Midterm 2 Review Problems - Hints / Sketched Solutions
(Note: all solutions, including examples, should be explained, unless indicated otherwise.)

Sketched solutions are not intended to be full proofs; you should give full proofs on the test.

1. *[With correction to τ and τ'; both were omitting the empty set originally.]* Let $X = \{0, 1, 2, 3\}$ and $Y = \{0, 1, 2\}$. Let $\tau = \{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2, 3\}\}$ and let $\tau' = \{\emptyset, \{0\}, \{0, 1\}, \{0, 2\}, \{0, 1, 2\}\}$. (Then (X, τ) and (Y, τ') are both topological spaces; you may assume this.)

Give an example of a function $f : X \to Y$ which is not continuous (with respect to these topologies).

Hint:
You need to define a function such that for some $U \in \tau'$, you have $f^{-1}(U) \notin \tau$. This can be arranged either with $U = \{0, 1\}$, or with $U = \{0, 2\}$.

2. Prove that if $\mathfrak{b}$ is a base for a topology τ on Y, and $f : X \to Y$, then

$$\{f^{-1}(U) \mid U \in \mathfrak{b}\}$$

is a base for a topology on τ' on X. Show, moreover, that f is continuous from (X, τ') to (Y, τ), and in fact that τ' is the smallest topology with this property. (I.e., if τ'' is another topology on X such that f is continuous from (X, τ'') to (Y, τ), then $\tau' \subseteq \tau''$.)

Hint:
Use the proposition proved in class, i.e. that a collection $\mathfrak{b}_1$ of subsets of X_1 is a base for a topology on X_1 iff: $\cup\mathfrak{b}_1 = X_1$ and for all $U_1, U_2 \in \mathfrak{b}_1$, we have that $U_1 \cap U_2$ can be formed as the union of some elements from $\mathfrak{b}_1$. I discussed the minimality property of τ' in class.

3. Let X be the collection of all functions $f : \mathbb{N} \to \mathbb{N}$. For each $n \in \mathbb{N}$, and each function $\sigma : \{0, 1, \ldots, n-1\} \to \mathbb{N}$, let $N_\sigma \subseteq X$ be the collection of functions

$$N_\sigma = \{f : \mathbb{N} \to \mathbb{N} \mid \sigma = f \restriction \{0, 1 \ldots, n-1\}\}.$$

So, for example, if σ_0 is the function with domain $\{0, 1, 2\}$, such that $\sigma_0(0) = 3$, $\sigma_0(1) = 5$ and $\sigma_0(2) = 0$, then

$$N_{\sigma_0} = \{f : \mathbb{N} \to \mathbb{N} \mid \sigma_0 = f \restriction \{0, 1, 2\}\}$$

$$= \{f : \mathbb{N} \to \mathbb{N} \mid f(0) = 3,\ f(1) = 5,\ f(2) = 0.\}.$$

Let $\mathfrak{b}$ be the collection of all sets of the form N_σ (ranging over all σ as above).

(a) Show that $\mathfrak{b}$ is a base, for a topology τ on X.

(b) Let C be the set of all functions $f : \mathbb{N} \to \mathbb{N}$ such that 5 is not in the range of f. Show that C is closed in this topology.

(c) Show that the set of all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 3$ is both open and closed in this topology.

(d) Prove that X is uncountable.

(This topological space is called *Baire space*.)

Hint:
(a) Use the same proposition as mentioned in the hint for problem 2. As part of this, let $n_1 \in \mathbb{N}$ and $\sigma_1 : \{0, 1, \ldots, n_1 - 1\} \to \mathbb{N}$ and similarly for n_2 and σ_2. You must show that $U = N_{\sigma_1} \cap N_{\sigma_2}$ is the union of some elements of $\mathfrak{b}$, i.e. the union of some collection of N_σ's. Show that in fact, either $U = \emptyset$, or $U = N_{\sigma_1}$, or $U = N_{\sigma_2}$; and that this implies the former statement. Which of these cases attains depends on whether $n_1 \leq n_2$ or $n_1 > n_2$, and whether $\sigma_1 \restriction \min(n_1, n_2) = \sigma_2 \restriction \min(n_1, n_2)$.

