THE GAUSS RADON TRANSFORM – A NEW INVERSION FORMULA FOR THE RADON TRANSFORM

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Abstract. In this work we extend the usual Radon transform [3] to the Gaussian measure. We develop a Support Theorem for this new Gauss-Radon transform and also several inversion formulas. We also prove a connection between the Gauss-Radon transform and a conditional expectation. Lastly, we develop a new inversion formula for the original Radon transform using the Gauss-Radon transform and Segal-Bargmann transforms.

1. Introduction

The Radon transform was invented by Johann Radon in 1917 [3]. The Radon transform of a suitable function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as a function $R_f$ on the set $\mathcal{P}^n$ of hyperplanes in $\mathbb{R}^n$ as follows

$$(1.1) \quad R_f(\alpha w + w^\perp) = \int_{\alpha w + w^\perp} f(x) \, dx,$$

where $dx$ is the Lebesgue measure on the hyperplane given by $\alpha w + w^\perp$. The Radon transform remains a useful and important tool even today because of it has applications to many fields, included tomography and medicine [1].

One of the important results related to the Radon transform is the Support Theorem. As the name implies, the Support Theorem makes use of the Radon transform to the describe the support of a suitable function $f$. The theorem allows one to demonstrate that Radon transforms are in fact unique.

The other primary results related to the Radon transform are the various inversion formulas. Using the Laplacian operator or the Fourier transform one can actually recover a function $f$ from the Radon transform $R_f$. [2] The is one of the main reasons the Radon transform has proved so useful in many applications.

In this work we discuss the consequences of replacing the Lebesgue measure $dx$ in (1.1) with the Gaussian measure on the hyperplane. In Section 2 we present (mostly without proof) the Support Theorem and inversion formulas for the Radon transform. In Section 3 we develop the Gauss Radon transform and discuss a connection between this transform and conditional expectations. Then in Section 4 we recast the inversion formulas and the Support Theorem in the context of the Gauss Radon transform. Lastly, in Section 5.1 we develop a new inversion formula for the Radon Transform using the Gauss Radon transform and the Segal Bargmann transform.

Date: August 5, 2010.

2000 Mathematics Subject Classification. Primary: 44A12; Secondary: 81P68.

Key words and phrases. Radon Transform, Gaussian Measure, Conditional Expectation.

The research of J. Becnel is supported by National Security Agency Young Investigators Grant MPO-BA331.
2. RADON TRANSFORM AND RESULTS

In the following we will denote the set of hyperplanes in $\mathbb{R}^n$ as $\mathbb{P}^n$. That is,
$$\mathbb{P}^n = \{ \alpha w + w^\perp : \alpha \in \mathbb{R}, w \in \mathbb{R}^n \text{ is a unit vector} \}.$$  

where in the above $w^\perp$ is the orthogonal complement of the singleton set $\{ w \}$ containing the unit vector $w$. Notice each hyperplane $\alpha w + w^\perp$ is specified by two parameters $\alpha$ and $w$. In this way $w$ represents the normal vector to the hyperplane and $|\alpha|$ represents the distance from the hyperplane to the origin. When convenient we also represent the hyperplane $\alpha w + w^\perp$ as follows
$$\alpha w + w^\perp = \{ x \in \mathbb{R}^n ; \langle x, w \rangle = \alpha \}.$$  

In this way we think of the hyperplane as the set of all vectors with “$w$ direction coordinate” as $\alpha$.

**Definition 2.1.** The Radon transform of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a function $\mathcal{R}_f$ on the set $\mathbb{P}^n$ given by
$$\mathcal{R}_f(\alpha w + w^\perp) = \int_{\alpha w + w^\perp} f(x) \, dx$$
where $dx$ is the Lebesgue measure on the hyperplane $\alpha w + w^\perp$.

Of course, with no restrictions on $f$, the function $\mathcal{R}_f$ may not exist for every element of $\mathbb{P}^n$. To ensure that the $\mathcal{R}_f$ is defined and finite for every element of $\mathbb{P}^n$, one usually assumes that $f$ is rapidly decreasing, i.e.
$$\sup_{x \in \mathbb{R}^n} |x|^k f(x) < \infty \quad \text{for all} \quad k > 0$$
or that $f$ is in the Schwartz space $S(\mathbb{R}^n)$.

2.1. **Support Theorem.** We begin examining the important results related the Radon transform. The first of which is the Support Theorem.

