1.

(a) Let A be the interval $(-2,-1) \cup (1,2)$. Prove that A is an open set (in $(\mathbb{R}, d_{\text{std}})$).

Solution.
Let $x \in A$. We find an open interval I such that $x \in I \subseteq A$ (to verify the original definition of openness for subsets of $\mathbb{R}$).

Now either $x \in (-2,-1)$ or $x \in (1,2)$.

If $x \in (-2,-1)$ then let $I = (-2,-1)$. Then I is an open interval, and $x \in I = (-2,-1) \subseteq A$, as required.

If $x \in (1,2)$ then let $I = (1,2)$, and similarly, I is an open interval, and $x \in I = (1,2) \subseteq A$, as required.

Therefore A is open.

Alternatively you can use the open ball definition of openness: given $x \in A$, if $x \in (-2,-1)$ then let $\varepsilon = \min(x-(-2), -1-x)$; note that since $-2 < x < -1$ then $\varepsilon > 0$, and that

$$\mathcal{B}(x,\varepsilon) = \{z \in \mathbb{R} \mid |x-z| < \varepsilon\}$$

$$= \{z \in \mathbb{R} \mid x-\varepsilon < z < x+\varepsilon\}$$

But $-2 \leq x-\varepsilon$ since $\varepsilon \leq 2+x$ (by definition of ε), and similarly $x+\varepsilon \leq -1$, and therefore

$$= \{z \in \mathbb{R} \mid -2 \leq x-\varepsilon < z < x+\varepsilon \leq -1\}$$

$$\subseteq \{z \in \mathbb{R} \mid -2 < z < -1\} = (-2,-1).$$

So $\mathcal{B}(x,\varepsilon) \subseteq (-2,-1)$, as required.

Do a similar thing when for when $x \in (1,2)$. (Or in fact by symmetry: just reflect everything across $x = 0$; d_{std} is symmetric under reflections.)

(b) Let A, B be non-empty sets such that $A \cap B = \emptyset$. Let $X = A \cup B$. For $x, y \in X$ define

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x,y \in A \ \& \ x \neq y, \\ 1 & \text{if } x,y \in B \ \& \ x \neq y, \\ 3 & \text{if } x \in A,\ y \in B, \\ 3 & \text{if } x \in B,\ y \in A. \end{cases}$$

(b1) Prove the triangle inequality for (X,d). (Hint: given $x,y,z \in X$, you'll need to consider different cases depending on which points are in A and which are in B.)

Solution.
We must prove that for all $x,y,z \in X$,

$$d(x,z) \leq d(x,y) + d(y,z).$$

We prove several claims. Note first that for all $a, b \in X$, $d(a, b) = 0$ or $= 1$ or $= 3$, so in particular $d(a, b) \geq 0$. And note that $d(a, a) = 0$ (by the first clause of the definition of d) and if $a \neq b$ then $d(a, b) \geq 1$ since $d(a, b) = 1$ or $= 3$, since one of the last 4 clauses of the definition of d must apply.

In the following, given a triple (x, y, z) in X^3, we say "the triple (x, y, z) satisfies the triangle inequality" to mean

$$d(x, z) \leq d(x, y) + d(y, z).$$

This is specific to the ordering of the elements in the triple: e.g. saying "(x, y, z) satisfies the triangle inequality" is different to asserting "(y, x, z) satisifies the triangle inequality", since the latter says

$$d(y, z) \leq d(y, x) + d(x, z)$$

instead. We just need to show that for all $(x, y, z) \in X^3$, the triple (x, y, z) satisfies the triangle inequality. We'll do this in 4 claims, which each prove that the triangle inequality holds for certain classes of triples (x, y, z).

Claim 1 will deal with triples (x, y, z) s.t. $x = z$; Claim 2 with those s.t. $x \neq z$ and x, z come from the same set, i.e. [$x, z \in A$ or $x, z \in B$]; Claims 3 and 4 with those s.t. $x \neq z$ and x, z come from different sets: Claim 3 with those s.t. $x \neq z$ and $x \in A$ and $z \in B$; Claim 4 with those s.t. $x \neq z$ and $x \in B$ and $z \in B$.

Note that any triple (x, y, z) will then be covered by one of the Claims, so this will prove the triangle inequality fully.

Claim 1. For all $x, y, z \in X$, if $x = z$, then (x, y, z) satisfies the triangle inequality.

Proof. Let $x, y, z \in X$ s.t. $x = z$. From the remarks above, $d(x, y) \geq 0$ and $d(y, z) \geq 0$, and therefore

$$0 \leq d(x, y) + d(y, z)$$

and so since $d(x, z) = 0$,

$$d(x, z) \leq d(x, y) + d(y, z)$$

as required.

