Math 1720, Midterm 1, Feb 24

Test rules/guidelines:

No calculators or any electronic devices.
Cel phones to be switched off.
Explain all your answers and show calculations, except where indicated otherwise.
Open exam only when instructed.
All work must be your own.
Exam duration is 50 minutes.
Submit exam to me within 51 minutes.
Starting from 51 minutes, late exams receive 5% reduction in credit for each minute or part thereof.
I'll announce when we're at 50 minutes.
There are 8 problems, comprising 76 points total.

Sign below to say you have read and understood these rules:
Name:
Signature:

Scores:

1.
2.
3.
4.
5.
6.
7.
8.

Total:

1.[8 points] Fill in bounds on the integral (so make it a definite integral) to make the equation correct.

No explanation necessary.

$$\ln(7) = \int \frac{1}{t} dt$$

Solution.

$$\ln(7) = \int_1^7 \frac{1}{t} dt$$

2.[8 points] Find the antiderivative:

$$\int 9^x dx.$$

Solution. Using the fact that

$$9^x = (e^{\ln(9)})^x = e^{x\ln(9)}$$

we have

$$\int 9^x dx = \int e^{x\ln(9)} dx,$$

and since $(e^{x\ln(9)})' = \ln(9)e^{x\ln(9)}$, and therefore $(\frac{1}{\ln(9)}e^{x\ln(9)})' = e^{x\ln(9)}$, we get

$$\int e^{x\ln(9)} dx = \frac{1}{\ln(9)} e^{x\ln(9)} + c = \frac{1}{\ln(9)} 9^x + c.$$

Alternatively using the differentiation rule

$$\frac{d}{dx} a^x = \ln(a)a^x,$$

we have

$$a^x + c = \int \ln(a)a^x dx,$$

so

$$\frac{1}{\ln(a)}(a^x + c) = \int a^x dx$$

but we can replace $c/\ln(a)$ with another arbitrary constant, since c was arbitrary:

$$\frac{1}{\ln(a)} a^x + d = \int a^x dx.$$

So with $a = 9$, we have

$$\frac{1}{\ln(9)} 9^x + d = \int 9^x dx.$$

3.[10 points] Compute the derivative

$$\frac{d}{dx}(x^{5x}).$$

Solution.
Write

$$x^{5x} = (e^{\ln(x)})^{5x} = e^{5x\ln(x)}$$

so the derivative we need to compute is

$$\frac{d}{dx}(e^{5x\ln(x)}).$$

Since $\frac{d}{dx}(e^x) = e^x$ and using the chain rule, we get:

$$= e^{5x\ln(x)}(5x\ln(x))'$$

By the product rule on the remaining derivative:

$$= e^{5x\ln(x)}((5x)'\ln(x) + 5x(\ln(x))')$$

$$= e^{5x\ln(x)}(5\ln(x) + 5x/x)$$

$$= e^{5x\ln(x)}(5\ln(x) + 5).$$

(Note the domain of the original function is $x > 0$, as the base x must be positive since the exponent is varying. And for all $x > 0$, $5x/x = 5$, so the previous step was correct.)

$$= x^{5x}(5\ln(x) + 5)$$

$$= 5x^{5x}(\ln(x) + 1)$$

4.[10 points] Compute and simplify the integral

$$\int_2^5 \frac{3}{1-x}dx$$

Solution. Substitute $u = 1 - x$. Then $du = -dx$, so $-du = dx$, and the integral becomes

$$\int_{x=2}^5 \frac{3}{u}(-du)$$

$$= -3\int_{x=2}^5 \frac{1}{u}du$$

$$= -3\ln(|u|)\Big|_{x=2}^5.$$

When $x = 2$, $u = 1 - x$ so $u = 1 - 2$ so $u = -1$, and when $x = 5$, similarly $u = -4$, so

$$= -3\ln(|-4|) - (-3\ln(|-1|))$$

$$= -3\ln(4) + 3\ln(1)$$

And $\ln(1) = 0$, so

$$= -3\ln(4).$$

Or you could rewrite this as

$$-3\ln(2^2) = -6\ln(2).$$

5.[10 points]

(i) Find a function $p(x)$ to make the equation correct.

$$\frac{1}{3}\ln(5-x) + \ln(x+1) = \ln(p(x))$$

(ii) For what values of x is the equation valid?

Solution.

(i) For all x such that $5 - x > 0$ and $x + 1 > 0$ (which is all x that make the left side make sense), we have

$$\frac{1}{3}\ln(5-x) + \ln(x+1)$$

$$= \ln((5-x)^{1/3}) + \ln(x+1)$$

$$= \ln((5-x)^{1/3}(x+1))$$

So setting

$$p(x) = (5-x)^{1/3}(x+1),$$

the equation holds for all x such that the left side makes sense.

(ii) The left side makes sense iff $5 - x > 0$ and $x + 1 > 0$, i.e. $x < 5$ and $-1 < x$, i.e. $-1 < x < 5$. The calcluation above shows that for all such x, the equation holds. So the equation is valid precisely when $-1 < x < 5$.

6.[10 points] A long term investment grows over time due to interest. Suppose the initial balance, at Jan 1, 1970, was 100 dollars. Suppose the balance on Jan 1, 1975 is 140 dollars. Let $B(t)$ be the function giving the balance at time t, where t is the number of years after Jan 1, 1970. Suppose the function $B(t)$ has exponential growth.

Find the formula for $B(t)$.

(Find the values of all constants in the formula you give; you can express them in terms of the natural log function.)

Solution.

