, F_2 + \dots$$

with $F_i \in B \otimes B$, satisfying

$$(\varepsilon \otimes \mathrm{id})\,(F) = 1 \otimes 1_B \qquad (\mathrm{id} \otimes \varepsilon)\,(F) = 1_B \otimes 1$$

and

$$[\,(\Delta \otimes \mathrm{id})\,(F)\,]\,(F \otimes 1_B) = [\,(\mathrm{id} \otimes \Delta)\,(F)\,]\,(1_B \otimes F)$$

Giaquinto and Zhang (1998)

Let A be an algebra and B a bialgebra. Let $m : A \otimes A \longrightarrow A$ be the multiplication of A, extended to be $\mathbb{C}[[t]]$-linear.

Proposition

If A is a left B-module algebra and F a universal deformation formula based on B, then there is a formal deformation of A given by

$$a * b = (m \circ F)\,(a \otimes b)$$

for all $a, b \in A$.

F is universal in the sense that it applies to any B-module algebra to yield a formal deformation.

The Hopf Algebra H_q

Let $q \in \mathbb{C}^\times$ and let H be the algebra generated by

$$D_1,\ D_2,\ \sigma,\ \sigma^{-1}$$

subject to the relations

$$\begin{aligned} D_1\ D_2 &= D_2\ D_1 \\ \sigma\ D_1 &= q^{-1}\ D_1\ \sigma \\ \sigma\ D_2 &= q^{-1}\ D_2\ \sigma \\ \sigma\ \sigma^{-1} &= \sigma^{-1}\ \sigma = 1_H \end{aligned}$$

The Hopf Algebra H_q

Then H is a Hopf algebra with

$$\begin{aligned}\Delta(D_1) &= D_1 \otimes \sigma + 1_H \otimes D_1\\ \Delta(D_2) &= D_2 \otimes 1_H + \sigma \otimes D_2\\ \Delta(\sigma) &= \sigma \otimes \sigma\end{aligned}$$

$$\begin{aligned}\varepsilon(D_1) &= 0 & S(D_1) &= -D_1\, \sigma^{-1}\\ \varepsilon(D_2) &= 0 & S(D_2) &= -\sigma^{-1}\, D_2\\ \varepsilon(\sigma) &= 1 & S(\sigma) &= \sigma^{-1}\end{aligned}$$

The Hopf Algebra H_q

If q is a primitive nth root of unity ($n \geq 2$), then the ideal $\mathcal{I}$ generated by D_1^n and D_2^n is a Hopf ideal, that is

$$\begin{aligned} \Delta(\mathcal{I}) &\subseteq \mathcal{I} \otimes H + H \otimes \mathcal{I} \\ \varepsilon(\mathcal{I}) &= 0 \\ S(\mathcal{I}) &\subseteq \mathcal{I} \end{aligned}$$

Thus, the quotient $H/\mathcal{I}$ is also a Hopf algebra.
Define

$$H_q = \begin{cases} H/\mathcal{I}, & \text{if } q \text{ is a primitive } n\text{th root of unity } (n \geq 2) \\ H, & \text{if } q = 1 \text{ or is not a root of unity} \end{cases}$$

The q-exponential function

Let A be an algebra.
If $q = 1$ or is not a root of unity, the q-exponential function is given by

$$\exp_q(y) = \sum_{i=0}^{\infty} \frac{1}{(i)_q!}\, y^i \quad \text{for } y \in A.$$

If q is a primitive nth root of unity ($n \geq 2$), the q-exponential function is given by

$$\exp_q(y) = \sum_{i=0}^{n-1} \frac{1}{(i)_q!}\, y^i \quad \text{for } y \in A.$$

Notation

$(i)_q = 1 + q + q^2 + \cdots + q^{i-1}$ *with* $(0)_q = 0$
$(i)_q! = (i)_q\,(i-1)_q \cdots (1)_q$ *with* $(0)_q! = 1$

Witherspoon (2006)

Theorem

Let $q \in \mathbb{C}^\times$. Then

$$\exp_q(t\ D_1 \otimes D_2)$$

is a universal deformation formula based on H_q.

Corollary

For every H_q-module algebra A,

$$m \circ \exp_q(t\ D_1 \otimes D_2)$$

gives a formal deformation of A.