(b) The complement of C is the set of all f such that $5 \in \mathrm{rg}(f)$. But $5 \in \mathrm{rg}(f)$ iff there's $n \in \mathbb{N}$ such that $f(n) = 5$, which is iff there's $k \in \mathbb{N}$ such that $5 \in \mathrm{rg}(\sigma)$, where $\sigma = f \restriction \{0, 1, \ldots, k - 1\}$, which is iff $f \in N_\sigma$ for some σ with $5 \in \mathrm{rg}(\sigma)$. The latter condition describes an open set.

(c) The proof of openness is similar to (b); in fact it's easier, as $f(0) = 3$ iff $f \in N_\sigma$, where $\sigma : \{0\}\mathbb{N}$ is the function such that $\sigma(0) = 3$. For closedness, note that $f(0) \neq 3$ iff $f(0) = k$ for some $k \in \mathbb{N}\backslash\{3\}$, which is iff $f \in N_\sigma$ for some $\sigma : \{0\} \to \mathbb{N}$, with $\sigma(0) = k$, some $k \in \mathbb{N}\backslash\{3\}$.

(d) Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be an enumeration of functions in X. Construct a function $g : \mathbb{N} \to \mathbb{N}$ such that $g \neq f_n$ for each n, by diagonalizing: define $g(n)$ such that $g(n) \neq f_n(n)$ for each n.

4. Let C be the Cantor set. (a) Let $x \in C$ and $\varepsilon > 0$. Show that there is some $y \in C$ such that $y \neq x$, but $|y - x| < \varepsilon$. (b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and f is constant on $\mathbb{R}\backslash C$. Prove that f is constant.

Hint:
(a) (I reversed the roles of the variables x, y.) Let $y \in C$ and $\varepsilon > 0$. Let $n \in \mathbb{N}$ be large enough that $1/3^n < \varepsilon$. Recall that C_n is the union of disjoint closed intervals of length $1/3^n$. (Here C_n, $n \in \mathbb{N}$, are the sets used to construct C, by $C = \cap_{n \in \mathbb{N}} C_n$.) Since $C \subseteq C_n$, $y \in C_n$, so $y \in I$ for one of the length $1/3^n$ closed intervals $I \subseteq C_n$. But the endpoints of I are both in C. Let x be the left end-point of I, unless y is that endpoint, in which case, let x be the right endpoint. Then $|x - y| \leq 1/3^n < \varepsilon$, $x \neq y$, and $x \in C$.

(b) This is just because the closure of $\mathbb{R}\backslash C$ is $\mathbb{R}$ (in the standard topology). I.e., let $c \in \mathbb{R}$ be the constant output of f on $\mathbb{R}\backslash C$. Let $x \in C$; we claim that $f(x) = c$ also. Since C contains no interval of positive length, for all $\varepsilon > 0$ there is $y \in \mathbb{R}\backslash C$ such that $|x - y| < \varepsilon$ and $y \in \mathbb{R}\backslash C$. But $f(y) = c$ for all such y. Since f is continuous and the standard topology comes from a metric, f is sequentially continuous. So for $n \in \mathbb{N}$ let $y_n \in \mathbb{R}\backslash C$, such that $y_n \to x$. Then $f(x) = \lim_{n \to \infty} f(y_n) = \lim_{x \to \infty} c = c$.

5. Let $f : [0, 1] \to (0, 1]$. Prove that there is some $n \in \mathbb{N}$ such that the set $f^{-1}((1/n, 1])$ is uncountable.

Hint:
We have $f^{-1}((0, 1]) = [0, 1]$, which is uncountable. Show that $[0, 1]$ is the union of the sets $f^{-1}((1/n, 1])$, $n \in \mathbb{N}$, and use this to prove that one of these sets is

uncountable.

6. Let X be the collection of all polynomial functions with domain $[0, 100]$. Let τ be the topology on X given by the max-metric on X. Construct a countable base for (X, τ) (and prove it is a base).

Hint:
Use the fact that a metric space with a countable dense set D has a countable base (given by taking open balls with centers $x \in D$ and radii $\varepsilon \in \mathbb{Q}$). And τ is the topology of a metric on X. To get a countable dense set D, consider the set of polynomials with rational coefficients. Show this is dense.

