Summary of topics covered in course:

Open/closed subsets of $\mathbb{R}$
Metric spaces
Countability and uncountability
Cantor set
Topological spaces
Bases
Interior, Closure
Continuity
Separation axioms
Product, subspace topologies
Connectedness
Compactness
Density (this came up in a few exercises; you should know the definition - it's in homework 7, problem 6 - that particular problem was not required for 4500, but you should still be comfortable with the definition).

Review Problems - I may add some more over the next few days

On Thursday, Steve will discuss (1) any questions that you have on the material and/or related exercises from homeworks or whatever; (2) a selection of these problems. I doubt there will be time to discuss them all so you should give the questions a look over and think about which ones you would prefer to discuss on Thursday, and in general, any questions that you want to ask. (The problems focus more on more recent material; all material covered will be examinable but you should look more at the older reviews/midterms and/or in Munkres for more problems on that material. There are a few problems here on the older material though, but some of them you've seen before.)

1. Let τ be the topology on $\mathbb{R}$ with base $\tau_{\text{std}} \cup \{\{q\} | q \in \mathbb{Q}\}$. Let $(\mathbb{R}^2, \rho)$ be the product of the spaces $(\mathbb{R}, \tau) \times (\mathbb{R}, \tau)$. How does the closure of a subset of $\mathbb{R}^2$, with respect to ρ, relate to its closure with respect to the standard topology on $\mathbb{R}^2$?

2. Let $C \subseteq \mathbb{R}$ and $A = \{(xy, y/\sin(x)) \mid (x, y) \in [\pi/4, 3\pi/4] \times C\}$. Show that if $C = [0, 5]$ then A is both compact and connected. Show that if C is the Cantor set then A is compact but not connected.

3. (a) Let $X = C([0, 1])$ with "max difference" metric topology. Let $A \subseteq X$ be the set of quadratic functions (with domain $[0, 1]$) and linear functions (with domain $[0, 1]$). Show that A is connected.
(b) Give an example of a non-connected subset of $C([0, 1])$.
(c) Same topology, show that it's $T4$.

4. Let (X, τ) be a T_2 topological space such that X is compact. Show that the space is T_3.

5. Let C be the Cantor set.

Is there
(a) an onto function $f : \mathbb{N} \to C$?
(b) an onto function $f : \mathbb{R} \to C$?
(c) an onto function $f : C \to \mathbb{R}$?
(d) a continuous onto function $f : \mathbb{N} \to C$? (where it's the discrete topology on $\mathbb{N}$ and restriction of standard on C)
(e) a continuous onto function $f : \mathbb{R} \to C$? (standard topology on $\mathbb{R}$)
(f) a continuous onto function $f : C \to \mathbb{R}$?

6. Let (X, d) be a metric space. Recall that a subset $D \subseteq X$ is *dense* iff for every non-empty open set U, $U \cap D \neq \emptyset$. Show that X has a countable dense subset iff X has a countable base.

7. Let X_1, X_2 be top spaces and $A_1 \subseteq X_1$, $A_2 \subseteq X_2$. Show that $\mathrm{Cl}(A_1 \times A_2) = \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$.

8. Let $A \subseteq \mathbb{R}$ be an open set. Show that there is some $J \subseteq \mathbb{N}$ and a family $\langle I_i \rangle_{i \in J}$ of pairwise disjoint open intervals (i.e. $i \neq j \in J \implies I_i \cap I_j = \emptyset$) such that $A = \cup_{i \in \mathbb{N}} I_i$ (note the family is required to be countable).

9. (a) Using the original definition of "closure" (i.e. the closure of a set A is the intersection of all closed sets B such that $A \subseteq B$), prove the characterization of closure given in class, i.e. prove that a point x is in the closure of A iff for every open set W such that $x \in W$, we have that $W \cap A \neq \emptyset$.

(b) The boundary of a set A is $B = \mathrm{Cl}(A) - \mathrm{Int}(A)$. Is it possible for the boundary to have non-empty interior, i.e. for $\mathrm{Int}(B) \neq \emptyset$?

10. Show that if A is connected subset of a topological space, and $A \subseteq B \subseteq \mathrm{Cl}(A)$, then B is also connected.

11.(a) Prove that in $\mathbb{R}^n$, every open ball is connected.

(b) Now if you did (a) by showing that every open ball is in fact path-connected, then do it again, without using path-connectedness. (Hint: show first that if $[a, b] \times [c, d]$ is connected for any reals $a < b$ and $c < d$. Combine this with Munkres' §23 exercise 2.)

12. Let $(X_1, \tau_1), (X_2, \tau_2)$ be two topological spaces such that $X_1 \cap X_2 = \emptyset$. Fix $x_1 \in X_1$ and $x_2 \in X_2$. We'll describe a way to "join" these two spaces, to form a single space, in which x_1 and x_2 represent the same point. We are joining the spaces "at" the points x_1 and x_2. Let a be some object such that $a \notin X_1 \cup X_2$.

Define a topological space (X_1', τ_1') to be essentially the same as (X_1, τ_1), except that we replace the point x_1 with a. That is, first define X_1' as:

$$X_1' = (X_1 - \{x_1\}) \cup \{a\}.$$

Let $f : X_1' \to X_1$ be the map given by $f(x) = x$ for $x \in X_1' - \{a\}$, and $f(a) = x_1$. Now define τ_1' by:

$$\tau_1' = \{f^{-1}(W) \mid W \in \tau_1\}.$$

This defines (X_1', τ_1'). Similarly, let (X_2', τ_2') be the space given by replacing $x_2 \in X_2$ with a.

It's straightforward to check that (X_1', τ_1') and (X_2', τ_2') are topological spaces, and $X_1' \cap X_2' = \{a\}$.

Now define a new topological space (X, τ) by "joining" the two spaces at their common point a. That is, let

$$X = X_1' \cup X_2',$$

and given $W \subseteq X$, let $W \in \tau$ iff both $(W \cap X_1') \in \tau_1'$ and $(W \cap X_2') \in \tau_2'$.

So (X, τ) is the result of "joining" (X_1, τ_1) to (X_2, τ_2), "at" the points x_1 and x_2. (Remark: in terms of the topologies, it doesn't matter what the underlying points actually *are*, it just matters what the topological structure is. Although we've changed the identities of x_1 and x_2 to a, we've preserved the topological structures of the original X_1 and X_2, on each of their "sides" of the space X. Now for the problems:

(a) Show that (X, τ) is a topological space.

(b) Describe the interior and closure operations of X in terms of those for X_1 and X_2.

(c) Show that X is compact (or connected, or path-connected) (w.r.t. τ) iff both X_1 and X_2 are compact (or connected, or path-connected) (w.r.t. τ_1 and τ_2)

(d) Suppose $X_1 = \mathbb{R}$ and $X_2 = \mathbb{R}^2$, $x_1 = 0$ and $x_2 = (0, 0)$. Give a subset of $A \subseteq \mathbb{R}^3$, and a bijection $f : X \to A$, and f is continuous, and f^{-1} is continuous. (This is a *homeomorphism*, a bijection which exactly preserves topological structure; i.e. U is open iff $f``U$ is open.)