Groups Acting On Restricted Lie Algebras and Centers of Deformations

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PBW Property

Example of an algebra that doesn't satisfy PBW

Connection Between PBW and Lie Algebras

$v^p - v^{[p]} \in Z(A)$

Smashing Restricted Lie Algebra with a Group

Poincare-Birkhoff-Witt (PBW Property)

Let $V = \mathbb{F} - span\{v_1, \cdots, v_n\} \cong \mathbb{F}^n$ vector space.

Let $\mathbb{F} < v_1, \cdots, v_n >$ be the Free Algebra on $v_1, \cdots, v_n$.

$A = \mathbb{F} < v_1, \cdots, v_n > /$ relations of the form vw-wv=something in V

A satisfies PBW means $\{v_1^{k_1} \cdots v_n^{k_n} : k \in \mathbb{Z}_{\geq 0}\}$ is $\mathbb{F}$-basis for A as $\mathbb{F}$-vector space (ie every $a \in A$ can be written UNIQUELY as finite sum of form $a = \sum \alpha v_1^{k_1} \cdots v_n^{k_n}$

Example that Does Not Satisfy PBW

$A = \mathbb{F} < x, y, z > / relations$

$Relations : yx = xy + x, \quad zy = yz + y, \quad zx = xz + x$

$zyx = (zy)x = (yz + y)x = yzx + yx = y(xz + x) + (xy + x) =$
$yxz + yx + xy + x = (xy + x)z + (xy + x) + xy + x =$
$xyz + xz + xy + x + xy + x = xyz + xz + 2xy + 2x$

$zyx = z(yx) = z(xy + x) = zxy + zx = (xz + x)y + (xz + x) =$
$xzy + xy + xz + x = x(yz + y) + xy + xz + x =$
$xyz + xy + xy + xz + x = xyz + xz + 2xy + x$

Not Unique Canonical Form

Example that Does Not Satisfy PBW - continued

Not Coming from a Lie Algebra!

If it were $[y, x] = x, \quad [z, x] = x, \quad [z, y] = y.$

$$\begin{aligned} [x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= [x,-y] + [y,x] + [z,-x] \\ &= x + x + -x \\ &= x \neq 0 \end{aligned}$$

Fails Jacobi Identity

Note: Smashing with a Group won't help.

Theorems

Theorem

An algebra that satisfies PBW is isomorphic to a deformation of a commutative polynomial ring. And the PBW property turns out to be equivalent to the commutator defining a Lie Bracket on a vector space V.

Theorem

For $L =$ Lie Algebra
$U = \mathbb{F} < v_1, \cdots, v_n > / < vw - wv - [v, w] >$.
U satisfies PBW.

Lie Algebra Definition

A Lie Algebra is a vector space V together with a multiplication(usually termed Lie Bracket) and denoted by $[x, y]$ such that

1. $[x, y]$ depends linearly on x and y.
2. $[x, x] = 0 \ \ \forall\ x \in V.$
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \ \ \forall\ x, y, z.$

Properties 1 and 2 imply that $[y, x] = -[x, y] \ \ \forall x, y \in V.$ Two elements x and y are said to commute if $[x, y] = 0.$

Theorem

If algebra A is coming from restricted Lie Algebra (in the sense that the commutator is the same as the Lie Bracket), then $v^p - v^{[p]} \in Z(A)$.

Proof:

We'll look at the case $p = 3$, and the general case follows similarly.
So let char $\mathbb{F} = 3$.
WTS $v^3 - v^{[3]}$ commutes with all $a \in A$.
Take $a \in A$.
Note: av=va-[v,a]
WTS $(v^3 - v^{[3]})a = a(v^3 - v^{[3]})$

$$v^3 a - v^{[3]} a = av^3 - av^{[3]}$$

$v^p - v^{[p]}$ continued

$$= (va - [v,a])v^2 - (v^{[3]}a - [v^{[3]},a])$$
$$= vav^2 - [v,a]v^2 - (v^{[3]}a - [v^{[3]},a])$$
$$= v(va - [v,a])v - [v,a]v^2 - (v^{[3]}a - [v^{[3]},a])$$
$$\dots$$
$$= v^3a - 3v^2[v,a] + 3v[v^{[2]},a] - v^{[3]}a$$
$$= v^3a - v^{[3]}a$$

In general, we will get coefficients of the form p choose r, which corresponds to Pascal's triangle. When p is prime, all coefficients except the first and last are divisible by p - and so in char p, they become 0. We are left with only the first and last terms. Thus $v^p - v^{[p]} \in Z(A)$ over char p. QED.

Skew Group Algebra

$\mathbb{F} < v_1, \cdots, v_n > \# G$, is the $\mathbb{F}$-algebra generated by $v \in V$ together with g in G such that
1. $\mathbb{F}[G]$ is subalgebra and
2. $gv = {}^{g}vg \quad \forall v \in V, g \in G$.

Skew Group Algebra Example $p \geq 3$

$R = \mathbb{F} < x, y, z > \#$ G/relations

Relations:

yx=xy+z	zy=yz+x	
zx=xz-y	gx=yg	$g^3 = 1$
gy=zg	gz=xg	

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$, G=g acts on V=$\mathbb{F}$-span$\{x, y, z\}$.

So $g^3 =$ identity matrix.

This example is an algebra coming from a Restricted Lie Algebra. $a \mapsto a^{[p]}$ is given by $a^{[p]} = (-1)^{(p-1)/2} a$ for $a = x, y, z$. Z(R) is generated by $x^2 + y^2 + z^2$ and $a^p - a^{[p]}$.

The center of an unsmashed algebra may contribute to the center of the corresponding smashed algebra.

Skew Group Algebra Example continued

Since $x^2 + y^2 + z^2$ is g-invariant it lies in the center of the smashed algebra.
Also $z^p + x^p + y^p - z^{[p]} - x^{[p]} - y^{[p]}$ is g-invariant.
Also $g^3 =$ identity is in the center of the smashed algebra.

In characteristic 3, this can be checked by hand.

Thank You

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