7. (a) Prove that in a topological space X, for any $A \subseteq X$, $\mathrm{Cl}(A)$ is the set of all $z \in X$ such that every open neighbourhood U of z is such that $U \cap A \neq \emptyset$.

(b) Let $f : X \to Y$ be continuous between top spaces. Does it follow that $f^{-1}(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f^{-1}(A))$ for every $A \subseteq Y$? What if "$\subseteq$" is replaced by "$\supseteq$"?

(c) Suppose (X, τ) is a top space such that $\mathrm{Int}(\mathrm{Cl}(U)) = U$ for every $U \in \tau$. Prove that every U in τ is closed (w.r.t. τ). (Hint: argue by contradiction. Start with an open set U which is not closed, and work with it to to construct an open set U' such that $\mathrm{Int}(\mathrm{Cl}(U_1)) \neq U_1$.)

Hint:
(a) Show that the set of all z satisfying the stated condition is closed, and in fact the smallest closed set which is a superset of A.

(b) For "$\subseteq$", consider examples in which $f : \mathbb{R} \to \mathbb{R}$ is constant and $A = \mathbb{Q}$. For "$\supseteq$", the resulting statement is true: use the fact that $A \subseteq \mathrm{Cl}(A)$, and therefore that $f^{-1}(A) \subseteq f^{-1}(\mathrm{Cl}(A))$, and that by continuity, $f^{-1}(\mathrm{Cl}(A))$ is closed.

(c) Continuing with the hint given following the statement of the problem, let U be as there, and note that $V = X \backslash \mathrm{Cl}(U)$ is open, so by hypothesis, $\mathrm{Int}(\mathrm{Cl}(V)) = V$. Show that $U \cup V \in \tau$, and $U \cup V \neq X$, but $\mathrm{Cl}(U \cup V) = X$, so $\mathrm{Int}(\mathrm{Cl}(U \cup V)) = X$, contradicting the fact that $\mathrm{Int}(\mathrm{Cl}(U \cup V)) = U \cup V$ (which follows from our hypothesis).

8. Prove that every closed set is sequentially closed in a topological space.

Hint:
Let A be closed and $x_n \in A$ for each $n \in \mathbb{N}$ and suppose that the sequence converges, and $x \in X$ is such that $x = \lim_{n\to\infty} x_n$. Prove that $x \in A$: otherwise $U = X \backslash A$ is open, and $x \in U$, and you can derive a contradiction from the sequence converging to x but consisting only of points in A.

9. For each $n \in \mathbb{N}$, let $D_n \subseteq \mathbb{R}^2$ be an open disc, $D_n = \mathcal{B}(p_n, \varepsilon_n)$, such that for each n, $\mathcal{B}(p_{n+1}, 2\varepsilon_{n+1}) \subseteq D_n$, and $0 < \varepsilon_n \leq 2^{-n}$. Prove that $\cap_{n\in\mathbb{N}} D_n$ consists of exactly one point. (Hint: we proved a related fact about $\mathbb{R}$.)

Hint:

The related fact was that given a sequence I_n, $n \in \mathbb{N}$, of closed intervals $\subseteq \mathbb{R}$, such that $I_0 = [0, 1]$ and $I_{n+1} \subseteq I_n$, the intersection $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$. Call this fact (*). (This was while we were discussing uncountability and the Cantor set.)

Show that for each $n \in \mathbb{N}$, there is a closed square $S_{n+1} = I_{n+1} \times J_{n+1}$ (where I_{n+1} and J_{n+1} are closed intervals $\subseteq \mathbb{R}$ each having the same length), such that $D_{n+1} \subseteq S_{n+1} \subseteq D_n$. Prove that $\cap_{n\in\mathbb{N}} D_n = \cap_{n\in\mathbb{N}} S_n$. Prove that $\cap_{n\in\mathbb{N}} S_n$ is non-empty, either by following the proof of (*), or by applying (*) to the sequence I_n, $n \in \mathbb{N}$, and to the sequence J_n, $n \in \mathbb{N}$ (the sides of S_n). Use the fact that $\varepsilon_n \to 0$ to show that the intersection has at most one point.