**Theorem 2.2** (Support Theorem). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying
\begin{enumerate}
  \item $f$ is rapidly decreasing
  \item there exists a constant $r > 0$ such that $\mathcal{R}_f(\alpha w + w^\perp) = 0$ when $\alpha w + w^\perp$ does not intersect the closed ball of radius $r$ about the origin, $D_r(0) = \{ x \in \mathbb{R}^n ; |x| \leq r \}$
\end{enumerate}
Then $f(x) = 0$ for all $|x| > r$.

Simply put, the Support Theorem says that for a suitable function $f$ the support of function lies inside a closed ball around the origin of radius $r$ when the Radon transform of the function is supported only on planes $\alpha w + w^\perp$ with $|\alpha| \leq r$. For a proof of the Support Theorem, the interested reader can refer to Theorem 2.6 on page 10 in Helgason’s book [2]. An immediate corollary of the Support Theorem is that the Radon transform of a function is unique.

**Corollary 2.3.** Suppose $g, h$ are continuous, rapidly decreasing functions defined on $\mathbb{R}^n$. If $\mathcal{R}_g(\alpha w + w^\perp) = \mathcal{R}_h(\alpha w + w^\perp)$ for all hyperplanes $\alpha w + w^\perp \in \mathbb{P}^n$, then $g = h$.

**Proof.** If $\mathcal{R}_g(\alpha w + w^\perp) = \mathcal{R}_h(\alpha w + w^\perp)$ for all hyperplanes $\alpha w + w^\perp \in \mathbb{P}^n$, then by linearity of the Radon transform we have that $\mathcal{R}_{g-h}(\alpha w + w^\perp) = 0$. Thus the Support Theorem is satisfied for any $r > 0$ and hence $g(x) - h(x) = 0$ for all $x$ except possibly at $x = 0$. However, continuity of $g, h$ give us that $g(0) = h(0)$. \(\square\)
2.2. Inversion Formulas. We now discuss the various inversion formulas associated with the Radon transform. As the name implies these formulas allow one to recover the original function from the Radon transform. The first inversion formula makes use of the Fourier transform. As is customary, we will denote the Fourier transform of a function \( f \in S(\mathbb{R}^n) \) as \( \hat{f} \). That is,

\[
\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} \, dx, \quad \text{for } y \in \mathbb{R}^n.
\]

And, of course, there is the Fourier Inversion formula given by

\[
(2.1) \quad f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i \langle x, y \rangle} \, dy, \quad \text{for } x \in \mathbb{R}^n.
\]

There are many good references for the Fourier transform and a proof of the (2.1). See, for instance, [5].

We now develop the inversion formula for the Radon transform using the Fourier transform. Since the proofs are relatively short, we provide them here. In the following it is convenient to think of \( f \) and \( R_f \) as functions of two variables, one from \( \mathbb{R} \) and one from the unit circle \( S^{n-1} \). In fact, for the following we adopt the notation:

\[
(2.2) \quad R_f(\alpha, w) \overset{\text{def}}{=} R_f(\alpha w + w^\perp).
\]

And for \( f(x) \) we represent the vector \( x \) as \( \alpha w \) where \( \alpha \in \mathbb{R} \) and \( w \in S^{n-1} \).

**Proposition 2.4.** The \( n \)-dimensional Fourier transform of a function \( f \in S(\mathbb{R}^n) \) is equal to the one-dimensional Fourier transform of \( R_f(\alpha w + w^\perp) \) (or \( R_f(\alpha, w) \)). That is, \( \hat{R}_f(\beta, w) = \hat{f}(\beta w) \).

Again the \( \hat{R}_f(\beta, w) \) is the Fourier transform only on \( \beta \) with \( w \) fixed. That is, \( \hat{R}_f(\beta, w) = \int_{\mathbb{R}} \hat{R}_f(\alpha w + w^\perp) e^{-2\pi i \beta \alpha} \, d\alpha \).

**Proof.** For any \( \beta \in \mathbb{R} \) and unit vector \( w \in S^{n-1} \) we have

\[
\hat{f}(\beta w) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \beta \langle w, x \rangle} \, dx \\
= \int_{\mathbb{R}} \int_{\alpha w + w^\perp} f(u) e^{-2\pi i \beta \langle w, u \rangle} \, du \, d\alpha
\]

and using that \( u \in \alpha w + w^\perp \) yields that \( \langle w, u \rangle = \alpha \), so the above becomes

\[
= \int_{\mathbb{R}} \int_{\alpha w + w^\perp} f(u) e^{-2\pi i \beta \alpha} \, du \, d\alpha \\
= \int_{\mathbb{R}} \hat{R}_f(\alpha w + w^\perp) e^{-2\pi i \beta \alpha} \, d\alpha \\
= \hat{R}_f(\beta, w).
\]

\( \square \)

