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Congruences of Hurwitz class numbers on square classes

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ABSTRACT

We extend a holomorphic projection argument of our earlier work to prove a novel divisibility result for non-holomorphic congruences of Hurwitz class numbers. This result allows us to establish Ramanujan-type congruences for Hurwitz class numbers on square classes, where the holomorphic case parallels previous work by Radu on partition congruences. We offer two applications. The first application demonstrates common divisibility features of Ramanujan-type congruences for Hurwitz class numbers. The second application provides a dichotomy between congruences for class numbers of imaginary quadratic fields and Ramanujan-type congruences for Hurwitz class numbers.

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Hurwitz class numbers $H(D)$ play a significant role in classical number theory. If $-D < -4$ is a negative fundamental discriminant, then $h(-D) = H(D)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$. Despite intensive studies, divisibility properties of these class numbers have remained mysterious.

In this work, we investigate Ramanujan-type congruences for Hurwitz class numbers (see Theorem A, B, and C). Furthermore, in Theorem D we connect Ramanujan-type congruences for Hurwitz class numbers $H(D)$ to congruences for class numbers $h(-D)$ in certain families of fundamental discriminants $-D$.

In [2], we explored Ramanujan-type congruences for Hurwitz class numbers $H(D)$ such as the following examples:

$$\begin{aligned} H(5^3n + 5^2) &\equiv 0 \pmod{5}, \\ H(7^3n + 3 \cdot 7^2) &\equiv 0 \pmod{7}, \\ H(11^3n + 7 \cdot 11^2) &\equiv 0 \pmod{11}. \end{aligned}$$

These congruences are of the form $H(an + b) \equiv 0 \pmod{\ell}$, where $\ell > 3$ is a prime and $a > 0$ and b are integers such that $-b$ is a square modulo a. We refer to such congruences as non-holomorphic Ramanujan-type congruences, because the generating series for $H(an + b)$ is a mock modular form, i.e., it has a non-holomorphic modular completion. In particular, one cannot access such congruences via standard techniques from the theory of holomorphic modular forms.

In our earlier work [2] we employed a holomorphic projection argument to prove that for such non-holomorphic congruences the divisibility $\ell \,|\, a$ holds. The above examples also suggest the divisibility $\ell \,|\, b$, and in the current paper we use another holomorphic projection argument to prove:

Theorem A. *Let $\ell > 3$ be a prime, $a \in \mathbb{Z}_{\geq 1}$, and $b \in \mathbb{Z}$. If $-b$ is a square modulo a and*

$$H(an + b) \equiv 0 \pmod{\ell}$$

for all integers n, then $\ell \,|\, b$.

There are also holomorphic Ramanujan-type congruences for Hurwitz class numbers, i.e., congruences $H(an + b) \equiv 0 \pmod{\ell}$ where $-b$ is not a square modulo a:

$$\begin{aligned} H(3^3n + 3^2) &\equiv 0 \pmod{5}, \\ H(5^3n + 2 \cdot 5^2) &\equiv 0 \pmod{7}, \\ H(2^9n + 3 \cdot 2^6) &\equiv 0 \pmod{11}. \end{aligned}$$

In these examples ℓ does not divide a or b. While such congruences can be studied with tools from the theory of holomorphic modular forms, the relation between a and b has not yet been resolved, either.

Our examples of holomorphic and non-holomorphic congruences indicate that $\mathrm{ord}_p\big(a/\gcd(a,b)\big) \le 1$ for odd primes p and $\mathrm{ord}_2\big(a/\gcd(a,b)\big) \le 3$. To prove these phenomena, we first establish the following result on congruences of square classes:

Theorem B. *Let $\ell > 3$ be a prime, $a \in \mathbb{Z}_{\ge 1}$, and $b \in \mathbb{Z}$. Suppose $H(an+b) \equiv 0 \pmod{\ell}$ for all integers n. Then $H(an+bu^2) \equiv 0 \pmod{\ell}$ for all integers u with $\gcd(u,a) = 1$ and $n \in \mathbb{Z}$.*

In the case of holomorphic Ramanujan-type congruences, the proof of Theorem B extends ideas of Radu's study of partition congruences [9,10]. The proof of Theorem B in the non-holomorphic case requires a deeper analysis, where Theorem A is an essential ingredient.

We give two applications of Theorem B. In our first application, we call a Ramanujan-type congruence for $H(D)$ modulo ℓ on $a\mathbb{Z}+b$ *maximal*, if $H(D)$ has no Ramanujan-type congruence modulo ℓ on any arithmetic progression $a'\mathbb{Z}+b'$ that is properly contained in $a\mathbb{Z}+b$.

Theorem C. *Let $\ell > 3$ be a prime. Suppose that we have a maximal Ramanujan-type congruence modulo ℓ for the Hurwitz class numbers on $a\mathbb{Z}+b$. Then for odd primes p,*

$$\mathrm{ord}_p\big(a/\gcd(a,b)\big) \le 1 \quad \textit{and} \quad \mathrm{ord}_2\big(a/\gcd(a,b)\big) \le 3.$$

As a further application of Theorem B, we provide a dichotomy between Ramanujan-type congruences for Hurwitz class numbers and congruences for class numbers of imaginary quadratic fields whose discriminant varies in a square class modulo a.

For the next statement, we require the usual Legendre symbol and also the divisor sum $\sigma_1(b) := \sum_{d \,|\, b} d$. For a prime p and an integer a, we call the largest p-power that divides a its p-part.

Theorem D. *Let $\ell > 3$ be a prime. Suppose that we have a Ramanujan-type congruence modulo ℓ for the Hurwitz class numbers on $a\mathbb{Z}+b$. For all odd primes $p \,|\, a$ assume that $\mathrm{ord}_p\big(a/\gcd(a,b)\big) \ge 1$ and if a is even, assume that $\mathrm{ord}_2\big(a/\gcd(a,b)\big) \ge 2$. Then either:*

(i) We have $h(-D) \equiv 0 \pmod{\ell}$ for all fundamental discriminants $-D < -4$ for which there is $f \in \mathbb{Z} \setminus \{0\}$ with $Df^2 \in a\mathbb{Z}+b$.

(ii) There is a prime p dividing a such that

$$\sigma_1(f_p) \equiv \left(\frac{-D}{p}\right) \sigma_1(f_p/p) \pmod{\ell}$$

for every fundamental discriminant $-D < 0$ and integer f satisfying $Df^2 \equiv b \pmod{a}$, where f_p is the p-part of f. Both $(-D/p)$ and f_p are uniquely determined by $a\mathbb{Z}+b$. In *this case, we have a Ramanujan-type congruence for Hurwitz class numbers on* $a_p\mathbb{Z} + b$*, where* a_p *is the* p*-part of* a*.*

Remark.

(1) The assumptions on the orders of $a/\gcd(a, b)$ can always be achieved by replacing a with a suitable multiple of it. They can be removed at the expense of a more technical statement involving the factorizations $Df^2 \in a\mathbb{Z} + b$ that appear in case (i).

(2) From the Hurwitz class number formula alone, one could deduce a statement similar to case (ii) for some prime p, not necessarily dividing a. To show that one must have $p \,|\, a$, we use Theorem B.

(3) One can verify that all congruences given in this introduction fall under case (ii). We do not expect that the first case in the theorem ever occurs, i.e., we expect that Ramanujan-type congruences for $H(D)$ modulo ℓ on $a\mathbb{Z} + b$ occur if and only if there is $p \,|\, a$ and $Df^2 \in a\mathbb{Z} + b$ satisfying the above condition in (ii). This belief is supported by extensive numerical evidence in addition to well-known theorems on the divisibility of class numbers, such as [14], which implies that if the first case occurs, we must have either $2 \nmid \mathrm{ord}_2(\gcd(a, b))$, $2 \nmid \mathrm{ord}_\ell(\gcd(a, b))$, $\mathrm{ord}_2(a/\gcd(a, b)) \geq 2$, $\mathrm{ord}_\ell(a/\gcd(a, b)) \geq 1$ or $p \equiv \pm 1 \pmod{\ell}$ for some odd p with $2 \nmid \mathrm{ord}_p(a)$ - in other words, $-Df^2 \in a\mathbb{Z} + b$ cannot be equivalent to requirements on the splitting behavior in $\mathbb{Q}(\sqrt{-D})$ of primes not congruent to $\pm 1 \pmod{\ell}$, other than $2, \ell$.

