Math 1720, Midterm 2, April 6

Test rules/guidelines:

No calculators or any electronic devices.
Cel phones to be switched off.
Explain all your answers and show calculations, except where indicated otherwise.
(*Explaining* is usually done by showing calculations; if you've shown calculations, you don't need to restate those calculations in other words.)
Open exam only when instructed.
All work must be your own.
Exam duration is 50 minutes.
Submit exam to me within 51 minutes.
Starting from 51 minutes, late exams receive 5% reduction in credit for each minute or part thereof.
I'll announce when we're at 50 minutes.
There are 8 problems, comprising 71 points total.

Sign below to say you have read and understood these rules:
Name:
Signature:

Scores:

1.
2.
3.
4.
5.
6.

Total:

Formulas

$\int sec^{n+2}(x)dx = \frac{1}{n+1}\sec^n(x)\tan(x) - \frac{n}{n+1}\int \sec^n(x)dx$
Unit Circle
Others...

1.[12 points] (a) Find

$$\frac{d}{dx}(\sec(x)\text{arcsec}(x)).$$

Solution.
Using the product rule,

$$= (\sec(x))'\text{arcsec}(x) + \sec(x)(\text{arcsec}(x))'$$

$$= \sec(x)\tan(x)\text{arcsec}(x) + \sec(x)\frac{1}{|x|\sqrt{x^2-1}}.$$

(Note that $\sec(x)\text{arcsec}(x) \neq 1$.)

(b) Find

$$\arccos(-\frac{1}{2}),$$

explaining your answer.
Solution.
$\arccos(x) = \alpha$ where α is the angle such that $0 \leq \alpha \leq \pi$ and $\cos(\alpha) = -1/2$. This is at $\alpha = 2\pi/3$.

(c) Simplify

$$\tan(\arcsin(x/3)),$$

to a formula in terms of x which does not involve any trig functions. You may assume $0 < x < 3$.

Solution.
Let $\alpha = \arcsin(x/3)$. We want to express $\tan(\alpha) = \sin(\alpha)/\cos(\alpha)$, in terms of x.
We have $\sin(\alpha) = \sin(\arcsin(x/3)) = x/3$.
And $\cos(\alpha)^2 + \sin(\alpha)^2 = 1$, so

$$\cos(\alpha)^2 = 1 - \sin(\alpha)^2 = 1 - (x/3)^2$$

so

$$|\cos(\alpha)| = \sqrt{1-(x/3)^2}$$

And arcsin always gives an angle in the interval $-\pi/2 \leq \alpha \leq \pi/2$, so $\cos(\alpha) \geq 0$, so

$$\cos(\alpha) = \sqrt{1-(x/3)^2}$$

So

$$\tan(\arcsin(x/3)) = \tan(\alpha) = \sin(\alpha)/\cos(\alpha) = \frac{x/3}{\sqrt{1-(x/3)^2}}.$$

2.[16 points] Use integration by parts to find

$$\int_0^{\pi/10} x^2 e^{5x} dx.$$

Solution.
For the integration by parts, we choose to differentiate the x^2 term and integrate the e^{5x} term. So setting $f(x) = e^{5x}$ and $g(x) = x^2$, we have

$$F(x) = \int e^{5x} dx = \frac{1}{5} e^{5x}$$

$$g'(x) = 2x$$

So

$$\int_0^{\pi/10} x^2 e^{5x} dx = Fg|_0^{\pi/10} - \int_0^{\pi/10} Fg' dx.$$

$$= \frac{1}{5} e^{5x} x^2 |_0^{\pi/10} - \int_0^{\pi/10} 2x(\frac{1}{5}) e^{5x} dx$$

$$= \frac{1}{5} e^{5x} x^2 |_0^{\pi/10} - \frac{2}{5} \int_0^{\pi/10} x e^{5x} dx$$