Claim 2. For all $x, y, z \in X$, if $x \neq z$ but [$x, z \in A$ or $x, z \in B$] then (x, y, z) satisfies the triangle inequality. (This case is iff the second or third clause in the definition of d applies to computing $d(x, z)$, i.e. iff $d(x, z) = 1$).

Proof. Let $x, y, z \in X$ as in the statement of the claim. Since $d(x, z) = 1$, we just need to show that

$$1 \leq d(x, y) + d(y, z).$$

But $x \neq z$. So either $x \neq y$ or $y \neq z$. (If $x = y$ and $y = z$ then $x = z$, contra.)
If $x \neq y$ then $d(x, y) \geq 1$ (as in the remarks above, since $x \neq y$, one of the last 4 clauses of the definition of d apply, so $d(x, y) = 1$ or $d(x, y) = 3$). And since $d(x, z) \geq 0$, we get

$$1 + 0 \leq d(x, y) + d(y, z)$$

and therefore

$$d(x, z) = 1 \leq d(x, y) + d(y, z),$$

as required.
If $y \neq z$ then similarly $d(y, z) \geq 1$, and $d(x, y) \geq 0$, and therefore

$$0 + 1 \leq d(x, y) + d(y, z),$$

so

$$d(x, z) = 1 \leq d(x, y) + d(y, z),$$

as required.

Claim 3. For all $x', y', z' \in X$ s.t. $x' \neq z'$ and $x' \in A$ and $z' \in B$, then (x', y', z') satisfies the triangle inequality. (This case is iff the 4th clause in the definition of d applies to computing $d(x', z')$, so implies $d(x', z') = 3$).

Proof. Let $x', y', z' \in X$ as in the statement of the claim. So $d(x', z') = 3$ by the def'n of d. So we need to show that

$$3 \leq d(x', y') + d(y', z').$$

Case 1. $y' \in A$.
Then $d(x', y') \geq 0$ and since $y' \in A$ and $z' \in B$, $d(y', z') = 3$. Hence

$$d(x', z') = 3 = 0 + 3 \leq d(x', y') + d(y', z').$$

Case 2. $y' \in B$.
Then since $x' \in A$ and $y' \in B$, $d(x', y') = 3$, and anyway $d(y', z') \geq 0$. Hence

$$d(x', z') = 3 = 3 + 0 \leq d(x', y') + d(y', z').$$

This covers all cases, finishing the proof of Claim 3.

Claim 4. For all $x, y, z \in X$ s.t. $x \neq z$ and $x \in B$ and $z \in A$, we have $d(x, z) \leq d(x, y) + d(y, z)$. (This case is iff the 5th clause in the definition of d applies to computing $d(x, z)$, so implies $d(x, z) = 3$).

Proof. Note that d is symmetric: for all $a, b \in X$, $d(a, b) = d(b, a)$: if $a = b$ then this is trivial; if $a, b \in A$ or $a, b \in B$ then $d(a, b) = 1 = d(b, a)$; if $[a \in A$ and $b \in B]$ or $[a \in B$ and $b \in A]$ then $d(a, b) = 3 = d(b, a)$).

We can use this to deduce Claim 4 from Claim 3, because: Let $x, y, z \in X$ as in the statement of Claim 4. Then by the symmetry of d,

$$d(x, z) = d(z, x)$$

and since $z \in A$ and $x \in B$, by Claim 3 applied to the triple $(x', y', z') = (z, y, x)$ (so here $x' = z$, $y' = y$, $z' = x$), we have

$$d(x', z') \leq d(x', y') + d(y', z'),$$

i.e.

$$d(z, x) \leq d(z, y) + d(y, x)$$

but then again by symmetry of d,

$$d(x, z) \leq d(y, z) + d(x, y) = d(x, y) + d(y, z)$$

by comm of $+$. This is the triangle inequality for the triple (x, y, z), as required.

The four claims prove the triangle inequality for d, since they cover all possible cases for $x, y, z \in X$.

Remark: I just used the symmetry method to prove Claim 4 for illustration. It would have been faster just to prove it in the same way Claim 3 was proven.

(b2) Assume that (X, d) (as defined above) is a metric space. Show that for every set $D \subseteq X$, D is open in (X, d).

Solution. Let $D \subseteq X$. We claim that D is open. For let $x \in D$. Let $\varepsilon = 1$. Then $\varepsilon > 0$ and

$$\mathcal{B}(x, \varepsilon) \subseteq D.$$

For if $z \in \mathcal{B}(x, \varepsilon)$ then

$$d(z, x) < 1,$$

but by the defintion of d, this implies $d(x, z) = 0$, and again by the definition of d (or using that d is a metric) this implies $z = x$. But $x \in D$ by hypothesis, so $z \in D$, as required.