(4) Theorem D also offers a partial explanation for the pattern in the examples of Ramanujan-type congruences given so far that there is always a prime $p \,|\, a$ for which $\mathrm{ord}_p(a) \geq 3$ and $\mathrm{ord}_p(b) \geq 2$. This comes from the fact that for the congruence in (ii) to hold, one must have $p|f_p$, and combined with the assumptions on a, b in Theorem D, this implies that $2 \leq \mathrm{ord}_p(b) \leq \mathrm{ord}_p(a) - 1$.

(5) For any congruences for Hurwitz class numbers that do not fall under case (ii), Theorem D implies the existence of fundamental discriminants $-D \equiv u^2 b \pmod{a}$ for which $h(-D) \equiv 0 \pmod{\ell}$ for some u co-prime to a.

The proof of Theorem D relies on the Hurwitz class number formula and Theorem B. In accordance with Theorem B, the case of holomorphic Ramanujan-type congruences is accessible via methods from the classical theory of modular forms, while the non-holomorphic case is not.

The paper is organized as follows. In Section 1, we review some tools from the theory of modular forms needed for our work. In Section 2, we establish Theorem A. In Section 3 we prove Theorem B. Finally, in Section 4 we settle Theorems C and D.

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1. Preliminaries

We introduce necessary notation to discuss modular forms (see for example [3]) and quasi-modular forms (see for example [8,16]). For odd D, set

$$\epsilon_D = \begin{cases} 1, & \text{if } D \equiv 1 \pmod 4; \\ i, & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{1.1}$$

Throughout the paper $\tau \in \mathbb{H}$ (the usual complex upper half plane), $y = \mathrm{Im}(\tau)$, and $e(s\tau) := \exp(2\pi i\, s\tau)$ for $s \in \mathbb{Q}$. Let $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$ be the standard congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathrm{M}_k(\Gamma)$ denote the space of modular forms of integral or half-integral weight k for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ with respect to the multiplier ν_θ^{2k} if $k \notin \mathbb{Z}$ (where ν_θ is the theta multiplier), and $\mathbb{M}_k(\Gamma)$ the corresponding space of harmonic Maass forms (satisfying the moderate growth condition at all cusps). We will make use of the fact that the space of quasi-modular forms of weight k for Γ is given by $\bigoplus_{j\geq 0} E_2^j\, M_{k-2j}(\Gamma)$, where $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sum_{d\,|\,n} d\, e(n\tau)$ is the quasi-modular Eisenstein series of weight 2.

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ with $\det(\gamma) > 0$, the weight-k slash operator is defined by

$$\big(f\big|_k\gamma\big)(\tau) \;=\; (\det\gamma)^{\frac{k}{2}}\,(c\tau+d)^{-k}\, f\Big(\frac{a\tau+b}{c\tau+d}\Big).$$

Recall that if $f(\tau) = \sum_{m\in\mathbb{Q}_{\geq 0}} c(f,m)e(m\tau) \in \mathrm{M}_{2-k}(\Gamma(N))$ is a holomorphic modular form of level $N \in \mathbb{Z}_{\geq 1}$ with $k \neq 1$, then its non-holomorphic Eichler integral is given by

$$\begin{aligned} f^*(\tau) \;&:=\; -(2i)^{k-1}\int_{-\overline{\tau}}^{i\infty} \frac{\overline{f(-\overline{w})}}{(w+\tau)^k}\,dw \\ &= \frac{\overline{c(f,0)}}{1-k}\,y^{1-k} \;-\; (4\pi)^{k-1}\sum_{m\in\frac{1}{N}\mathbb{Z}_{<0}} \overline{c(f,|m|)}\,|m|^{k-1}\Gamma(1-k,4\pi|m|y)e(m\tau), \end{aligned} \tag{1.2}$$

where Γ represents the upper incomplete Gamma-function.

1.1. Generating series of Hurwitz class numbers

Zagier [15] investigated the generating series $\sum_D H(D)e(D\tau)$ of Hurwitz class numbers, and proved that it has a modular completion:

$$E_{\frac{3}{2}}(\tau) \;:=\; \sum_{D=0}^{\infty} H(D)e(D\tau) \;+\; \frac{1}{16\pi}\theta^*(\tau) \;\in\; \mathbb{M}_{\frac{3}{2}}(\Gamma_0(4)), \tag{1.3}$$

where

$$\theta \;:=\; \theta_{1,0} \;\in\; \mathrm{M}_{\frac{1}{2}}(\Gamma_0(4)) \;\text{ with}$$
$$\theta_{a,b}(\tau) \;:=\; \sum_{\substack{n\in\mathbb{Z}\\ n\equiv b \pmod a}} e\big(\tfrac{n^2\tau}{a}\big) \;\in\; \mathrm{M}_{\frac{1}{2}}(\Gamma(4a)), \quad a\in\mathbb{Z}_{\geq 1}, b\in\mathbb{Z}. \tag{1.4}$$

For $a \in \mathbb{Z}_{\geq 1}$ and $b \in \mathbb{Z}$, we recall the operators $\mathrm{U}_{a,b}$ from our earlier work [2], which act on Fourier series expansions of non-holomorphic modular forms by:

$$\mathrm{U}_{a,b} \sum_{n\in\mathbb{Z}} c(f;\, n;\, y)e(n\tau) \;:=\; \sum_{\substack{n\in\mathbb{Z}\\ n\equiv b \pmod a}} c\big(f;\, n;\, \tfrac{y}{a}\big)e\big(\tfrac{n\tau}{a}\big). \tag{1.5}$$

In particular, the holomorphic part of $\mathrm{U}_{a,b}\, E_{\frac{3}{2}}(\tau)$ is the generating series of Hurwitz class numbers $H(an+b)$ for $n \in \mathbb{Z}$, and one finds that (see also [4,7] for the holomorphic case)

$$\mathrm{U}_{a,b}\, E_{\frac{3}{2}} \;\in\; \mathbb{M}_{\frac{3}{2}}(\Gamma(4a)). \tag{1.6}$$

The action of the U-operators on theta series can be described by

$$\mathrm{U}_{a,b}\, \theta \;=\; \sum_{\beta^2\equiv b \pmod a} \theta_{a,\beta} \quad\text{and}\quad \mathrm{U}_{a,b}\, \theta^* \;=\; \sum_{\beta^2\equiv -b \pmod a} \sqrt{a}\, \theta^*_{a,\beta}. \tag{1.7}$$

Note that if $-b$ is not a square modulo a, then $\mathrm{U}_{a,b}\, E_{\frac{3}{2}}$ is a holomorphic modular form.

