Cohomology of Quotients of Quantum Symmetric Algebras

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Outline

Let B be a PBW algebra generated by $x_1, \cdots, x_\theta, \cdots, x_n$ and $A = B/(x_1^{N_1}, \cdots, x_\theta^{N_\theta})$.
To show $H^*(A, k) = Ext_A^*(k, k)$ is finitely generated.
Notation: $H^r(A, k) = Ext_A^r(k, k)$ and $H^*(A, k) = \bigoplus_{r \geq 0} H^r(A, k)$.

So let us ask the question. Is $H^*(A, k)$ finitely generated?

- M. Mastnak, J. Pevtsova, P. Schauenburg and S. Witherspoon in 2010 proved for nilpotent generators.
- V. Ginzburg and S. Kumar in 1993 proved for non-nilpotent generators in case for quantum groups at roots of unity.
- For mixed case the work is in progress.

Poincaré Birkhoff Witt Algebra

Definition

A PBW algebra, R over a field k, is a k-algebra together with elements $x_1, \cdots, x_n \in R$ and an a monomial order on $\mathbb{N}^n$ for which there are scalars $q_{ij} \in k^*$ such that
1) $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n\}$ is a basis of R as a k-vector space.
2) $x_i x_j = q_{ij} x_j x_i + p_{ij}$ for $p_{ij} \in R$ with $exp(p_{ij}) < \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq n)$ where $\varepsilon_i = (0, \cdots, 0, 1_i, 0, \cdots, 0) \in \mathbb{N}^n$.

Quantum Symmetric Algebras

Definition

Let k be a field. Let $\mathbb{N}$ be positive integer and for each pair i, j of elements in $\{1, \cdots, n\}$, let q_{ij} be a nonzero scalar such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for i, j. Denote by $\mathbf{q}$ the corresponding tuple of scalars, $\mathbf{q} := (q_{ij})_{1 \leq i < j \leq n}$. Let V be a vector space with basis $x_1, \cdots, x_n$, and let

$$S_{\mathbf{q}}(V) := k\langle x_1, \ldots, x_n \mid x_i x_j = q_{ij} x_j x_i \text{ for all } 1 \leq i < j \leq n\rangle,$$

the quantum symmetric algebra determined by $\mathbf{q}$.

Let $S := k\langle x_1, \cdots, x_\theta, \cdots, x_n \mid x_i x_j = q_{ij} x_j x_i$ for all $i < j$ and
$x_i^{N_i} = 0$ for $1 \leq i \leq \theta\rangle$

Let $K_\bullet$ be the following complex of free S-modules.

For each n-tuple $(a_1, \cdots, a_n)$ of non-negative integers with $a_i = 0$ or 1 for each $i, \theta + 1 \leq i \leq n$, let $\Phi(a_1, \cdots, a_n)$ be a free generator in degree $a_1 + \cdots + a_n$.

Let

$$K_m = \oplus_{a_1+\cdots+a_n=m} S\Phi(a_1, \cdots, a_n).$$

For each $i, 1 \leq i \leq \theta$, let $\sigma_i, \tau_i : \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\sigma_i(a) = \begin{cases} 1, & \text{if } a \text{ is odd} \\ N_i - 1, & \text{if } a \text{ is even,} \end{cases}$$

and

$$\tau_i(a) = \begin{cases} \sum_{j=1}^{a} \sigma_i(j), & \text{for } a \geq 1 \\ 0, & \text{if } a = 0. \end{cases}$$

For each i, $\theta + 1 \leq i \leq n$ we define $\sigma_i(a) = 1$ and $\tau_i(a) = a$.

We define our differential as follows:

$$d_i(\Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n))$$

$$= \begin{cases} \prod_{i<\ell} (-1)^{a_\ell} q_{\ell i}^{\sigma_i(a_i)\tau_\ell(a_\ell)} x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n), & \text{if } a_i > 0 \\ 0, & \text{if } a_i = 0 \end{cases}$$

Next we give a contracting homotopy:
Let $\eta \in S$, and fix $\ell, 1 \leq \ell \leq n$. Write

$$\eta = \begin{cases} \sum_{j=0}^{N_i - 1} \eta_j x_\ell^j, & \text{for } 1 \leq \ell \leq \theta \\ \sum_j \eta_j x_\ell^j, & \text{for } \theta + 1 \leq \ell \leq n \end{cases}$$

where η_j is in the subalgebra of S generated by the x_i with $i \neq \ell$.

