Intro to Topology Homework 3 - due Friday Feb 10 (you can hand it in at my office, or slide it under my office door, 416 GAB)

Any problem is good for homework discussion.

1.
(a) Let $X = \mathbb{R}^3$ and for $p = (x, y, z)$ and $q = (a, b, c)$ in $\mathbb{R}^3$, let

$$d(p, q) = \max(|x - a|, |y - b|, |z - c|).$$

Show that (X, d) is a metric space.

(b) Let $X = C([0, 1])$, i.e.,

$$X = \{f | f : [0, 1] \to \mathbb{R},\ f \text{ continuous}\}.$$

(i) Fix some $x_0 \in [0, 1]$, and let d_0 be defined by

$$d_0(f, g) = |f(x_0) - g(x_0)|.$$

Show that (X, d_0) is *not* a metric space. Which of the metric space axioms *are* true of (X, d_0)?

(ii) Now define $d_\int$ as follows: for $f, g \in X$, let

$$d_\int(f, g) = \int_0^1 |f(x) - g(x)| dx.$$

Show that $(X, d_\int)$ *is* a metric space.

(c) Let Y be any set, and let X be the set of finite subsets of $\mathbb{N}$.
(i) For $A, B \in X$, let

$$d(A, B) = \min\{|a - b| \mid a \in A,\ b \in B\}.$$

Is (X, d) a metric space? (Give proof.)

(ii) For A, B any two sets, define their *symmetric difference* $A\Delta B$ as

$$A\Delta B = (A \backslash B) \cup (B \backslash A).$$

Let

$$d_2(A, B) = \text{the number of elements in } A\Delta B.$$

Is (X, d_2) a metric space? (Give proof.)

(d) For the metric space in (a), give a geometric description of the open ball $\mathcal{B}((-1, 0, 3), 0.2)$ for that space. If either of (c)(i) or (c)(ii) gave a metric space, give a description of the open ball $\mathcal{B}(\{2\}, 2)$ for that space.

2. Let (X, d) be the metric space with $X = C([0, 1])$ and d the "max absolute difference metric". I.e.,

$$X = \{f | f : [0, 1] \to \mathbb{R},\ f \text{ continuous}\},$$

and for $f, g \in X$,

$$d(f,g) = \max\{|f(x) - g(x)| \bigm| x \in [0,1]\}.$$

Let $f, g \in X$ be defined by $f(x) = x$ and $g(x) = x^2$. Give an explicit definition for a function h such that

$$h \in \mathcal{B}(f, \frac{1}{3}) \cap \mathcal{B}(g, \frac{1}{3}).$$

Explain why h is in this intersection. Find some $\varepsilon > 0$ such that

$$\mathcal{B}(f, \varepsilon) \cap \mathcal{B}(g, \varepsilon) = \emptyset,$$

proving that your choice works. (All open balls here are defined from the metric d.)

3. Suppose (X, d) is a metric space. Prove that the intersection of finitely many open subsets of X is open.

4.

(a) Prove Theorem L1.7, i.e. that if (X, d) is a metric space, and if $x \in X$ and $\varepsilon \in \mathbb{R}$, then $\mathcal{B}(x, \varepsilon)$ is open.

(b) Let (X, d) be a metric space, $x \in X$ and $\varepsilon \in \mathbb{R}$. Define the *closed ball*

$$\bar{\mathcal{B}}(x, \varepsilon) = \{z \in X | d(x, z) \leq \varepsilon\}.$$

Show that every closed ball is a closed set. Use this result and problem 3 to deduce that every finite subset of X is closed.

5. Let $X = \mathbb{R}^2$ and let d_s be the standard metric (given by the distance formula) on $\mathbb{R}^2$, and let d_t be the taxicab metric on $\mathbb{R}^2$, i.e. given by

$$d_t((a,b),(c,d)) = |a - c| + |b - d|.$$

Show that for any $A \subseteq \mathbb{R}^2$, A is open in $(\mathbb{R}^2, d_s)$ iff A is open in $(\mathbb{R}^2, d_t)$. (You may assume that d_s and d_t are both metrics on $\mathbb{R}^2$.)

6(not required). Consider the metric space $(\mathbb{R}^2, d_s)$ (with d_s the standard metric). Give an example of an open subset A, such that A is *not* a disjoint union of open balls. (This is in contrast to the fact that for the standard metric in $\mathbb{R}^1$, every open set is a disjoint union of open intervals.)