Call the above line (*). Now the remaining integral within (*) is

$$\int_0^{\pi/10} x e^{5x} dx$$

which we again use integration by parts for. We proceed in the same direction, i.e. differentiate the x term and integrate the e^{5x} term. This gives

$$= \frac{1}{5} e^{5x} x|_0^{\pi/10} - \int_0^{\pi/10} \frac{1}{5} e^{5x} dx$$

$$= \frac{1}{5} e^{5x} x|_0^{\pi/10} - \frac{1}{25} e^{5x}|_0^{\pi/10}$$

$$= \frac{1}{5} e^{5x} (x - \frac{1}{5})|_0^{\pi/10}$$

Now plugging this back in for the remaining integral in (*), we get that the original integral equals

$$= \frac{1}{5} e^{5x} x^2 |_0^{\pi/10} - \frac{2}{5} \left(\frac{1}{5} e^{5x} (x - \frac{1}{5}) \right) |_0^{\pi/10}$$

$$= \frac{1}{5} e^{5x} \left(x^2 - \frac{2}{5}(x - \frac{1}{5}) \right) |_0^{\pi/10}$$

$$= \frac{1}{5} e^{5x} \left(x^2 - \frac{2}{5} x + \frac{2}{25} \right) |_0^{\pi/10}$$

$$= \frac{1}{5} e^{5(\pi/10)} \left((\pi/10)^2 - \frac{2}{5}(\pi/10) + \frac{2}{25} \right) - \frac{1}{5} e^0 (0 - 0 + \frac{2}{25})$$

$$= \frac{1}{5} e^{\pi/2} \left((\pi/10)^2 - \frac{2}{5}(\pi/10) + \frac{2}{25} \right) - \frac{2}{125}$$

3.[13 points] Find

$$\int \sqrt{\sin(x)} \cos^3(x) dx.$$

Solution.
Note the integrand is a product of powers of $\sin(x)$ and $\cos(x)$ (since $\sqrt{x} = (\sin(x))^{1/2}$). The power of $\cos(x)$ is 3, odd. So we will substitute $u = \sin(x)$. This gives $du = \cos(x)dx$, so separating $\cos(x)$:

$$= \int (\sin(x))^{1/2} \cos^2(x) \cos(x) dx$$

$$= \int (\sin(x))^{1/2} (1 - \sin^2(x)) \cos(x) dx$$

$$= \int u^{1/2}(1 - u^2) du$$

$$= \int u^{1/2} - u^{5/2} du$$

$$= \frac{2}{3} u^{3/2} - \frac{2}{7} u^{7/2} + c$$

$$= \frac{2}{3} (\sin(x))^{3/2} - \frac{2}{7} (\sin(x))^{7/2} + c$$

4.[7 points] Consider the functions

$$f(x) = 5^x$$

and

$$g(x) = x^7 + x\ln(x).$$

(a) Write down the limit which must be found in order to compare the growth rates of f vs g. (I.e. to determine whether $f << g$, $f \approx g$ or $f >> g$.)

Solution.

Either

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

$$= \lim_{x\to\infty} \frac{5^x}{x^7 + x\ln(x)}$$

or

$$\lim_{x\to\infty} \frac{g(x)}{f(x)}$$

$$= \lim_{x\to\infty} \frac{x^7 + x\ln(x)}{5^x}.$$

(b) Compare the growth rates of f vs g. (I.e., determine whether $f << g$, $f \approx g$, or $g << f$.) Explain your answer.

Solution. Computing the second limit written above (i.e. of $g(x)/f(x)$),

$$= \lim_{x\to\infty} \frac{x^7 + x\ln(x)}{5^x}.$$

$$= \lim_{x\to\infty} \frac{x^7}{5^x} + \lim_{x\to\infty} \frac{x\ln(x)}{5^x}$$

(assuming both these limits exist; let's check this). The limit $\lim_{x\to\infty}(x^7/5^x) = 0$ since $x^7 << 5^x$ since $x^r << b^x$ whenever $b > 1$ (by standard comparisons in text). And $\lim_{x\to\infty}(x\ln(x)/5^x) = 0$ since $x\ln(x) << 5^x$ (by standard comparisons in text, $x^r\ln(x) << 5^x$ whenever $1 < b$. So both limits do exist and both equal 0, so the above assumption was correct, and so the overall limit is

$$0 + 0 = 0.$$

Therefore $g << f$.