1.2. Holomorphic projection

Holomorphic projection plays an important role in our proofs of Theorems A and B. We briefly review the holomorphic projection operator from [6] in the scalar-valued case (see also [5,13]). Note that [5] treats the case of congruence subgroups $\Gamma_0(N)$, but the generalization to $\Gamma(N)$ that we require here follows from the vector-valued case in [6] when using the induction of the trivial representation of $\Gamma(N)$ to $\mathrm{SL}_2(\mathbb{Z})$. Let $k \in \mathbb{Z}$, $k \geq 2$, $N \in \mathbb{Z}_{\geq 1}$, and $f : \mathbb{H} \to \mathbb{C}$ an N-periodic continuous function with Fourier series expansion

$$f(\tau) \;=\; \sum_{n\in\frac{1}{N}\mathbb{Z}} c(f;\, n;\, y)e(\tau n)$$

satisfying the conditions: (i) For some $a > 0$ and all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, there are coefficients $\tilde{c}(f|_k\,\gamma;\, 0) \in \mathbb{C}$, such that $(f|_k\,\gamma)(\tau) = \tilde{c}(f|_k\,\gamma;\, 0) + \mathcal{O}(y^{-a})$ as $y \to \infty$; (ii) For all $n \in \frac{1}{N}\mathbb{Z}_{>0}$, we have $c(f;\, n;\, y) = \mathcal{O}(y^{2-k})$ as $y \to 0$. Then

$$\begin{aligned} \pi_k^{\mathrm{hol}}(f) \;&:=\; \tilde{c}(f;\,0) \,+\, \sum_{n \in \frac{1}{N}\mathbb{Z}_{>0}} c\big(\pi_k^{\mathrm{hol}}(f);\,n\big)\, e(n\tau) \quad \text{with} \\ c\big(\pi_k^{\mathrm{hol}}(f);\,n\big) \;&:=\; \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \lim_{s \to 0} \int_0^\infty c(f;\,n;\,y) \exp(-4\pi n y) y^{s+k-2}\, dy. \end{aligned} \tag{1.8}$$

Recall the following key properties of the holomorphic projection operator in (1.8): Proposition 4 of [6] states that if f is holomorphic, then $\pi_k^{\mathrm{hol}}(f) = f$. Theorem 5 of [6] asserts that if f transforms like a modular form of weight 2 for the group $\Gamma_1(N)$, then $\pi_2^{\mathrm{hol}}(f)$ is a quasi-modular form of weight 2 for $\Gamma_1(N)$.

1.3. A theorem of Serre

We conclude this Section with a result of Serre, which is required for our proof of Theorem A.

Theorem 1.1 *(Serre [11,12]). Fix positive integers k and N, and an odd prime number ℓ. Then there exist infinitely many primes $p \equiv 1 \pmod{\ell N}$ such that for all $f \in M_k(\Gamma_1(N))$ with ℓ-integral Fourier coefficients, we have*

$$c(f;np^r) \equiv (r+1)\, c(f;n) \pmod{\ell} \tag{1.9}$$

for all $n \in \mathbb{Z}$ coprime to ℓ and all non-negative integers r.

Finally, we recall the extension of Serre's result to quasi-modular forms.

Corollary 1.2 *(Corollary 1.2 of [2]). Fix positive integers k and N, and a prime number $\ell > 3$. Then there exist infinitely many primes $p \equiv 1 \pmod{\ell N}$ such that for all quasi-modular forms f of weight k for $\Gamma_1(N)$ with ℓ-integral Fourier coefficients, we have the congruence (1.9).*

2. Conditions on non-holomorphic congruences

We have already proved in [2] that non-holomorphic congruences modulo ℓ for Hurwitz class numbers on an arithmetic progression $a\mathbb{Z} + b$ include the divisibility $\ell \,|\, a$. For the purpose of this work, we need to extend this result.

We will prove Theorem A by contradiction. Proposition 2.2 provides us with explicit congruences for specific Fourier series coefficients, which we then use to derive a contradiction. The next lemma allows us to pass from a given arithmetic progression $\tilde{a}\mathbb{Z} + \tilde{b}$ to a more convenient one.

Lemma 2.1. *Let $\tilde{a} \in \mathbb{Z}_{\geq 1}$ and $\tilde{b} \in \mathbb{Z}$. Then there exist integers a, b, and β such that*

(i) We have $\tilde{a}\,|\,a$ *and* $b \equiv \tilde{b} \pmod{\tilde{a}}$.
(ii) We have $-b \equiv \beta^2 \pmod{a}$.
(iii) For every prime $p\,|\,a$*, writing* a_p *for the* p*-part of* a*, we have that* $\gcd(a_p, 2\beta)$ *is a proper divisor of* a_p*.*
(iv) There is a prime $p\,|\,a$ *such that* $0 < a < p^2$ *and* $0 \le 2\beta < p$*.*

Proof. First, we fix any integer $\beta \ge 0$ with $-\tilde{b} \equiv \beta \pmod{\tilde{a}}$. Then we choose an appropriate multiple of $\tilde{a}$:

$$a' := \prod_{p\,|\,\tilde{a}} p^{\max\{\mathrm{ord}_p(\tilde{a}),\, \mathrm{ord}_p(2\beta)+1\}},$$

where a_{sf} is the maximal square-free divisor of $\tilde{a}$. We let $p > \max\{a', 2\beta\}$, and set $a := a' \cdot p$. Then if b is any integer congruent to $-\beta^2$ modulo a, the four requirements in the lemma are met. □

Proposition 2.2. *Let* $a \in \mathbb{Z}_{\ge 1}$*,* $b \in \mathbb{Z}$*, and* $\ell > 3$ *be a prime. Assume that for all* $n \in \mathbb{Z}$ *we have* $H(an+b) \equiv 0 \pmod{\ell}$*. Assume further that* $b \not\equiv 0 \pmod{\ell}$ *and that* a*,* b*, and* β *satisfy Conditions (ii)–(iv) in Lemma 2.1. Set* $a' = \gcd(a, 2\beta)$*.*

Then $\pi_2^{\mathrm{hol}}\big((\mathrm{U}_{a,b}\, E_{\frac{3}{2}}) \cdot (\theta_{a,\beta} + \theta_{a,-\beta})\big)$ *is a quasi-modular form for* $\Gamma_1(4a)$ *and*

$$\pi_2^{\mathrm{hol}}\big((\mathrm{U}_{a,b}\, E_{\frac{3}{2}}) \cdot (\theta_{a,\beta} + \theta_{a,-\beta})\big) \;=\; \sum_{n=0}^{\infty} c(n) e(n\tau),$$

where

$$c(a'p) \;\equiv\; 0 \pmod{\ell}, \quad c(a'pp') \;\equiv\; -2\beta \pmod{\ell} \qquad \text{if } 2\beta \not\equiv a' \pmod{a}$$

for any primes p, p' *with*

$$a'p \equiv 2\beta \pmod{a}, \quad p > a/a', \qquad p' \equiv 1 \pmod{a}, \quad p' > a'p/a;$$

And

$$c(a') \;\equiv\; c(a'p') \;\equiv\; -2\beta \pmod{\ell} \qquad \text{if } 2\beta \equiv a' \pmod{a}$$

for any prime p' *with* $p' \equiv 1 \pmod{a}$*.*

Remark 2.3. Our Proposition 2.2 is the analogue of Proposition 2.5 of [2]. The assumption $\ell \nmid a$ was accidentally omitted from Proposition 2.5 of [2]. The analogue to it in our current Proposition 2.2 is the condition $\beta \not\equiv 0 \pmod{\ell}$.

There was another issue in the proof of Proposition 2.5 of [2]. Namely, when arguing that we may assume that d_1 and d_2 are positive (as we will do in the present proof), this is only legitimate when considering the sum over $\pm\beta$ and all $\tilde{\beta}$, but not on the level of individual terms.

Finally, on a related note, we remark that the condition that $b \equiv \tilde{b} \pmod{\tilde{a}}$ was incorrectly omitted from Lemma 2.3 of [2].