(c) Let (X', d') be an arbitrary metric space. Suppose $x_1, x_2, x_3, x_4 \in X$ are such that $d(x_1, x_3) = 8$, $d(x_2, x_3) = 10$, $d(x_4, x_2) = 7$. Prove that $d(x_1, x_4) \leq 25$.

Solution.
By the triangle inequality,

$$d(x_1, x_4) \leq d(x_1, x_3) + d(x_3, x_4)$$

and again by the triangle inequality, $d(x_3, x_4) \leq d(x_3, x_2) + d(x_2, x_4)$, so

$$d(x_1, x_4) \leq d(x_1, x_3) + d(x_3, x_2) + d(x_2, x_4).$$

By symmetry, $d(x_3, x_2) = d(x_2, x_3)$ and $d(x_2, x_4) = d(x_4, x_2)$, so

$$d(x_1, x_4) \leq d(x_1, x_3) + d(x_2, x_3) + d(x_4, x_2),$$

so by the hypothesis,

$$d(x_1, x_4) \leq 8 + 10 + 7 = 25,$$

as required.

2. Suppose $A \subseteq \mathbb{R}$ is closed and $\mathbb{Q} \cap [3, 4] \subseteq A$. Prove that $\pi \in A$. (Work directly from the definitions of closed/open subsets of $\mathbb{R}$, using the standard metric on $\mathbb{R}$. State any standard facts about $\mathbb{Q}$ that you use (you don't need to prove such facts).)

Hint: one method is to assume $\pi \notin A$ and derive a contradiction.

Solution.
Suppose $\pi \notin A$. Then $\pi \in \mathbb{R}\backslash A$. Since A is closed, $\mathbb{R}\backslash A$ is open. Thus there is $\varepsilon > 0$ such that $\mathcal{B}(\pi, \varepsilon) \subseteq \mathbb{R}\backslash A$. Let $\varepsilon' = \min(\varepsilon, 0.1)$; then $\varepsilon' > 0$ since $\varepsilon > 0$ and $0.1 > 0$; and $\varepsilon' \leq \varepsilon$ and $\varepsilon' \leq 0.1$. Then

$$\mathcal{B}(\pi, \varepsilon') \subseteq \mathcal{B}(\pi, \varepsilon) \subseteq \mathbb{R}\backslash A,$$

since $\varepsilon' \leq \varepsilon$, so

$$\mathcal{B}(\pi, \varepsilon') \subseteq \mathbb{R}\backslash A,$$

and in other words,

$$\mathcal{B}(\pi, \varepsilon') \cap A = \emptyset.$$

Now $\mathcal{B}(\pi, \varepsilon')$ is the interval $(\pi - \varepsilon', \pi + \varepsilon')$, and by the density of $\mathbb{Q}$ in $\mathbb{R}$, there is $q \in \mathbb{Q}$ such that $q \in (\pi - \varepsilon', \pi + \varepsilon')$. But $3.1 < \pi < 3.2$, and $\varepsilon' \leq 0.1$, so $3 < \pi - \varepsilon' < \pi < \pi + \varepsilon' < 3.3$. Therefore $3 < q < 3.3$, so $q \in [3, 4]$. Therefore $q \in \mathbb{Q} \cap [3, 4]$, but therefore by hypothesis (that $\mathbb{Q} \cap [3, 4] \subseteq$), we have $q \in A$. But we had $q \in \mathcal{B}(\pi, \varepsilon')$. Therefore $q \in \mathcal{B}(\pi, \varepsilon') \cap A$, contradicting the fact that the latter set is empty.

So $\pi \in A$, as required.

3. Is the following statement true?

"Let (X, d) be a metric space and let $\langle C_i \rangle_{i \in J}$ be a family of closed subsets of X. (Closed with respect to d.) Then the union of the family,

$$\bigcup_{i \in J} C_i,$$

is also closed."

Either prove the statement or give a counterexample (if you give a counterexample, you needn't prove the counterexample works, but make your example completely specific).

Solution. The statement is false. We work with $\mathbb{R}$ and the standard metric.

E.g. let $J = \mathbb{N}$, and for $n \in \mathbb{N}$, let $C_n = [1/n, 1]$. Then C_n is closed for each $n \in J$, but $\cup_{n\in\mathbb{N}} C_n = (0, 1]$, and this set is not closed.

