Intro to Topology Homework 7 - due Friday Mar 9.
(Note: all solutions, including examples, should be explained, unless stated otherwise.)

1. Suppose $f : X \to Y$ is a bijection. Show that X is countable iff Y is countable, and X is countably infinite iff Y is.

2. Let X be countable. Show the following.
(i) Let $A \subseteq X$. Then A is countable.
(ii) Suppose $f : Y \to X$, and f is 1-1. Show that Y is countable.
(iii) Suppose Y is uncountable, and let $f : Y \to X$. Prove that for some $x \in X$, $f^{-1}(\{x\})$ is uncountable.
(iv) Let

$$\mathcal{P}_{<\mathbb{N}}(X) = \{F \subseteq X \mid F \text{ is finite}\}.$$

Show that $\mathcal{P}_{<\mathbb{N}}(X)$ is countable.

3. A number $z \in \mathbb{R}$ is *algebraic* iff there is a polynomial p such that $p(z) = 0$, and p has integer coefficients. (I.e., p is defined by

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n,$$

where $a_i \in \mathbb{Z}$ for each $i \leq n$.) Prove that the set of all algebraic numbers is countable. (Hint: use the fact that a deree n polynomial has at most n zeros.)

4.
(i) Prove that if $a < b \in \mathbb{R}$ then the interval (a, b) is uncountable.
(ii) The *power set* of $\mathbb{N}$, written $\mathcal{P}(\mathbb{N})$, is defined by

$$\mathcal{P}(\mathbb{N}) = \{X \mid X \subseteq \mathbb{N}\}.$$

Show that $\mathcal{P}(\mathbb{N})$ is uncountable.

5. Let $X \subseteq \mathbb{R}$ be uncountable. Show that there is $x_0 \in X$ which is an *accumulation point* of X; that is, for every open interval I such that $x_0 \in I$, we have $I \cap X \backslash \{x_0\} \neq \emptyset$.

6*. (For 4500 students, this problem is not required, but will contribute extra credit if completed.)

Let (X, d) be a metric space. Given $D \subseteq X$, we say D is *dense* iff for every open $U \subseteq X$, we have $D \cap A \neq \emptyset$. A metric space is called *separable* iff there is a countable dense subset of X.
(i) Prove that $(\mathbb{R}^n, d_{\text{std}})$ is separable.
(ii) Let d be the discrete metric on $\mathbb{R}^n$. Prove that $(\mathbb{R}^n, d)$ is not separable.
(iii) Prove that $(X, d_{\max})$ is separable, where $X = C_{\text{Lip}}([0, 1])$, the collection of all Lipschitz continuous functions $f : [0, 1] \to \mathbb{R}$. (See homework 5 for Lipschitz continuity.)
(Remark: Also, $(Y, d_{\max})$ is separable, where $Y = C([0, 1])$. This might be a future homework problem, after we've discussed some other ideas.)