Proof of Proposition 2.2. As in Proposition 2.5 of [2], we compute the holomorphic projection of $(\mathrm{U}_{a,b}\, E_{\frac{3}{2}}) \cdot (\theta_{a,\beta} + \theta_{a,-\beta})$. We first remark that the conditions required for holomorphic projection as stated in Section 1.2 hold for $(\mathrm{U}_{a,b}\, E_{\frac{3}{2}}) \cdot (\theta_{a,\beta} + \theta_{a,-\beta})$ and we can apply Theorem 5 of [6]. Condition (i) is true because $E_{\frac{3}{2}}$ has moderate growth at cusps and $\theta_{a,\pm\beta}$ is a cusp form, since $\beta \not\equiv 0 \pmod{a}$ by assumption. Condition (ii) is met because $\lim_{y\to 0} \Gamma(4\pi|n|y, \frac{1}{2}) = \Gamma(\frac{1}{2})$, which implies that the coefficients are $\mathcal{O}(1) = \mathcal{O}(y^{2-2})$ as $y \to 0$. Finally, $\mathrm{U}_{a,b}\, E_{\frac{3}{2}} \cdot (\theta_{a,\beta} + \theta_{a,-\beta})$ is modular of weight 2 on $\Gamma_1(4a)$ (to see this, both factors are modular on $\Gamma(4a)$ and the Fourier expansion of their product is supported on integer exponents; it follows that the product is modular with respect to $\Gamma_1(4a)$). It then follows from Theorem 5 of [6] that $\pi_2^{\mathrm{hol}}\big((\mathrm{U}_{a,b}\, E_{\frac{3}{2}})\cdot(\theta_{a,\beta}+\theta_{a,-\beta})\big)$ is a weight 2 quasi-modular form for $\Gamma_1(4a)$.

While we have remarked that the assumption $\ell \nmid a$ was incorrectly omitted from Proposition 2.5 of [2], the first part of its proof never makes use of it. The following computation is verbatim the one in [2], and we offer additional explanations on subtleties that were not mentioned in our previous work. As in that paper, we note that

$$\pi_2^{\mathrm{hol}}\big((\mathrm{U}_{a,b}\, E_{\frac{3}{2}}) \cdot (\theta_{a,\beta} + \theta_{a,-\beta})\big) \;\equiv\; \frac{1}{16\pi}\, \pi_2^{\mathrm{hol}}\big((\mathrm{U}_{a,b}\, \theta^{*}) \cdot (\theta_{a,\beta} + \theta_{a,-\beta})\big) \pmod{\ell}. \tag{2.1}$$

To justify this claim, we first note that the difference between the left and right hand side of (2.1) is the holomorphic projection of the product of the holomorphic generating series of $H(an+b)$ for $n \in \mathbb{Z}$ with a sum of theta series, and thus by assumption has Fourier coefficients divisible by ℓ. Second, the Fourier coefficients of the right hand side of (2.1) are ℓ-integral due to (1.8) and the following Equality (2.2), in which each summand $an/(|m|+|\tilde{m}|)$ on the right hand side of (2.1) is integral. We compute the sum over $\pm\beta$ and $\tilde{\beta}^2 \equiv -b \pmod{a}$ of

$$c\Big(\pi_2^{\mathrm{hol}}\big(\sqrt{a}\,\theta^{*}_{a,\tilde{\beta}}(\tau)\cdot\theta_{a,\beta}(\tau)\big);\, n\Big) \;=\; -4\pi\, an \sum_{\substack{m\equiv\beta \pmod{a}\\ \tilde{m}\equiv\tilde{\beta} \pmod{a}\\ \tilde{m}\neq 0\\ an=m^2-\tilde{m}^2}} \frac{1}{|m|+|\tilde{m}|}. \tag{2.2}$$

The term with $\delta_{\tilde{\beta}\equiv 0}$ in Equation [16] of [2] does not appear here for the following reason: First, the assumption $H(an+b) \equiv 0 \pmod{\ell}$ for all integers n implies $\ell \mid a$ by the main theorem of [2]. Second, we have $\tilde{\beta} \not\equiv 0 \pmod{\ell}$, since $-\beta^2 \equiv b \not\equiv 0 \pmod{\ell}$.

We can still proceed as in [2], factoring $an = d_1 d_2$ to arrive at the analogue of Equation [17] of [2]. We treat only the positive case, $d_1, d_2 > 0$; the negative case yields the same sum, after applying the summation over $\pm\beta$ and all $\tilde{\beta}$. We account for this suppressing the sum over $\pm\beta$ and multiplying with 2. As in [2], we assume that an is not a square, and obtain after taking the factors $1/16\pi$ and -4π from our previous expressions into account that

$$c(n) \;=\; -\frac{1}{2} \sum_{\tilde{\beta}^2 = -b \pmod a} \;\; \sum_{\substack{an = d_1 d_2 \\ d_1, d_2 > 0 \\ d_1 \equiv \beta + \tilde{\beta} \pmod a \\ d_2 \equiv \beta - \tilde{\beta} \pmod a}} \big(d_1 \delta_{d_1 < d_2} + d_2 \delta_{d_2 < d_1} \big). \tag{2.3}$$

Only now we diverge from [2], where we separated the archimedean and nonarchimedean conditions in this sum. This is no longer possible in the present setting, but we can still separate all nonarchimedian conditions away from ℓ from the archimedean ones. We repeatedly use the fact that $\ell \,|\, a$, which is the statement of the main theorem of [2].

In the following discussion, q will always denote a prime. To ease the discussion, we introduce additional notation: Recall from Lemma 2.1 that, given $q \,|\, a$, we denote by a_q the maximal q-power that divides a, and that $\beta \not\equiv -\beta \pmod{a_q}$ by Condition (iii) of that lemma. We write Q_a for the set of all prime divisors of a. Given a subset $Q \subseteq Q_a$, we let a_Q be the product of all a_q for $q \in Q$, and set $a_Q^{\#} := a/a_Q$. Likewise, we define a'_q to be the maximal q power dividing a', and we define $a'_Q := \prod_{q \in Q} a'_q$ and $a'^{\#}_Q := a'/a'_Q$.

If $\tilde{\beta}^2 \equiv \beta^2 \pmod a$, then for each $q \in Q_a$, $\tilde{\beta} \equiv \pm\beta \pmod{a_q}$. Moreover, by the Chinese Remainder Theorem, we can associate to each subset $Q \subseteq Q_a$ a residue class $\tilde{\beta}_Q \pmod a$ such that $\tilde{\beta}_Q \equiv \beta \pmod{a_q}$ for each $q \in Q$ and $\tilde{\beta}_Q \equiv -\beta \pmod{a_q}$ for each $q \in Q_a \backslash Q$. Using this correspondence between subsets of Q_a and residue classes $\tilde{\beta} \pmod a$ such that $\tilde{\beta}^2 \equiv -b \pmod a$, we rewrite our expression for $c(n)$:

$$c(n) \;=\; -\frac{1}{2} \sum_{Q \subseteq Q_a} \;\; \sum_{\substack{an = d_1 d_2 \\ d_1, d_2 > 0 \\ d_1 \equiv \beta + \tilde{\beta}_Q \pmod a \\ d_2 \equiv \beta - \tilde{\beta}_Q \pmod a}} \big(d_1 \delta_{d_1 < d_2} + d_2 \delta_{d_2 < d_1} \big). \tag{2.4}$$

We examine the inner sum more closely. For each $q \in Q$, the congruence condition on d_2 tells us that $d_2 \equiv \beta - \tilde{\beta}_Q \equiv 0 \pmod{a_q}$. Hence $a_q \,|\, d_2$. On the other hand, for each $q \in Q_a \backslash Q$, the congruence condition on d_1 implies that $d_1 \equiv 0 \pmod{a_q}$. Hence $a_Q^{\#} \,|\, d_1$.

Recall that $a' = \gcd(a, 2\beta)$ and hence a' must be a divisor of both d_1 and d_2. This forces the divisibility requirements $a_Q^{\#} a'_Q \,|\, d_1$ and $a_Q a'^{\#}_Q \,|\, d_2$.

We first consider the case $2\beta \not\equiv a' \pmod a$. We restrict n as in the statement of the proposition, by fixing a prime $p > a/a'$ with $a'p \equiv 2\beta \pmod a$ and a further prime $p' > a'p/a$ with $p' \equiv 1 \pmod a$. Our aim is to calculate $c(a'p)$ and $c(a'pp')$. The observations in the previous paragraph imply that any d_1, d_2 appearing in the sum in (2.4) are of the form $d_1 = a_Q^{\#} a'_Q k_1$ and $d_2 = a_Q a'^{\#}_Q k_2$, with $k_1 k_2 = p$ if $n = a'p$ and $k_1 k_2 = pp'$ if $n = a'pp'$.