Since $B(t)$ has exponential growth, there are constants A, k, with $k > 0$, such that

$$B(t) = Ae^{kt}.$$

Since the initial balance is 100 dollars, at time $t = 0$ (0 years after Jan 1, 1970, i.e. when the time is in fact equal to Jan 1, 1970), we have

$$B(0) = 100,$$

so

$$B(0) = Ae^{k0} = 100$$

$$A * 1 = Ae^{0} = 100$$

So $A = 100$. And Jan 1, 1975 is $t = 5$ years after Jan 1, 1970, so

$$B(5) = 140 = 100e^{k5},$$

so

$$140/100 = e^{5k}$$

$$\ln(1.4) = \ln(e^{5k}) = 5k$$

$$k = \ln(1.4)/5.$$

Thus the function is:

$$B(t) = 100e^{\frac{\ln(1.4)}{5}t}.$$

7.[12 points] Let

$$f(x) = \sqrt{x^2 + 4}.$$

Let D be the interval $(-\infty, 0]$. The range of f over D is $[2, \infty)$ (you may assume this).

(i) Find the formula for the inverse of f over D.

(ii) Find the domain and range of the inverse you found in (i).

Solution.

(Remark: f does have an inverse over D; by the graph given in the exam, f is decreasing over $(-\infty, 0]$, so is 1-1 over that interval, so has an inverse there. Or you could take the derivative:

$$f'(x) = \frac{2x}{2\sqrt{x^2 + 4}},$$

And note that for $x < 0$, $2x < 0$ and $\sqrt{x^2+4} > 0$, so $f'(x) < 0$. So f is decreasing over $(-\infty, 0)$, and since f is continuous, it's therefore decreasing over $(-\infty, 0]$.)

(i) Now to find the inverse: For x in D (that is, $x \leq 0$) and y any number, we have

$$y = \sqrt{x^2+4}$$

iff

$$y^2 = x^2 + 4$$

iff

$$y^2 - 4 = x^2$$

iff

$$\sqrt{y^2-4} = \sqrt{x^2} = |x| = -x,$$

the last equation because we're dealing only with $x \leq 0$. So this if iff

$$x = -\sqrt{y^2-4}.$$

So let g be the inverse of f over D. Then for y in the domain of g,

$$g(y) = -\sqrt{y^2-4}.$$

(ii) Since we found the inverse over the interval $D = (-\infty, 0]$ (which is included within f's domain), the range of the inverse is $D = (-\infty, 0]$. The domain of the inverse is f's range over this interval, i.e. $[2, \infty)$ (this was given).

8.[8 points] Suppose f is a 1-1 function such that

$$f(1) = 2, \quad f'(1) = 1$$
$$f(2) = 3, \quad f'(2) = 2$$
$$f(3) = 6, \quad f'(3) = 4.$$

Let $g = f^{-1}$.

(i) Find $g(3)$ and $g'(3)$, or state that there's insufficient information.
(ii) Find $g(1)$ and $g'(1)$, or state that there's insufficient information.

(Hint: it's a good idea to sketch a possible graph for $y = f(x)$, and its inverse, and use this to guide your calculations.)

Solution.

(Sketch a possible graph...)

(i) We're looking for $g(3) = f^{-1}(3)$, so here 3 is the input to the inverse, which corresponds to being an output to f. We have $f(2) = 3$ is given. This implies $g(3) = 2$. And also $f'(2) = 2$ is given, so we have

$$g'(3) = \frac{1}{f'(2)} = \frac{1}{2}.$$

(ii) To find $g(1) = f^{-1}(1)$, we would need our data to tell us $f(a) = 1$ for some value a, i.e. for 1 to be an f-output in the list. But the only "f-outputs" we know about are $2, 3, 6$ (from $f(1) = 2$, $f(2) = 3$, $f(3) = 6$). These do not include 1. So we don't have sufficient information to find $g(1)$. We also don't have sufficient information to find $g'(1)$. For to determine $g'(1)$ from the data, we'd need an a with $f(a) = 1$, and then we'd also need to know $f'(a)$, in order to then use the fact

$$g'(1) = \frac{1}{f'(a)},$$

with that value of a. Since we don't have a value a with $f(a) = 1$, we can't even get started.

(Remark: the discussion above doesn't actually prove that we don't have sufficient information to determine $g(1)$ and/or $g'(1)$. The discussion only explains why we don't have sufficient information to determine these values using the same approach as we used to find $g(3)$ and $g'(3)$. But maybe with some other approach we might be able to determine these other values? In fact we cannot. For one can give examples of two distinct one-to-one differentiable functions f_1 and f_2, such that f_1 satisfies the six equations listed above (i.e. $f_1(1) = 2$, $f_1'(1) = 1$, etc), and also f_2 satisfies the six equations above, and such that also $f_1(\frac{1}{2}) = 1$, and $f_1'(\frac{1}{2}) = 1$, and such that $f_2(0) = 1$ and $f_2'(0) = \frac{1}{2}$. Then by the methods we used earlier, we would have $f_1^{-1}(1) = \frac{1}{2}$ and $(f_1^{-1})'(1) = 1$, but $f_2^{-1}(1) = 0$ and $(f_2^{-1})'(1) = 2$. But from the information given about f, it's possible that $f = f_1$ or that $f = f_2$ (or that f is some other function), we don't know. So it's possible that $f^{-1}(1) = 0$ (e.g. if $f = f_2$) and it's possible that $f^{-1}(1) = \frac{1}{2}$ (e.g. if $f = f_1$). So there are at least two possibilities, so we can't possibly determine $g(1) = f^{-1}(1)$. (In fact, if we're just requiring that f be differentiable, one-to-one, and satisfy those six equations, then any number < 1 is a possible value for $f^{-1}(1)$.) And similarly, we have at least two possibilities for $g'(1) = (f^{-1})'(1)$, so it cannot be determined either.)