Proposition 2.4 leads us directly into the following inversion formula.
Theorem 2.5. The inversion formula for a function \( f \in S(\mathbb{R}^n) \) in terms of the Radon transform is

\[
(2.3) \quad f(x) = \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \mathcal{R}_f(\alpha w + w^\perp)e^{-2\pi i\beta(\alpha - (w,x))}\beta^{n-1}d\alpha d\beta d\sigma(w).
\]

Proof. We start by writing the Fourier inversion formula (2.1) for \( f \) in polar coordinates

\[
f(x) = \int_{\mathbb{R}^n} \hat{f}(y)e^{2\pi i(x,y)}dy
\]

and using Proposition 2.4 above we arrive at

\[
\text{which yields the desired result} \quad \square
\]

The other primary inversion formula for the Radon transform depends on the dimension of the space. We present the formulas here without proof. The interested reader can refer to [1, 2, 4].

Theorem 2.6. Let \( f \in S(\mathbb{R}^n) \) with Radon Transform \( R_f \). If \( n \) is odd then

\[
f(x) = \frac{\pi (-1)^n}{(2\pi)^n} \int_{S^{n-1}} \mathcal{R}_f^{(n-1)}(\langle w, x \rangle, w) d\sigma(w), \quad n = 2m + 1.
\]

If \( n \) is even, then

\[
f(x) = \frac{(-1)^{n/2}(n-1)!}{(2\pi)^n} \int_{S^{n-1}} \int_{-\infty}^\infty \frac{\mathcal{R}_f(\alpha, w)}{(\alpha - (w, x))^n} d\alpha d\sigma(w), \quad n = 2m
\]

which, after integration by parts with respect to \( \alpha \), yields

\[
f(x) = \frac{(-1)^{n/2+1}}{(2\pi)^n} \int_{S^{n-1}} \int_{-\infty}^\infty \frac{\mathcal{R}_f^{(n-1)}(\alpha, w)}{\langle w, x \rangle - \alpha} d\alpha d\sigma(w), \quad n = 2m
\]

where \( \mathcal{R}_f^{n-1} \) represents the \( n-1 \)th derivative of \( R_f \) with respect to the real variable.

3. Gauss Radon Transform

To construct the Gauss Radon Transform we first must construct a Gaussian measure on a hyperplane \( \alpha w + w^\perp \). We will denote such a measure by \( \mu_{\alpha w + w^\perp} \). Suppose \( d\mu_{\alpha w + w^\perp}(x) = c_\alpha e^{-|x|^2/2} dx \), where \( dx \) is the Lebesgue measure on the hyperplane. We need to determine the constant \( c_\alpha \) such that \( \int_{\alpha w + w^\perp} d\mu_{\alpha w + w^\perp} = 1. \)
To this end, note that by a change of variables we have

\[
\int_{\alpha w + w^\perp} e^{-|x|^2/2} \, dx = \int_{w^\perp} e^{-|\alpha w + x|^2/2} \, dx
\]

\[
= \int_{w^\perp} e^{-|x|^2/2} \, dx \quad \text{since } |w| = 1 \text{ and } \langle w, x \rangle = 0
\]

\[
= e^{-\alpha^2/2} \int_{w^\perp} e^{-|x|^2/2} \, dx
\]

\[
= e^{-\alpha^2} (2\pi)^{(n-1)/2}.
\]

Using this calculation we define the Gaussian measure on \(\alpha w + w^\perp\) as follows.

**Definition 3.1.** The Gaussian measure \(\mu_{\alpha w + w^\perp}\) on the hyperplane \(\alpha w + w^\perp\) is defined by

\[
(3.1) \quad d\mu_{\alpha w + w^\perp}(x) = e^{\alpha^2/2} \frac{(2\pi)^{(n-1)/2}}{2\pi} e^{-|x|^2/2} \, dx
\]

where \(dx\) is the Lebesgue measure on the hyperplane \(\alpha w + w^\perp\).

**Remark 3.2.** Using that for \(x \in \alpha w + w^\perp\) we have that

\[
|x - \alpha w|^2 = |x|^2 - 2\langle x, \alpha w \rangle + |\alpha w|^2 = |x|^2 - 2\alpha^2 + \alpha^2 = |x|^2 - \alpha^2
\]

allows us also to express the \(d\mu_{\alpha w + w^\perp}\) as follows

\[
d\mu_{\alpha w + w^\perp}(x) = e^{-|x - \alpha w|^2/2} \frac{dx}{(2\pi)^{(n-1)/2}}.
\]

### 3.1. Properties of Gaussian measure on the hyperplane

We know describe and prove some properties for this measure. We begin by computing the measures characteristic function.