Example (Taft Algebra)

Let A be the algebra generated by a, b, x, y subject to the relations

$$\begin{aligned} a^2 &= a & b^2 &= b \\ ab &= 0 & ba &= 0 \\ x^2 &= 0 & y^2 &= 0 \\ xy &= 0 & yx &= 0 \\ ax &= 0 & by &= 0 \\ ay &= y & bx &= x \\ xa &= x & yb &= y \\ xb &= 0 & ya &= 0 \\ a + b &= 1_A \end{aligned}$$

Example (Taft Algebra)

Let $q = -1$. Then H_{-1} is generated by

$$D_1,\ D_2,\ \sigma,\ \sigma^{-1}$$

subject to the relations

$$\begin{aligned} D_1 D_2 &= D_2 D_1 \\ -\sigma D_1 &= D_1 \sigma \\ -\sigma D_2 &= D_2 \sigma \\ \sigma\sigma^{-1} &= \sigma^{-1}\sigma = 1_H \\ D_1^2 &= 0 \\ D_2^2 &= 0 \end{aligned}$$

Example (Taft Algebra)

Define an action of H_{-1} on the generators of A by

$$\begin{aligned} D_1(a) &= 0 & D_1(b) &= 0 \\ D_2(a) &= 0 & D_2(b) &= 0 \\ \sigma(a) &= b & \sigma(b) &= a \\ D_1(x) &= b & D_1(y) &= a \\ D_2(x) &= a & D_2(y) &= b \\ \sigma(x) &= -y & \sigma(y) &= -x \end{aligned}$$

Extend this action to all of A under the conditions

$$\begin{aligned} D_1(fg) &= D_1(f)\,\sigma(g) + f\,D_1(g) \\ D_2(fg) &= D_2(f)\,g + \sigma(f)\,D_2(g) \\ \sigma(fg) &= \sigma(f)\,\sigma(g) \end{aligned}$$

for all $f, g \in A$.

Example (Taft Algebra)

A is an H_{-1}-module algebra:

- the relations of H_{-1} are preserved by the generators of A:

Example

Check that $D_1 D_2 = D_2 D_1$ is preserved by x:

$$D_1 D_2(x) = D_1(a) = 0 = D_2(b) = D_2 D_1(x).$$

- the relations of A are preserved by the generators of H_{-1}:

Example

Check that $xy = 0$ is preserved by D_1:

$$D_1(xy) = D_1(x)\,\sigma(y) + x\ D_1(y) = -bx + xa = -x + x = 0.$$

Example (Taft Algebra)

Since A is an H_{-1}-module algebra, by Corollary, we have that

$$\begin{aligned} m \circ \exp_q(t\ D_1 \otimes D_2) &= m \circ \left(\sum_{i=0}^{n-1} \frac{1}{(i)_q!} \, (t\ D_1 \otimes D_2)^i \right) \\ &= m \circ (1 + t\ D_1 \otimes D_2) \end{aligned}$$

yields a formal deformation of A.
Recall: a formal deformation of A has multiplication given by

$$a * b = ab + \mu_1(a \otimes b)\ t + \mu_2(a \otimes b)\ t^2 + \cdots$$

for all $a, b \in A$.
In this case, $\mu_1 = m \circ (D_1 \otimes D_2)$ and $\mu_j = 0$ for all $j \geq 2$.

Example (Taft Algebra)

To find the new relations in the deformed algebra $A[[t]]$, consider

$$\begin{aligned} x * y &= (m \circ (1 + t\ D_1 \otimes D_2))\ (x \otimes y) \\ &= m\ (x \otimes y) + m\ ((t\ D_1 \otimes D_2)\ (x \otimes y)) \\ &= xy + m\ (t\ D_1(x) \otimes D_2(y)) \\ &= xy + m\ (t\ b \otimes b) \\ &= xy + tb^2 \\ &= tb \end{aligned}$$

Similarly, $y * x = yx + ta^2 = ta$.

Example (Taft Algebra)

The deformation of A is generated by a, b, x, y subject to the new relations

$$\begin{aligned} a^2 &= a & b^2 &= b \\ ab &= 0 & ba &= 0 \\ x^2 &= 0 & y^2 &= 0 \\ xy &= tb & yx &= ta \\ ax &= 0 & by &= 0 \\ ay &= y & bx &= x \\ xa &= x & yb &= y \\ xb &= 0 & ya &= 0 \\ a + b &= 1_A & & \end{aligned}$$

Thank you!!!