Define $s_\ell(\eta\Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n))$

$$= \begin{cases} \displaystyle\sum_{j=0}^{N_i-1} s_\ell(\eta_j x_\ell^j \Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n)), & \text{for } 1 \leq \ell \leq \theta \\ \\ \displaystyle\sum_{j} s_\ell(\eta_j x_\ell^j \Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n)), & \text{for } \theta + 1 \leq \ell \leq n \end{cases}$$

where
$s_\ell(\eta_j x_\ell^j \Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n))$

$$= \begin{cases} \delta_{j>0}(\prod_{\ell<m}(-1)^{a_m} q_{m\ell}^{-\sigma_\ell(a_\ell+1)\tau_m(a_m)})\eta_j x_\ell^{j-1}\Phi(a_1, \cdot\cdot, a_\ell+1, \cdot\cdot a_\theta, \cdot\cdot, a_n), \\ \qquad\qquad \text{if } a_\ell \text{ is even with } 1 \le \ell \le \theta \\ \delta_{j,N_\ell-1}(\prod_{\ell<m}(-1)^{a_m} q_{m\ell}^{-\sigma_\ell(a_\ell+1)\tau_m(a_m)})\eta_j \Phi(a_1, \cdot\cdot, a_\ell+1, \cdot\cdot a_\theta, \cdot\cdot, a_n), \\ \qquad\qquad \text{if } a_\ell \text{ is odd with } 1 \le \ell \le \theta \\ \omega \eta_j x_\ell^{j-1}\Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_\ell+1, \cdots, a_n), \quad \text{if } \theta+1 \le \ell \le n \end{cases}$$

Where $\delta_{j>0} = 1$ if $j > 0$ and 0 if $j = 0$ and $\omega = \dfrac{1}{\prod_{\ell<u}(-1)^{a_u} q_{u\ell}^{a_u}}$.

Exactness

Calculations show that for all i, $1 \leq i \leq n$
$(s_i d_i + d_i s_i)(\eta_j x_i^j \Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n))$

$$= \begin{cases} \eta_j x_i^j \Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n), & \text{if } j > 0 \text{ or } a_i > 0 \\ 0, & \text{if } j = 0 \text{ and } a_i = 0 \end{cases}$$

For all i, ℓ when $i \neq \ell$, we get $s_\ell d_i + d_i s_\ell = 0$.
For each $x_1^{j_1} \cdots x_n^{j_n} \Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n)$, let $C = c_{j_1, \cdots, j_n, a_1, \cdots, a_n}$ be the cardinality of the set of all $i (1 \leq i \leq n)$ such that $j_i a_i = 0$.

Define $s(x_1^{j_1} \cdots x_n^{j_n} \Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n)) =$

$$\frac{1}{n-C}(s_1 + \cdots + s_n)(x_1^{j_1} \cdots x_n^{j_n} \Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n))$$

and letting $d = d_1 + \cdots + d_n$, we have $sd + ds = id$ on each $K_m, m > 0$. That is, $K_\bullet$ is exact in positive degrees. Exactness at $K_0 = S$, can be seen by looking at the kernel of augmentation map $\varepsilon : S \to k$ and the image of $d(x_i^{j_i - 1} \cdots x_n^{j_n} \Phi(0, \cdots, 1, \cdots, 0))$.

Current Work

Let $\xi_i \in Hom_S(K_2, k)$ be the function dual to $\Phi(0, \cdots, 0, 2, 0, \cdots, 0)$ and $\eta_i \in Hom_S(K_1, k)$ be the function dual to $\Phi(0, \cdots, 0, 1, 0, \cdots, 0)$.
Identify these functions with the corresponding elements in $H^2(S, k)$ and $H^1(S, k)$, respectively. We would like to show that the ξ_i, η_i generate $H^*(S, k)$, and determine the relations among them.

In order to do this we define chain maps $\xi_i : K_n \to K_{n-2}$ and $\eta_i : K_n \to K_{n-1}$ by

$$\xi_i(\Phi(a_1, \cdots, a_\theta)) = \prod_{\ell > i} q_{i\ell}^{N_i \tau_\ell(a_\ell)} \Phi(a_1, \cdots, a_i - 2, \cdots, a_\theta)$$

$$\eta_i(\Phi(a_1, \cdots, a_n)) = \prod_{\ell < i} q^{(\sigma_i(a_i)-1)\tau_\ell(a_\ell)} \prod_{\ell > i} (-1)^{a_\ell} q_{i\ell}^{\tau_\ell(a_\ell)} \cdot x_i^{\sigma_i(a_i)-1} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)$$

Thus we conjecture the following:

Conjecture: Let S be the k-algebra generated by $x_1, \cdots, x_\theta, \cdots, x_n$, subject to relations $x_i x_j = q_{ij} x_j x_i$ for all $i < j$, $x_i^{N_i} = 0$ for $1 \leq i \leq \theta$. Then $H^*(S, k)$ is generated by $\xi_i (i = 1, \cdots, \theta)$ and $\eta_i (i = 1, \cdots, n)$ where deg $\xi_i = 2$ and deg $\eta_i = 1$, subject to the relations

$$\xi_i \xi_j = q_{ji}^{N_i N_j} \xi_j \xi_i, \ \eta_i \xi_j = q_{ji}^{N_j} \xi_j \eta_i, \quad \text{and} \quad \eta_i \eta_j = -q_{ji} \eta_j \eta_i.$$

Reference

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2. V. Ginzburg and S. Kumar, *Cohomology of quantum groups at roots of unity,* Duke Math. J. 69(1993), no. 1, 179-198.
3. Deepak Naidu, Piyush Shroff and Sarah Witherspoon, *Hochschild cohomology of group extensions of quantum symmetric algebras,* Proc. Amer. Math. Soc. 139 (2011), 1553-1567.

Thank You!