**Proposition 3.3.** The characteristics function (or Fourier transform) of the Gaussian measure on the hyperplane \(\alpha w + w^\perp\) is

\[
\int_{\alpha w + w^\perp} e^{i\langle x, y \rangle} d\mu_{\alpha w + w^\perp}(x) = e^{i\alpha \langle w, y \rangle} - \frac{1}{2}|y_w^\perp|^2
\]

for any \(y \in \mathbb{R}^n\)

where \(y_w^\perp\) denote the orthogonal projection of \(y\) onto the subspace \(w^\perp\).

**Proof.** Observe that

\[
\int_{\alpha w + w^\perp} e^{i\langle x, y \rangle} d\mu_{\alpha w + w^\perp}(x) = \int_{\alpha w + w^\perp} e^{i\langle x, y \rangle} e^{-|x - \alpha w|^2/2} \frac{dx}{(2\pi)^{(n-1)/2}}
\]

\[
= \int_{w^\perp} e^{i\langle x + \alpha w, y \rangle} e^{-|x|^2/2} \frac{dx}{(2\pi)^{(n-1)/2}} \quad \text{changing variables}
\]

\[
= e^{i\alpha \langle w, y \rangle} \int_{w^\perp} e^{i\langle x, y_w^\perp \rangle} e^{-|x|^2/2} \frac{dx}{(2\pi)^{(n-1)/2}}
\]

\[
= e^{i\alpha \langle w, y \rangle} e^{-|y_w^\perp|^2/2}
\]

which yield the desired result. \(\square\)
A useful and fundamental characteristic about the probability measure $\mu_{\alpha w + w^\perp}$ is that every vector in $\mathbb{R}^n$ can be understand as a normally (or Dirac) distributed random variable in its probability space. Consider the following:

**Proposition 3.4.** There is a mapping given by

$$\mathbb{R}^n \to L^2(\mu_{\alpha w + w^\perp})$$

$$u \mapsto \tilde{u}(x) = \langle u, x \rangle$$

such that $\tilde{u}$ is normally (or Dirac) distributed with mean $\alpha \langle w, u \rangle$ and variance $|u w^\perp|^2$.

Note that in the above with $u w^\perp = 0$ the random variable $\tilde{u}$ takes on the Dirac distribution. In all other cases $\tilde{u} \sim N(\alpha \langle w, u \rangle, |u w^\perp|^2)$.

**Proof.** To prove this we simply compute the characteristic function of the random variable $\tilde{u}$. Observe for any $t \in \mathbb{R}$ we have

$$\phi_{\tilde{u}}(t) = E[e^{it\tilde{u}}] = \int_{\alpha w + w^\perp} e^{it\langle u, x \rangle} d\mu_{\alpha w + w^\perp}(x).$$

We can apply Proposition 3.3 with $y = tu$ to the above to see that

$$\phi_{\tilde{u}}(t) = e^{i\alpha \langle w, tu \rangle - \frac{1}{2} |tu w^\perp|^2} = e^{i\alpha \langle w, u \rangle - \frac{1}{2} |u w^\perp|^2}$$

Comparing the above to the characteristic function of the normal distribution we immediately see that $\tilde{u} \sim N(\alpha \langle w, u \rangle, |u w^\perp|^2)$. □

As we continue Proposition 3.4 will provide us a convenient way of studying many functions in terms of the Gauss Radon transform. Before we begin, we make one more observation about the random variables $\tilde{u}$.

**Proposition 3.5.** If $u, v \in \mathbb{R}^n$ satisfy $\langle u w^\perp, v w^\perp \rangle = 0$, then the random variables $\tilde{u}, \tilde{v}$ are independent with respect to $\mu_{\alpha w + w^\perp}$.

**Proof.** Proposition 3.3 and Proposition 3.4 help simplify this proof through use of the characteristic function again. Observe that for any $t, s \in \mathbb{R}$ we have

$$E[e^{it\tilde{u} + is\tilde{v}}] = \int_{\alpha w + w^\perp} e^{it\langle tu + sv, x \rangle} d\mu_{\alpha w + w^\perp}(x).$$