We must determine which subsets $Q \subseteq Q_a$ contribute to the sum in (2.4). If we have $n = a'p$, note that $Q = \emptyset$ indeed yields a factorization $d_1 = a$, $d_2 = a'p$ that satisfies the given congruence conditions by the assumptions on p as $2\beta \not\equiv a' \pmod{a}$. Since $a'p > a$, its contribution $-a/2$ to $c(n) \pmod{\ell}$ vanishes by the main theorem of [2]. Similarly $Q = Q_a$ yields the same contribution coming from the factorization $d_1 = a'p$, $d_2 = a$.

For $n = a'pp'$, we have two factorizations for $Q = \emptyset$ and $Q = Q_a$ each that appear. The factorizations $d_1 = ap'$, $d_2 = a'p$ and $d_1 = a$, $d_2 = a'pp'$ associated with $Q = \emptyset$ contribute $-a'p/2$ and $-a/2$ to the sum, since $p' > a'p/a$ and $a'pp' > ap' > a$. For $Q = Q_a$, the factorizations $d_1 = a'p$, $d_2 = ap'$ and $d_1 = a'pp'$, $d_2 = a$ give the same contribution.

We claim that no other Q contributes to the sum for $n = a'p$ or $n = a'pp'$. To show this, we employ the prime $q_a \,|\, a$ with $0 < a < q_a^2$ and $0 \leq 2\beta < q_a$ whose existence is asserted by Condition (iv) of Lemma 2.1.

In the case that $q_a \notin Q$ we have $\tilde{\beta}_Q \equiv -\beta \pmod{q_a}$. We examine the condition

$$d_2 \equiv \beta - \tilde{\beta} \equiv 2\beta \pmod{q_a}.$$

Since $p' \equiv 1 \pmod{q_a}$, we know that $d_2 \equiv a_Q a_Q'^{\#} p \pmod{q_a}$ or $d_2 \equiv a_Q a_Q'^{\#} \pmod{q_a}$. In the first case, since $a'_Q a_Q'^{\#} p = a'p \equiv 2\beta \pmod{q_a}$ by our assumptions on p and since $2\beta < q_a$ is a unit modulo q_a, this implies the congruence $a_Q \equiv a'_Q \pmod{q_a}$. Since further $a_Q/a'_Q \leq a_Q \leq a/q_a < q_a$ and $a'_Q \,|\, a_Q$, we find that $a_Q = a'_Q$, and hence $Q = \emptyset$ by Condition (iii) of Lemma 2.1.

Similarly, in the second case $a_Q \equiv a'_Q p \pmod{q_a}$, that is, $a_Q/a'_Q \equiv p \pmod{q_a}$. We have $a'p \equiv 2\beta \pmod{a}$, and since $q_a \nmid a'$, we find $p \equiv 2\beta/a' \pmod{q_a}$. This yields the congruence $a_Q/a'_Q \equiv 2\beta/a' \pmod{q_a}$. Since $a_Q/a'_Q < q_a$ as in the first case and further $2\beta/a' \leq 2\beta < q_a$, we can strengthen it to the equality $a_Q/a'_Q = 2\beta/a'$. We conclude that a_Q/a'_Q is co-prime to a, which by Condition (iii) of Lemma 2.1 implies that $Q = \emptyset$ (hence $2\beta = a'$, which cannot occur in the present case $2\beta \not\equiv a' \pmod{a}$).

Assuming on the other hand that $q_a \in Q$, i.e., $Q \neq \emptyset$. We inspect the condition

$$d_1 \equiv \tilde{\beta} + \beta \equiv 2\beta \pmod{q_a}.$$

Suppose that $d_1 = a_Q^{\#} a'_Q$ or $d_1 = a_Q^{\#} a'_Q p' \equiv a_Q^{\#} a'_Q \pmod{q_a}$. We infer $a_Q^{\#} a'_Q = 2\beta$, since $0 < a_Q^{\#} a'_Q \leq a/q_a < q_a$ and $0 \leq 2\beta < q_a$. From here we have $d_1 = a_Q^{\#} a'_Q = 2\beta$ or $d_1 = 2\beta p'$. Since $\beta \not\equiv -\beta \pmod{a_q}$, this means $Q = Q_a$.

Similarly, if $d_1 = a_Q^{\#} a'_Q p$ or $d_1 = a_Q^{\#} a'_Q pp'$, we obtain $a_Q^{\#} a'_Q \equiv 2\beta/p \equiv a' \pmod{q_a}$. The same inequalities as above imply $a_Q^{\#} a'_Q = a'$. By definition we have $a' = a'_Q a_Q'^{\#}$ so we have $a_Q^{\#} = a_Q'^{\#}$. Since a'_q is a strict divisor of a_q for every $q \in Q_a$, this implies that $Q = Q_a$.

Summarizing our discussion, for $2\beta \not\equiv a' \pmod{a}$, only $Q = \emptyset$ and $Q = Q_a$ contribute to $c(a'p)$ and $c(a'pp')$, and we have

$$c(a'p) \;=\; -a \;\equiv\; 0 \pmod{\ell},$$

$$c(a'pp') \;=\; -(a'p+a) \;\equiv\; -2\beta \pmod{\ell}.$$

The case $2\beta \equiv a' \pmod{a}$ is a bit simpler, since we do not need p as in the above discussion. We can adopt the previous argument to see that the only contributions to the sum over $Q \subset Q_a$ arise from $Q = \emptyset$ and $Q = Q_a$. For $n = a'$ there is one factorization each associated with $Q = \emptyset$ and $Q = Q_a$. Specifically, the factorizations $d_1 = a$, $d_2 = a'$ and $d_1 = a'$, $d_2 = a$ contribute $-a'/2$ each. For $n = a'p'$, we have two factorizations each associated with $Q = \emptyset$ and $Q = Q_a$. The contributions of $d_1 = ap'$, $d_2 = a'$ (associated with $Q = \emptyset$) and $d_1 = a'$, $d_2 = ap'$ (associated with $Q = Q_a$) equal $-a'/2$, and those of $d_1 = a$, $d_2 = a'p'$ and $d_1 = a'p'$, $d_2 = a$ and equal $-a/2$. In summary, we find that

$$c(a'p') \;=\; -(a'+a) \;\equiv\; -a' \pmod{\ell},$$

where in the last congruence we again have invoked the fact that $\ell \mid a$ from [2]. □

Proof of Theorem A. We establish the theorem by contradiction. Assume that $\ell \nmid b$. We have $\ell \mid a$ by the main theorem of [2]. Lemma 2.1 allows us to replace a and b in such a way that we can apply Proposition 2.2 with an integer β also provided by Lemma 2.1.

We can now proceed as in [2] and apply Corollary 1.2 to deduce a contradiction. Let $a' = (2\beta, a)$. By assumption, $\ell \nmid a'$. First suppose $a' \not\equiv 2\beta \pmod{\ell}$. We choose a prime $p > a/a'$ such that $a'p \equiv 2\beta \pmod{a}$. Then we choose a prime $p' > a'p/a$ with $p' \equiv 1 \pmod{4\ell a}$. The first part of Proposition 2.2 says that we must have

$$c(a'p) \;\equiv\; 0 \pmod{\ell} \quad \text{and} \quad c(a'pp') \;\equiv\; -a' \not\equiv 0 \pmod{\ell},$$

but Corollary 1.2 leads to

$$c(a'pp') \;\equiv\; 2c(a'p') \pmod{\ell},$$

and hence the contradiction $c(a'pp') \equiv 0 \pmod{\ell}$.