Another example: every singleton in $\mathbb{R}$ is closed. Therefore if unions of families of closed sets are always closed, we'd get that every subset of $\mathbb{R}$ is closed, which is false. E.g. take $J = (0, 1]$, and for $x \in J$, let $C_x = \{x\}$. Then C_x is closed for each $x \in J$, and $\cup_{x\in(0,1]} C_x = (0, 1]$, which is not closed.

4. Let (X, d) be a metric space. Let $\langle x_i \rangle_{i\in\mathbb{N}}$ be a sequence such that $x_i \in X$ for all $i \in \mathbb{N}$. Prove that the sequence converges to *at most* one point $x \in X$.

Solution. Suppose $x, y \in X$ are such that the sequence converges to both x and y. We'll prove that $x = y$.

Let $\varepsilon > 0$. We'll show that $d(x, y) < \varepsilon$. Since ε is an arbitrary positive, this will prove that $d(x, y) \leq 0$, and since d is a metric, therefore $d(x, y) = 0$, and $x = y$.

So to see $d(x, y) < \varepsilon$. Since $x_i \to x$, we can fix $N_x \in \mathbb{N}$ such that for all $n \geq N_x$, $d(x, x_n) < \varepsilon/2$. Since $x_i \to y$, we can fix $N_y \in \mathbb{N}$ such that for all $n \geq N_y$, $d(y, x_n) < \varepsilon/2$.

Let $n = \max(N_x, N_y)$. Then by choice of N_x, N_y, we have $d(x, x_n) < \varepsilon/2$ and $d(y, x_n) < \varepsilon/2$. But combining this with the triangle inequality and symmetry,

$$d(x, y) \leq d(x, x_n) + d(x_n, y) = d(x, x_n) + d(y, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

showing $d(x, y) < \varepsilon$, as required.

5. Let $X = C([0, 1])$ and $d = d_{\max}$; that is,

$$d(f, g) = \max\{|f(x) - g(x)| \mid x \in [0, 1]\}.$$

Let $F : X \to \mathbb{R}$ be the function defined by

$$F(f) = f(0.3) + f(0.7).$$

Show that F is continuous as a function from $(X, d_{\max})$ to $(\mathbb{R}, d_{\text{std}})$.

Solution. Let $f \in X$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$F``\mathcal{B}_{\max}(f, \delta) \subseteq \mathcal{B}_{\text{std}}(F(f), \varepsilon),$$

or in other words, such that

$$d_{\text{std}}(F(g), F(f)) < \varepsilon$$

whenever

$$d_{\max}(g, f) < \delta.$$

Note that for any $g \in X$,

$$d_{\text{std}}(F(g), F(f))$$

$$= |g(0.3) + g(0.7) - (f(0.3) + f(0.7))|$$

$$= |g(0.3) - f(0.3) + g(0.7) - f(0.7)|$$
$$\leq |g(0.3) - f(0.3)| + |g(0.7) - f(0.7)|,$$

the latter by the triangle inequality on $\mathbb{R}$. So

$$d_{\text{std}}(F(g), F(f)) \leq |g(0.3) - f(0.3)| + |g(0.7) - f(0.7)|. \tag{1}$$

But

$$d_{\max}(g, f) = \max\{|g(x) - f(x)| \mid x \in [0, 1]\},$$

and $0.3 \in [0, 1]$ and $0.7 \in [0, 1]$, so the "max" is taken over a collection of values including both $|g(0.3) - f(0.3)|$ and $|g(0.7) - f(0.7)|$, so

$$|g(0.3) - f(0.3)| \leq d_{\max}(g, f),$$

and

$$|g(0.7) - f(0.7)| \leq d_{\max}(g, f).$$

So the last two lines combined with (1) give

$$d_{\text{std}}(F(g), F(f)) \leq d_{\max}(g, f) + d_{\max}(g, f) = 2d_{\max}(g, f). \tag{2}$$

So if we make g "close" to f in $d_{\max}$, then $F(g)$ will be at most twice this distance from $F(f)$ in d_{std}, which is still quite close. And in fact then, if we set $\delta = \varepsilon/2$, then for all

$$g \in \mathcal{B}_{\max}(f, \delta) = \mathcal{B}_{\max}(f, \varepsilon/2),$$

we have $d_{\max}(g, f) < \varepsilon/2$, so then by (2),

$$d_{\text{std}}(F(g), F(f)) \leq 2d_{\max}(f, g) < 2(\varepsilon/2) = \varepsilon.$$

So

$$F``\mathcal{B}_{\max}(f, \delta) \subseteq \mathcal{B}_{\text{std}}(F(f), \varepsilon),$$

and since $\varepsilon > 0$, $\delta = \varepsilon/2 > 0$ also, as required.

So F is continuous.