The we can apply Proposition 3.3 to the above with $y = tu + sv$ to get

$$E[e^{it\tilde{u} + is\tilde{v}}] = e^{i\alpha \langle w, tu + sv \rangle - \frac{1}{2} |tu w^\perp + sv w^\perp|^2}$$

$$= e^{it\alpha \langle w, u \rangle} e^{is\alpha \langle w, v \rangle} e^{-\frac{1}{2} |tu w^\perp|^2} e^{-\frac{1}{2} |sv w^\perp|^2}$$

$$= e^{it\alpha \langle w, u \rangle} e^{is\alpha \langle w, v \rangle} e^{-\frac{1}{2} |tu w^\perp|^2} e^{-\frac{1}{2} |sv w^\perp|^2} \quad \text{using that } \langle u w^\perp, v w^\perp \rangle = 0$$

$$= E[e^{it\tilde{u}}] E[e^{is\tilde{v}}].$$

Therefore $\tilde{u}, \tilde{v}$ are independent. □
3.2. **Definition of Gauss Radon Transform.** With the definition of the Gaussian measure on the hyperplane securely behind us, we can turn our attention to defining the Gauss Radon transform. Just as the Radon transform finds the integral of a function over a hyperplane using the Lebesgue measure for the hyperplane, the Gauss Radon transform will output the integral of a function over a hyperplane using the Gaussian measure for the hyperplane.

**Definition 3.6.** The Gauss Radon transform of a function \( f : \mathbb{R}^n \to \mathbb{R} \) is a function \( \mathcal{G}_f \) on the set \( \mathbb{P}^n \) given by

\[
\mathcal{G}_f(\alpha w + w^\perp) = \int_{\alpha w + w^\perp} f(x) \, d\mu_{\alpha w + w^\perp}(x)
\]

where \( \mu_{\alpha w + w^\perp} \) is the Gaussian measure on the hyperplane \( \alpha w + w^\perp \).

Again, with no restrictions on \( f \), the function \( \mathcal{G}_f \) may not exist for every element of \( \mathbb{P}^n \). But it is important to note that using the Gaussian measure in place of the Lebesgue measure ensures the existence of \( \mathcal{G}_f \) for a much broader class of function, including polynomials. We will discuss this more in Section 4. We will now compute some examples of the Gauss Radon Transform for common functions.

**Example 3.7.** Consider a polynomial function \( p : \mathbb{R}^n \to \mathbb{R} \). For simplicity we first just assume that \( p(x_1, x_2, \ldots, x_n) = x_k^m \). The key observation is to observe that \( x_k^m = (x, e_k)^m \) where \( e_k \) is the from the standard orthonormal basis on \( \mathbb{R}^n \) (i.e. \( e_k \) is the vector of all zeros except for a single 1 at the \( k^{th} \) position). Note that

\[
\mathcal{G}_p(\alpha w + w^\perp) = \int_{\alpha w + w^\perp} x_k^m \, d\mu_{\alpha w + w^\perp}(x)
\]

\[
= \int_{\alpha w + w^\perp} (x, e_k)^m \, d\mu_{\alpha w + w^\perp}(x)
\]

\[
= E[\hat{e}_k^m].
\]

We now combine the observation above and use that \( \hat{e}_k \sim N(\alpha(e_k, w), |P_{w^\perp} e_k|^2) \) (here \( P_{w^\perp} \) is the orthogonal projection onto \( w^\perp \)) to see that \( \mathcal{G}_p(\alpha w + w^\perp) = E[\hat{e}_k^m] \) is nothing more than the \( m^{th} \) raw moment of \( \hat{e}_k^m \). Therefore

\[
\mathcal{G}_p(\alpha w + w^\perp) = \sum_{j=0}^{m/2} \binom{m}{2j} (\alpha(w, e_k))^{m-2j}(2j)! |P_{w^\perp} e_k|^{2j}.
\]

Using that \( \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n \) are independent this argument can be extended to compute \( \mathcal{G}_p(\alpha w + w^\perp) \) for any polynomial function \( p : \mathbb{R}^n \to \mathbb{R} \).

**Example 3.8.** In this example we will compute the Gauss Radon transform of an exponential function \( f(x) = e^{t\langle x \rangle} \) for a fixed \( t \in \mathbb{R}^n \). Observe that

\[
\mathcal{G}_f(\alpha w + w^\perp) = \int_{\alpha w + w^\perp} e^{t\langle x \rangle} \, d\mu_{\alpha w + w^\perp}(x) = E[e^{\alpha t}].
\]

Using Proposition 3.4 we have that \( t \sim N(\alpha(t, w), |t_{w^\perp}|^2) \). Thus the above can be computed using the moment generating function for \( t \). As such, we have that

\[
\mathcal{G}_f(\alpha w + w^\perp) = e^{\alpha(t, w) + \frac{1}{2}|t_{w^\perp}|^2}.
\]

In both of the above examples we have used functions for which the Radon transform does not exist and easily computed the Gauss Radon transform.
3.3. Gauss Radon Transform as a Conditional Expectation. In this section we develop an interesting result relating the the Gauss Radon transform to a conditional expectation. In the following $\mu$ represents the standard Gaussian measure on $\mathbb{R}^n$.

