Midterm 2 Review Problems
(Note: all solutions, including examples, should be explained, unless indicated otherwise.)

1. *[With correction to τ and τ'; both were omitting the empty set originally.]* Let $X = \{0, 1, 2, 3\}$ and $Y = \{0, 1, 2\}$. Let $\tau = \{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2, 3\}\}$ and let $\tau' = \{\emptyset, \{0\}, \{0, 1\}, \{0, 2\}, \{0, 1, 2\}\}$. (Then (X, τ) and (Y, τ') are both topological spaces; you may assume this.)

Give an example of a function $f : X \to Y$ which is not continuous (with respect to these topologies).

2. Prove that if $\mathfrak{b}$ is a base for a topology τ on Y, and $f : X \to Y$, then

$$\{f^{-1}(U) \mid U \in \mathfrak{b}\}$$

is a base for a topology on τ' on X. Show, moreover, that f is continuous from (X, τ') to (Y, τ), and in fact that τ' is the smallest topology with this property. (I.e., if τ'' is another topology on X such that f is continuous from (X, τ'') to (Y, τ), then $\tau' \subseteq \tau''$.)

3. Let X be the collection of all functions $f : \mathbb{N} \to \mathbb{N}$. For each $n \in \mathbb{N}$, and each function $\sigma : \{0, 1, \ldots, n-1\} \to \mathbb{N}$, let $N_\sigma \subseteq X$ be the collection of functions

$$N_\sigma = \{f : \mathbb{N} \to \mathbb{N} \mid \sigma = f \restriction \{0, 1 \ldots, n-1\}\}.$$

So, for example, if σ_0 is the function with domain $\{0, 1, 2\}$, such that $\sigma_0(0) = 3$, $\sigma_0(1) = 5$ and $\sigma_0(2) = 0$, then

$$N_{\sigma_0} = \{f : \mathbb{N} \to \mathbb{N} \mid \sigma_0 = f \restriction \{0, 1, 2\}\}$$

$$= \{f : \mathbb{N} \to \mathbb{N} \mid f(0) = 3,\ f(1) = 5,\ f(2) = 0.\}.$$

Let $\mathfrak{b}$ be the collection of all sets of the form N_σ (ranging over all σ as above).

(a) Show that $\mathfrak{b}$ is a base, for a topology τ on X.

(b) Let C be the set of all functions $f : \mathbb{N} \to \mathbb{N}$ such that 5 is not in the range of f. Show that C is closed in this topology.

(c) Show that the set of all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 3$ is both open and closed in this topology.

(d) Prove that X is uncountable.

(This topological space is called *Baire space*.)

4. Let C be the Cantor set. (a) Let $x \in C$ and $\varepsilon > 0$. Show that there is some $y \in C$ such that $y \neq x$, but $|y - x| < \varepsilon$. (b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and f is constant on $\mathbb{R} \backslash C$. Prove that f is constant.

5. Let $f : [0, 1] \to (0, 1]$. Prove that there is some $n \in \mathbb{N}$ such that the set $f^{-1}((1/n, 1])$ is uncountable.

6. Let X be the collection of all polynomial functions with domain $[0, 100]$. Let τ be the topology on X given by the max-metric on X. Construct a countable base for (X, τ) (and prove it is a base).

7. (a) Prove that in a topological space X, for any $A \subseteq X$, $\mathrm{Cl}(A)$ is the set of all $z \in X$ such that every open neighbourhood U of z is such that $U \cap A \neq \emptyset$.

(b) Let $f : X \to Y$ be continuous between top spaces. Does it follow that $f^{-1}(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f^{-1}(A))$ for every $A \subseteq Y$? What if "$\subseteq$" is replaced by "$\supseteq$"?

(c) Suppose (X, τ) is a top space such that $\mathrm{Int}(\mathrm{Cl}(U)) = U$ for every $U \in \tau$. Prove that every U in τ is closed (w.r.t. τ). (Hint: argue by contradiction. Start with an open set U which is not closed, and work with it to to construct an open set U' such that $\mathrm{Int}(\mathrm{Cl}(U_1)) \neq U_1$.)

8. Prove that every closed set is sequentially closed in a topological space.

9. For each $n \in \mathbb{N}$, let $D_n \subseteq \mathbb{R}^2$ be an open disc, $D_n = \mathcal{B}(p_n, \varepsilon_n)$, such that for each n, $\mathcal{B}(p_{n+1}, 2\varepsilon_{n+1}) \subseteq D_n$, and $0 < \varepsilon_n \leq 2^{-n}$. Prove that $\cap_{n \in \mathbb{N}} D_n$ consists of exactly one point. (Hint: we proved a related fact about $\mathbb{R}$.)