Now suppose $a' \equiv 2\beta \pmod{a}$. Let $p' \equiv 1 \pmod{4\ell a}$ be a prime. Then

$$c(a') \;\equiv\; c(a'p') \;\equiv\; -a' \;\not\equiv\; 0 \pmod{\ell}$$

by the second part of Proposition 2.2, but by Corollary 1.2 we have

$$2c(a') \;\equiv\; c(a'p) \pmod{\ell}.$$

Hence $c(a') \equiv 0 \pmod{\ell}$, a contradiction. □

3. Congruences on square-classes

The proof of Theorem B is split into two parts. Both require the following lemma.

Lemma 3.1. *Let $a \in \mathbb{Z}_{\geq 1}$ and $b, u \in \mathbb{Z}$ with $\gcd(u, a) = 1$. Let*

$$\gamma = \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{smallmatrix}\right) \in \Gamma_0(4a) \quad \textit{satisfy} \quad \gamma \equiv \left(\begin{smallmatrix} u & 0 \\ 0 & \overline{u} \end{smallmatrix}\right) \pmod{4a},$$

where $\overline{u} \pmod{4a}$ is a multiplicative inverse of u modulo $4a$. Then there exists $\omega(\gamma) \in \{\pm 1, \pm i\}$ such that

$$\mathrm{U}_{a,b}\, E_{\frac{3}{2}}\big|_{\frac{3}{2}}\, \gamma \;=\; \omega(\gamma)\, \mathrm{U}_{a,bu^2}\, E_{\frac{3}{2}} \quad \textit{and} \quad \mathrm{U}_{a,b}\, \theta\big|_{\frac{1}{2}}\, \gamma \;=\; \overline{\omega(\gamma)}\, \mathrm{U}_{a,bu^2}\, \theta.$$

Proof. We give the argument only in the case of the Eisenstein series. The case of the theta series follows from almost literally the same calculation, where ϵ_D from (1.1) that appears later needs to be replaced by ϵ_D^{-1}, since the weight is $\frac{1}{2}$ as opposed to $\frac{3}{2}$. We can write $\mathrm{U}_{a,b}$ as a double coset operator:

$$\mathrm{U}_{a,b}\, E_{\frac{3}{2}} \;=\; a^{\frac{3}{4}-1} \sum_{\lambda \pmod{a}} e\Big(\frac{-\lambda b}{a}\Big)\, E_{\frac{3}{2}}\big|_{\frac{3}{2}} \left(\begin{smallmatrix} 1 & \lambda \\ 0 & a \end{smallmatrix}\right).$$

For γ as in the statement of the lemma, we have the following factorization, where the two matrices on the right both have integer entries:

$$\begin{pmatrix} 1 & \lambda \\ 0 & a \end{pmatrix} \gamma \;=\; \begin{pmatrix} a_\gamma + c_\gamma \lambda & \frac{1}{a}(-\lambda \overline{h}^2 a_\gamma + b_\gamma) + \frac{\lambda}{a}(-c_\gamma \lambda \overline{u}^2 + d_\gamma) \\ a c_\gamma & d_\gamma - \overline{u}^2 \lambda c_\gamma \end{pmatrix} \begin{pmatrix} 1 & \lambda \overline{u}^2 \\ 0 & a \end{pmatrix}.$$

Combining this with the previous equation, we find that

$$\begin{aligned}
\big(\mathrm{U}_{a,b}\, E_{\frac{3}{2}}\big)\big|_{\frac{3}{2}}\, \gamma \;&=\; a^{\frac{3}{4}-1} \sum_{\lambda \pmod{a}} e\Big(\frac{-\lambda b}{a}\Big)\, E_{\frac{3}{2}}\big|_{\frac{3}{2}} \left(\begin{smallmatrix} 1 & \lambda \\ 0 & a \end{smallmatrix}\right)\gamma \\
&=\; a^{\frac{3}{4}-1} \sum_{\lambda \pmod{a}} e\Big(\frac{-(\lambda \overline{u}^2)(bu^2)}{a}\Big)\, \epsilon_{d_\gamma - \lambda \overline{u}^2 c_\gamma} \left(\frac{a c_\gamma}{d_\gamma - \lambda \overline{u}^2 c_\gamma}\right) E_{\frac{3}{2}}\big|_{\frac{3}{2}} \begin{pmatrix} 1 & \lambda \overline{u}^2 \\ 0 & a \end{pmatrix}.
\end{aligned}$$

Since c_γ is divisible by $4a$, we have $d_\gamma - \lambda \overline{u}^2 c_\gamma \equiv d_\gamma \pmod{4}$, hence

$$\epsilon_{d_\gamma - \lambda \overline{u}^2 c_\gamma} \;=\; \epsilon_{d_\gamma}.$$

Let $c'_\gamma := c_\gamma / 4a$. Using quadratic reciprocity and the fact that $4c'_\gamma \mid \lambda \overline{u}^2 c_\gamma$, we obtain

$$\left(\frac{a c_\gamma}{d_\gamma - \lambda \overline{u}^2 c_\gamma}\right) = \left(\frac{c'_\gamma}{d_\gamma - \lambda \overline{u}^2 c_\gamma}\right) = \left(\frac{c'_\gamma}{d_\gamma}\right) = \left(\frac{a c_\gamma}{d_\gamma}\right).$$

Let

$$\omega_\gamma \;:=\; \epsilon_{d_\gamma} \left(\frac{a c_\gamma}{d_\gamma}\right) \;\in\; \{\pm 1, \pm i\}.$$

Then we have

$$\begin{aligned}\big(U_{a,b}\, E_{\frac{3}{2}}\big)\big|_{\frac{3}{2}}\gamma \;&=\; a^{\frac{3}{4}-1}\omega(\gamma) \sum_{\lambda \pmod{a}} e\left(\frac{-(\lambda\overline{u}^2)(bu^2)}{a}\right)\, E_{\frac{3}{2}}\big|_{\frac{3}{2}}\begin{pmatrix} 1 & \lambda\overline{u}^2 \\ 0 & a\end{pmatrix} \\ &= \omega(\gamma)\, U_{a,bu^2}\, E_{\frac{3}{2}}. \quad \square\end{aligned}$$

Part 1 of the proof of Theorem B. We prove the theorem in the case that $-b$ is not a square modulo a, which implies that

$$0 \;\equiv\; \sum_{n\in\mathbb{Z}} H(an+b)e\big((an+b)\tau\big) \;=\; \mathrm{U}_{a,b}\, E_{\frac{3}{2}} \in \mathrm{M}_{\frac{3}{2}}\big(\Gamma(4a)\big).$$

Fix u as in the statement. By replacing u by $u+a$, if needed, we can and will assume that $\gcd(u,2a)=1$. Let $\gamma\in\Gamma_0(4a\ell)$ satisfy $\gamma \equiv \left(\begin{smallmatrix} u & 0 \\ 0 & \overline{u}\end{smallmatrix}\right) \pmod{4a}$ where $\overline{u} \pmod{4a}$ is a multiplicative inverse of u modulo $4a$. Since $\gamma\in\Gamma_0(\ell)$, we can combine the q-expansion principle (see Lemma 2.3, [1]) with Lemma 3.1 to find that

$$0 \;\equiv\; \big(U_{a,b}\, E_{\frac{3}{2}}\big)\big|_{\frac{3}{2}}\gamma = \omega(\gamma)\, U_{a,bu^2}\, E_{\frac{3}{2}} \pmod{\ell}, \tag{3.1}$$

where the congruence is to be understood in the ring of Gaussian integers if $\omega(\gamma)$ does not lie in $\mathbb{Q}$. We obtain the statement from the Fourier expansion of the right hand side of (3.1). $\square$

The second part of our proof of Theorem B requires two further lemmas.