**Theorem 3.9.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a integrable function on the probability space $(\mathbb{R}^n, \mu)$ such that $G_f$ exists for all hyperplane in $\mathbb{R}^n$. Then the Gauss-Radon transform of $f : \mathbb{R}^n \to \mathbb{R}$ on a hyperplane $\alpha w + w^\perp$ is equal to the conditional expectation of $f$ under the condition $\tilde{w} = p$, i.e.

$$G_f(\alpha w + w^\perp) = E[f \mid \tilde{w} = \alpha].$$

**Proof.** Let $B = \{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Starting with the unit vector $w$ we will define another basis for $\mathbb{R}^n$, say $B' = \{w, v_1, \ldots, v_{n-1}\}$, where $v_1, \ldots, v_{n-1}$ is an orthonormal basis for $w^\perp$. By Proposition 3.5, we know $\tilde{w}, \tilde{v}_1, \ldots, \tilde{v}_{n-1}$ are independent random variables and, by Proposition 3.4, each one of them has the standard normal distribution. We will also make use of the orthonormal transformation (change of basis) matrix $A = (w, v_1, \ldots, v_{n-1})$.

Now consider the vector $x'$ defined by $x' = A^{-1}x$. To prove (3.2), let us start from the right. Observe

$$E[f \mid \tilde{w} = \alpha] = E[f \circ A | x'_1 = \alpha]$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f \circ A((v_1', v_2', \ldots, v_n')) \frac{e^{-\frac{1}{2}(x_2'^2 + \cdots + x_n'^2)}}{(2\pi)^{(n-1)/2}} dx_2' \cdots dx_n'.$$

On the other hand, for the $G_f(\alpha w + w^\perp)$, we have

$$G_f(\alpha w + w^\perp) = \int_{\alpha w + w^\perp} f(x) d\mu_{\alpha w + w^\perp}(x).$$

We know $d\mu_{\alpha w + w^\perp}(x) = \frac{e^{-\frac{1}{2}(x_1'^2 + \cdots + x_n'^2)}}{(2\pi)^{n/2}} dx_2' \cdots dx_n'$, and $dx = dx_2' \cdots dx_n'$. So

$$d\mu_{\alpha w + w^\perp}(x) = \frac{e^{-\frac{1}{2}(x_1'^2 + \cdots + x_n'^2)}}{(2\pi)^{(n-1)/2}} dx_2' \cdots dx_n'.

Then (3.3) yields

$$G_f(\alpha w + w^\perp) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f \circ A((v_1', v_2', \ldots, v_{n-1}')) \frac{e^{-\frac{1}{2}(x_2'^2 + \cdots + x_n'^2)}}{(2\pi)^{(n-1)/2}} dx_2' \cdots dx_n'$$

$$= E[f \mid \tilde{w} = \alpha].$$

\[\square\]

4. Support Theorem and Inversion Formula

In this section we will develop a Support Theorem and several inversion formulas for the Gauss Radon transform. These will be based primarily off the existing Support Theorem (Theorem 2.2) and inversion formulas (Theorem 2.5 and 2.6), for the Radon transform. We begin by introducing a convenient class of function to work with.
4.1. Gaussian Bounded Functions. In order to develop the Support Theorem and inversion formula for the Gauss Radon transform we must first decide on a class of function to use. The rapidly decreasing Schwartz space functions used in the Radon transform seem too restrictive for the Gauss Radon transform. We would like to expand on these classes. With this in mind we give the following definition:

Definition 4.1. A function $f \in C^\infty(\mathbb{R}^n)$ is $\delta$-Gaussian bounded if for any polynomial function $p$, there exists a constants $M_p > 0$ and $0 < k_p < \delta$ such that

$$|p(\partial_1, \partial_2, \ldots, \partial_n)f(x)| \leq M_p e^{k_p|x|^2} \quad \text{for all } x \in \mathbb{R}^n.$$ 

Remark 4.2. From the definition it is clear that if $f$ is $\frac{1}{2}$-Gaussian bounded, then $f(x) e^{-\frac{|x|^2}{2}} \in S(\mathbb{R}^n)$.

Notice that the class of $\delta$-Gaussian bounded functions holds a much broader class of functions than the usual Schwartz space. In particular polynomial functions are $\delta$-Gaussian bounded.

4.2. Relationship between Radon and Gauss Radon Transform. The relationship between the Radon transform and the Gauss Radon transform is really the key to developing a Support Theorem and inversion formulas for the Gauss Radon Transform. In the results to come we will extensively use the following proposition.