(c) Based on your answer to (b) (or otherwise), what is the value of the limit you wrote down in (a)? Explain.

Solution.

I computed the limit $\lim_{x\to\infty} g(x)/f(x) = 0$ in (b). For the limit $\lim_{x\to\infty} f(x)/g(x)$, since $g(x) << f(x)$, i.e. $f(x) >> g(x)$, the latter limit is ∞.

5.[10 points] Compute
$$\lim_{x\to 0^+} (1-x)^{2/x},$$
explaining your answer. *Solution.*
As $x \to 0^+$, $1 - x \to 1$ and $2/x \to \infty$. (Note for $x \to 0^+$ we have $x > 0$ so $2/x > 0$.)
So the limit has form 1^∞, which is indeterminate.

We convert to the term in the limit to base e:
$$\lim_{x\to 0^+} (e^{\ln(1-x)})^{2/x}$$
$$= \lim_{x\to 0^+} e^{\frac{2}{x}\ln(1-x)}$$
$$= e^{\lim_{x\to 0^+} \frac{2}{x}\ln(1-x)}$$
$$= e^L$$
where
$$L = \lim_{x\to 0^+} \frac{2}{x}\ln(1-x)$$
$$L = 2\lim_{x\to 0^+} \frac{\ln(1-x)}{x}.$$
Now as $x \to 0^+$, the denominator $\to 0$, and
$$\lim_{x\to 0^+} \ln(1-x) = \ln(\lim_{x\to 0^+}(1-x))$$
since ln is continuous,
$$= \ln(1) = 0.$$
So we have form $0/0$, which is indeterminate, but we may attempt to apply L'Hopital's rule. This gives
$$2\lim_{x\to 0^+} \frac{(\ln(1-x))'}{(x)'}$$
$$= 2\lim_{x\to 0^+} \frac{\frac{1}{1-x}(-1)}{1}$$
$$= 2\lim_{x\to 0^+} \frac{1}{x-1} = 2(-1) = -2.$$
So the limit existed, so L'Hopital's rule did apply correctly, so by L'Hopital's rule,
$$L = 2\lim_{x\to 0^+} \frac{\ln(1-x)}{x} = -2.$$
So the final answer is $e^L = e^{-2}$.

6.[13 points] *You aren't required to fully compute the following antiderivative.*
(a) Make an appropriate trig substitution on the antiderivative

$$\int \frac{1}{(x^2+9)^3} dx.$$

(I.e., appropriate in that it leads to finding the antiderivative.)
Solution.
The form $x^2 + a^2$ appears in the denominator, with $a = 3$. So an appropriate trig sub is $x = a\tan(\theta) = 3\tan(\theta)$. This gives $dx = 3\sec^2(\theta)d\theta$, so subbing,

$$= \int \frac{1}{(9\tan^2(\theta)+9)^3} 3\sec^2(\theta)d\theta$$

(b) After making the substitution in (a), show that the antiderivative simplifies to one of the following forms: either

$$k\int \cos^4(\theta)d\theta,$$

OR the form

$$k\int \sin^4(\theta)d\theta,$$

for some constant k. Also find k.

Solution. Well, after the sub we had

$$\int \frac{1}{(9\tan^2(\theta)+9)^3} 3\sec^2(\theta)d\theta$$

$$= \int \frac{1}{9^3(\tan^2(\theta)+1)^3} 3\sec^2(\theta)d\theta$$

$$= \frac{3}{9^3}\int \frac{1}{(\sec^2(\theta))^3} \sec^2(\theta)d\theta$$

$$= \frac{3}{9^3}\int \frac{1}{(\sec^2(\theta))^2} d\theta.$$

and $(\sec^2(\theta))^2 = \sec^4(\theta) = 1/\cos^4(\theta)$, so the antiderivative is

$$= \frac{3}{9^3}\int \cos^4(\theta)d\theta,$$

as required. And $k = 3/9^3$.