Lemma 3.2. *Fix a prime $\ell > 3$. Let $a > 0$ and β, β' be integers that are all divisible by ℓ. Then we have*

$$\frac{\sqrt{a}}{\pi}\pi_2^{\mathrm{hol}}\big(\theta^*_{a,\tilde{\beta}}\cdot\theta_{a,\beta}\big) \;\equiv\; 0 \pmod{\ell}.$$

In particular, for integers b, b' that are divisible by ℓ, we have

$$\frac{1}{\pi}\pi_2^{\mathrm{hol}}\big(\mathrm{U}_{a,b'}\,\theta^*\cdot\mathrm{U}_{a,b}\,\theta\big) \;\equiv\; 0 \pmod{\ell}$$

and

$$\pi_2^{\mathrm{hol}}\big(\mathrm{U}_{a,b}\, E_{\frac{3}{2}}\cdot\mathrm{U}_{a,b'}\,\theta\big) \equiv \mathrm{U}_{a,b}\, E^{\mathrm{hol}}_{\frac{3}{2}}\cdot\mathrm{U}_{a,b'}\,\theta \pmod{\ell}.$$

Proof. We first note that the Fourier coefficients in the three congruences in the statement of the lemma are ℓ-integral by the same argument as the one following (2.1).

The second part follows from the first part, in light of (1.7). The third part follows from the second one and (1.3):

$$\pi_2^{\mathrm{hol}}\big(\mathrm{U}_{a,b}\,E_{\frac{3}{2}}\cdot \mathrm{U}_{a,b'}\,\theta\big) = \pi_2^{\mathrm{hol}}\big(\mathrm{U}_{a,b}\,\big(E_{\frac{3}{2}}^{\mathrm{hol}} + \frac{1}{16\pi}\theta^*\big)\cdot \mathrm{U}_{a,b'}\,\theta\big)$$
$$= \pi_2^{\mathrm{hol}}\big(\mathrm{U}_{a,b}\,E_{\frac{3}{2}}^{\mathrm{hol}}\cdot \mathrm{U}_{a,b'}\,\theta\big) + \frac{1}{16\pi}\pi_2^{\mathrm{hol}}\big(\mathrm{U}_{a,b}\,\theta^*\cdot \mathrm{U}_{a,b'}\,\theta\big) \equiv \mathrm{U}_{a,b}\,E_{\frac{3}{2}}^{\mathrm{hol}}\cdot \mathrm{U}_{a,b'}\,\theta \pmod{\ell}.$$

We employ the calculations from the proof of Proposition 2.5 in [2], in the same way as we used them in the proof of Proposition 2.2, to establish the first congruence. We have (compare Equation [16] of [2])

$$\frac{\sqrt{a}}{\pi}\pi_2^{\mathrm{hol}}\big(\theta^*_{a,\tilde\beta}(\tau)\cdot\theta_{a,\beta}(\tau)\big)$$
$$= -4\Big(\delta_{\tilde\beta\equiv 0 \pmod a}\sum_{\substack{m\equiv\beta \pmod a\\ m\neq 0}}|m|e\big(\frac{m^2\tau}{a}\big) + \sum_{\substack{m\equiv\beta \pmod a\\ \tilde m\equiv\tilde\beta \pmod a\\ \tilde m\neq 0}}\frac{m^2-\tilde m^2}{|m|+|\tilde m|}e\Big(\frac{(m^2-\tilde m^2)\tau}{a}\Big)\Big).$$

Since $\ell \,|\, a, \beta$, the Fourier coefficients in the first summand are all divisible by ℓ. Consider a term in the second sum for fixed m and $\tilde m$. The ratio in this term is divisible by either $m+\tilde m$ or $m-\tilde m$. Since $\ell \,|\, m, \tilde m$, this proves the lemma. $\square$

Lemma 3.3. *Let N be a positive integer and f be a quasi-modular form of weight 2 for a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion*

$$f(\tau) = \sum_{\substack{n=0\\ N\nmid n}}^{\infty} c\big(f;\tfrac{n}{N}\big)e\big(\tfrac{n}{N}\tau\big).$$

Then f is a modular form.

Proof. We decompose f as a sum $cE_2 + g$ for a constant $c \in \mathbb{C}$ and a modular form g for Γ, where E_2 is again the quasi-modular Eisenstein series of weight 2. From the Fourier expansion of f, we infer that

$$0 = \frac{1}{N}\sum_{m=1}^{N} f\big|_2\big(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\big) = cE_2 + \frac{1}{N}\sum_{m=1}^{N} g\big|_2\big(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\big).$$

Since the second summand is a modular form for the finite index subgroup

$$\bigcap_{m=1}^{N}\big(\begin{smallmatrix}1 & -m\\ 0 & 1\end{smallmatrix}\big)\Gamma\big(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\big) \subseteq \mathrm{SL}_2(\mathbb{Z}),$$

we conclude that $c = 0$ as desired. $\square$

Part 2 of the proof of Theorem B. We now prove the theorem in the case that $-b$ is a square modulo a. We start with the congruences

$$\mathrm{U}_{a,b}\, E^{\mathrm{hol}}_{\frac{3}{2}} \cdot \mathrm{U}_{a,\tilde{b}}\, \theta \;\equiv\; 0 \pmod{\ell},$$

which hold for all integers $\tilde{b}$, since the first factor vanishes modulo ℓ by our assumptions. The main theorem of [2] informs us that $\ell \,|\, a$, and Theorem A asserts that $\ell \,|\, b$. If $a \,|\, b$, there is nothing to show. We assume the opposite. Then there is some integer b' with $\ell \,|\, b'$ and $b + b' \not\equiv 0 \pmod{a}$ and b' is a square modulo a, for instance, $b' = 0$. Lemma 3.2 now yields the following congruence of quasi-modular forms:

$$\pi^{\mathrm{hol}}_{2}\big(\mathrm{U}_{a,b}\, E_{\frac{3}{2}} \cdot \mathrm{U}_{a,b'}\, \theta\big) \;\equiv\; 0 \pmod{\ell}. \tag{3.2}$$

Let $\gamma \in \Gamma_0(4a\ell)$ be as in the first part of the proof. By Lemma 3.3, the left hand side is modular, so we may apply Lemma 2.3 from [1] as in the first part of the proof to deduce

$$\pi^{\mathrm{hol}}_{2}\big(\mathrm{U}_{a,b}\, E_{\frac{3}{2}} \cdot \mathrm{U}_{a,b'}\, \theta\big)\big|_{2}\, \gamma \;\equiv\; 0 \pmod{\ell}.$$

To determine the left hand side of this congruence, recall that the slash action intertwines with the holomorphic projection (see [6]). We have

$$\pi^{\mathrm{hol}}_{2}\Big(\big(\mathrm{U}_{a,b}\, E_{\frac{3}{2}}\big|_{\frac{3}{2}}\, \gamma\big) \cdot \big(\mathrm{U}_{a,b'}\, \theta\big|_{\frac{1}{2}}\, \gamma\big)\Big) \;\equiv\; 0 \pmod{\ell}.$$

Lemma 3.1 yields the congruence

$$\pi^{\mathrm{hol}}_{2}\big(\mathrm{U}_{a,bu^2}\, E_{\frac{3}{2}} \cdot \mathrm{U}_{a,b'u^2}\, \theta\big) \;\equiv\; 0 \pmod{\ell}.$$

Observe that bu^2 and $b'u^2$ are divisible by ℓ, so that we can apply Lemma 3.2 to find that

$$\mathrm{U}_{a,bu^2}\, E^{\mathrm{hol}}_{\frac{3}{2}} \cdot \mathrm{U}_{a,b'u^2}\, \theta \;\equiv\; 0 \pmod{\ell}. \tag{3.3}$$

Since b' is a square modulo a by assumption, the second factor on the left hand side of (3.3) is not congruent to zero modulo ℓ, and more specifically its first non-zero Fourier coefficient equals one or two. From this we infer that $\mathrm{U}_{a,bu^2}\, E^{\mathrm{hol}}_{\frac{3}{2}} \equiv 0 \pmod{\ell}$. $\square$

4. Proofs of Theorems C and D

Proof of Theorem C. For simplicity, we say that (a, b) is a mod ℓ Hurwitz congruence pair if we have the Ramanujan-type congruence $H(an + b) \equiv 0 \pmod{\ell}$ for all $n \in \mathbb{Z}$. Furthermore, we say that a mod ℓ Hurwitz congruence pair (a, b) is maximal if the corresponding Ramanujan-type congruence is maximal.