Proposition 4.3. Suppose $f$ is an $\frac{1}{2}$-Gaussian bounded function. Then

$$G_f(\alpha w + w^\perp) = e^{\frac{\alpha^2}{2}} R_g(\alpha w + w^\perp)$$

where $g(x) = f(x) e^{-\frac{|x|^2}{2}}$.

Proof. Starting from the left we have

$$G_f(\alpha w + w^\perp) = \int_{\alpha w + w^\perp} f \, d\mu_{\alpha w + w^\perp}$$

$$= e^{\frac{\alpha^2}{2}} \int_{\alpha w + w^\perp} f(x) \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{n-1/2}} \, dx \quad \text{by (3.1)}$$

$$= e^{\frac{\alpha^2}{2}} \int_{\alpha w + w^\perp} g(x) \, dx$$

$$= e^{\frac{\alpha^2}{2}} R_g(\alpha w + w^\perp)$$

yielding the result. \qed

4.3. Support Theorem for Gauss Radon Transform. Proposition 4.3 will ease our work in developing the Support Theorem for the Gauss Radon transform. In fact, it will allow us to appeal directly to Theorem 2.2.

Theorem 4.4 (Support Theorem for Gauss Radon Transform). Let $f$ be a $\frac{1}{2}$-Gaussian bounded function. If there exists a constant $r > 0$ such that $G_f(\alpha w + w^\perp) = 0$ when $\alpha w + w^\perp$ does not intersect the closed ball of radius $r$ about the origin, $D_r(0) = \{ x \in \mathbb{R}^n ; |x| \leq r \}$, then $f(x) = 0$ for all $|x| > r$. 

expressed in terms of the conditional expectations $E$. Theorem 4.8.

We attempt adapt them to the Gauss Radon Transform.

g$(y)$ for any $y$ terms of its Gauss Radon transform is $\text{Theorem 2.5 and Theorem 2.6 using Proposition 4.3. The first inversion formula}$

involves the Fourier transform.

The inversion formula for a $1/2$-Gaussian bounded function $f$ in terms of its Gauss Radon transform is

scored. The theorem gives one a means of determining the support of a random variable using only its conditional expectations.

4.4. Inversion Formulas for the Gauss Radon Transform. We now present inversion formulas for the Gauss Radon transform and the corresponding inversion formula involving conditional expectations. The formula will be derived from Theorem 2.5 and Theorem 2.6 using Proposition 4.3. The first inversion formula involves the Fourier transform.

**Theorem 4.6.** Let $X$ be a $1/2$-Gaussian bounded random variable on the probability space $(\mathbb{R}^n, \mu)$ where $\mu$ is the standard Gaussian measure. If there exists a constant $r > 0$ such that $\text{Theorem 3.9 we can recast the Support theorem in probabilistic set-}$

In light of Theorem 3.9 we can recast the Support theorem in probabilistic setting.

**Corollary 4.5.** Let $X$ be a $1/2$-Gaussian bounded random variable on the probability space $(\mathbb{R}^n, \mu)$ where $\mu$ is the standard Gaussian measure. If there exists a constant $r > 0$ such that $|X| > r$, then $X(y) = 0$ when $|y| > r$.

The proof is similar to the proof of Theorem 2.2.

In light of Theorem 3.9 we can recast the Support theorem in probabilistic setting.

**Corollary 4.5.** Let $X$ be a $1/2$-Gaussian bounded random variable on the probability space $(\mathbb{R}^n, \mu)$ where $\mu$ is the standard Gaussian measure. If there exists a constant $r > 0$ such that $|X| > r$, then $X(y) = 0$ when $|y| > r$.

The above corollary gives one a means of determining the support of a random variable using only its conditional expectations.

The inversion formula for a $1/2$-Gaussian bounded function $f$ in terms of its Gauss Radon transform is

$$f(x) = (2\pi)^{\frac{n-1}{2}} \frac{w^2}{x^2} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} G_f(\alpha w + w^+) e^{-2\pi i \beta (\alpha - \langle w, x \rangle) - \frac{x^2}{2} \beta^{n-1}} d\alpha d\beta d\sigma(w).$$

for any $x \in \mathbb{R}^n$.

**Proof.** Simply replace $f$ in (2.3) with $g(x) = f(x) \frac{e^{-|x|^2/2}}{(2\pi)^{n-1/2}}$. Then use Proposition 4.3 to replace $R_g(\alpha w + w^+)$ with $e^{-a^2/2} G_f(\alpha w + w^+)$.

We now recast the above theorem in terms of a conditional expectation.