Let p be a prime, let $k = \mathrm{ord}_p(\gcd(a, b))$ and $r = \mathrm{ord}_p(a/\gcd(a, b))$. After replacing b by $b + a$, if needed, we then have $a = p^{k+r}a'$ and $b = p^k b'$ for integers a' and b' with $\gcd(a'b', p) = 1$.

First, we assume that p is odd, that $r \geq 2$, and that (a, b) is a Hurwitz congruence pair modulo ℓ. We will show that (a, b) is not a maximal Hurwitz congruence pair modulo ℓ. If $m \equiv b \pmod{a/p}$, then $p^k \| m$ and $m/p^k \equiv b' \pmod{p^{r-1}}$. From Hensel's Lemma, there exists $u \in \mathbb{Z}$ with $\gcd(u, p) = 1$ such that $m/p^k \equiv b'u^2 \pmod{p^r}$. Using the Chinese Remainder Theorem, one can find such a u with $u \equiv 1 \pmod{a'}$ so that we have $m \equiv bu^2 \pmod{a}$. By Theorem B, we have $H(m) \equiv 0 \pmod{\ell}$. Hence $(a/p, b)$ is a Hurwitz congruence pair modulo ℓ, so (a, b) is not a maximal Hurwitz congruence pair.

The $p = 2$ case is almost identical. We assume $r \geq 4$ and we will show that (a, b) cannot be a maximal Hurwitz congruence pair modulo ℓ. Suppose $m \equiv b \pmod{a/2}$. Then $2^k | m$, and using a Hensel's lemma type argument, one easily checks that there exists an integer u which is relatively prime to a such that $m \equiv bu^2 \pmod{a}$ (this is where we require $r \geq 4$ rather than $r \geq 2$). By Theorem B, we have $H(m) \equiv 0 \pmod{\ell}$, which means (a, b) is not a maximal Hurwitz congruence pair. $\square$

Proof of Theorem D. Assume that (i) of Theorem D does not hold. That is, there is a fundamental discriminant $-D$ and a positive integer f such that $Df^2 \in a\mathbb{Z} + b$ and $H(D) \not\equiv 0 \pmod{\ell}$. Given a prime $p \,|\, f$ we write f_p for its p-part. We will show that there is a prime $p \,|\, \gcd(f, a)$ such that

$$\sigma_1(f_p) - \sigma_1\left(\frac{f_p}{p}\right)\left(\frac{-D}{p}\right) \equiv 0 \pmod{\ell}. \tag{4.1}$$

We factor $f = f_a u$ into the product of two positive integers f_a and u, where every prime dividing f_a also divides a and u is co-prime to a. In particular, there is an inverse $\overline{u}$ of u modulo a.

Theorem B asserts that we have a Ramanujan-type congruence modulo ℓ for Hurwitz class numbers on $a\mathbb{Z} + \overline{u}^2 b \ni Df_a^2$. The Hurwitz class number formula asserts that

$$\begin{aligned} H(Df_a^2) &= H(D)\,\frac{\omega(-Df_a^2)}{\omega(-D)} \sum_{d \,|\, f_a} d \prod_{p \,|\, d} \left(1 - \frac{1}{p}\left(\frac{-D}{p}\right)\right) \\ &= H(D)\,\frac{\omega(-Df_a^2)}{\omega(-D)} \prod_{p \,|\, f_a} \left(\sigma_1(f_p) - \sigma_1\left(\frac{f_p}{p}\right)\left(\frac{-D}{p}\right)\right), \end{aligned}$$

where $\omega(-Df_a^2)$ is the number of units in the imaginary quadratic order of discriminant $-Df_a^2$. By assumption on D, we have $H(D) \not\equiv 0 \pmod{\ell}$. Further, we note that $\omega(-Df_a^2) \not\equiv 0 \pmod{\ell}$, since $\ell > 3$ and $\omega(-Df_a^2) \,|\, 6$. Therefore, we conclude the existence of some prime $p \,|\, \gcd(f, a)$ satisfying (4.1) as desired.

To finish the proof, let $-D'$ be another fundamental discriminant such that there is an integer f' with $D'f'^2 \in a\mathbb{Z} + b$. We will show that we have $f_p = f'_p$ and $(\frac{-D'}{p}) = (\frac{-D}{p})$. Case (ii) follows immediately from these two claims and (4.1).

First, assume we know $f_p = f'_p$, and we will explain why $(\frac{-D'}{p}) = (\frac{-D}{p})$ follows. From

$$D'f'^2 \equiv b \equiv Df^2 \pmod{a},$$

we have

$$\frac{D' f'^2}{f_p'^2} \equiv \frac{D f^2}{f_p^2} \pmod{a/f_p^2}.$$

Since we have assumed $\mathrm{ord}_p(a/(a,b)) > 1$, we have $p \,|\, (a/f_p^2)$, and it follows that D' and D are in the same square class modulo p.

Finally, we will show $f_p = f_p'$. By assumption $\mathrm{ord}_p(a/\gcd(a,b)) > 0$, and hence $a_p \nmid b$ and $\mathrm{ord}_p(b) = \mathrm{ord}_p(Df^2) = \mathrm{ord}_p(D' f'^2)$. In other words, we have

$$\mathrm{ord}_p(D') + 2\,\mathrm{ord}_p(f') = \mathrm{ord}_p(b) = \mathrm{ord}_p(D) + 2\,\mathrm{ord}_p(f). \tag{4.2}$$

Now consider the case of odd p. Since $-D'$ and $-D$ are fundamental discriminants, $\mathrm{ord}_p(D') \le 1$ and $\mathrm{ord}_p(D) \le 1$ are given by the parity of $\mathrm{ord}_p(b)$ and hence $f_p = f_p'$ as required.

Next, consider the case of $p = 2$. We have $\mathrm{ord}_2(D) \in \{0, 2, 3\}$. If $\mathrm{ord}_2(D) = 3$ or $\mathrm{ord}_2(D') = 3$, the argument for odd p extends. From now on, we assume that $\mathrm{ord}_2(D)$ and $\mathrm{ord}_2(D')$ are both in $\{0, 2\}$. If $\mathrm{ord}_2(D) = 0$ and $\mathrm{ord}_2(D') = 2$, then from (4.2) we have

$$2 + 2\,\mathrm{ord}_2(f') = \mathrm{ord}_2(b) = 2\,\mathrm{ord}_2(f).$$

From $D' f'^2, Df^2 \in a\mathbb{Z} + b$ and $\mathrm{ord}_2(a/\gcd(a,b)) \ge 2$, we obtain

$$\frac{D' f'^2}{4 f_2'^2} \equiv \frac{D f^2}{f_2^2} \pmod{4},$$

from which we have $D \equiv D'/4 \pmod{4}$. Since $-D'$ is a fundamental discriminant and $\mathrm{ord}_2(D') = 2$, we would have $D'/4 \equiv 1 \pmod{4}$. Therefore $D \equiv 1 \pmod{4}$. This is a contradiction, since $-D$ is a discriminant. The case of $\mathrm{ord}_2(D) = 2$ and $\mathrm{ord}_2(D') = 0$ is excluded by a symmetric argument. We conclude that we must have $\mathrm{ord}_2(D') = \mathrm{ord}_2(D)$, which implies $f_2 = f_2'$ as desired.

Regarding the last claim in the statement of the theorem, we note that a Ramanujan-type congruence on $a_p\mathbb{Z} + b$ follows from the Hurwitz class number formula, (4.1), and the fact just shown, that f_p is fixed for all $D \in a\mathbb{Z} + b$. $\square$

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