**Corollary 4.6.** Let $X$ be a $1/2$-Gaussian bounded random variable on the probability space $(\mathbb{R}^n, \mu)$ where $\mu$ is the standard Gaussian measure. Then $X$ can be expressed in terms of the conditional expectations $E[X | \tilde{w} = \alpha]$ as follows:

$$X(y) = (2\pi)^{\frac{n-1}{2}} \frac{w^2}{x^2} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} E[X | \tilde{w} = \alpha] e^{-2\pi i \beta (\alpha - \langle w, y \rangle) - \frac{x^2}{2} \beta^{n-1}} d\alpha d\beta d\sigma(w).$$

for any $y \in \mathbb{R}^n$.

We now turn our attention to the inversion formula found in Theorem 2.6 and attempt adapt them to the Gauss Radon Transform.

**Theorem 4.8.** Let $f \in S(\mathbb{R}^n)$ with Radon Transform $R_f$. If $n = 2m + 1$ is odd then

$$f(x) = \frac{\pi w^2}{(2\pi)^{n+1}} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{k+m} \int_{S^{n-1}} \int e^{-a^2/2} h_k(\langle w, x \rangle) G_f^{n-1-k}(\langle w, x \rangle, w) d\sigma(w).$$

for any $x \in \mathbb{R}^n$. The above corollary gives one a means of determining the support of a random variable using only its conditional expectations.
If \( n = 2m \) is even, then

\[
(4.2) \quad f(x) = \frac{(-1)^{n/2}(n-1)!}{(2\pi)^{m+1}} e^{\frac{|x|^2}{2}} \int_{S^{n-1}} \int_{-\infty}^{\infty} \frac{G_f(\alpha, w)}{(\alpha - \langle w, x \rangle)^n} e^{-\alpha^2/2} \, d\alpha \, d\sigma(w)
\]

which, after integration by parts with respect to \( \alpha \), yields

\[
(4.3) \quad f(x) = \frac{e^{\frac{|x|^2}{2}}}{(2\pi)^{m+1}} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{k+m+1} \int_{S^{n-1}} \int_{-\infty}^{\infty} e^{-\alpha^2/2} h_k(\alpha) G_f^{n-1-k}(\alpha, w) \, d\alpha \, d\sigma(w)
\]

where \( R_f^{n-1} \) represents the \( n-1 \)th derivative of \( R_f \) with respect to the real variable and \( h_k \) is the Hermite polynomial given by \( h_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2} \).

**Proof.** We begin with (4.1). For odd \( n = 2m + 1 \) we have by Theorem 2.6 that

\[
(4.4) \quad g(x) = \frac{\pi(-1)^m}{(2\pi)^n} \int_{S^{n-1}} R_g^{(n-1)}(\langle w, x \rangle) \, d\sigma(w)
\]

where \( g \) is again given by \( g(x) = f(x) e^{-\frac{|x|^2}{2}} (2\pi)^{n-1} \). Now we use that \( R_g(\alpha, w) = e^{-\alpha^2/2} G_f(\alpha, w) \) by Proposition 4.3. Thus using the Leibniz Rule we have

\[
R_g^{(n-1)}(\alpha, w) = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k e^{-\alpha^2/2} h_k(\alpha) G_f^{n-1-k}(\alpha, w).
\]

Now we bring in the Hermite polynomials \( h_k(\alpha) = (-1)^k e^{\alpha^2/2} \frac{d^k}{d\alpha^k} e^{-\alpha^2/2} \) and substitute \( \frac{d^k}{d\alpha^k} e^{-\alpha^2/2} = (-1)^k e^{-\alpha^2/2} h_k(\alpha) \) into the above equation to get

\[
(4.5) \quad R_g^{(n-1)}(\alpha, w) = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k e^{-\alpha^2/2} h_k(\alpha) G_f^{n-1-k}(\alpha, w).
\]

Substituting the above in (4.4) we have that

\[
g(x) = \frac{\pi(-1)^m}{(2\pi)^n} \int_{S^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k e^{-\frac{|w,x|^2}{2}} h_k(\langle w, x \rangle) G_f^{n-1-k}(\langle w, x \rangle) \, d\sigma(w)
\]

which yields

\[
f(x) = \frac{\pi e^{\frac{|x|^2}{2}}}{(2\pi)^{m+1}} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{k+m} \int_{S^{n-1}} \int_{-\infty}^{\infty} e^{-\frac{|w,x|^2}{2}} h_k(\langle w, x \rangle) G_f^{n-1-k}(\langle w, x \rangle) \, d\alpha \, d\sigma(w).
\]

For \( n = 2m \), in order to get (4.2) we use \( g \) as given above again to get

\[
(4.6) \quad g(x) = \frac{(-1)^{n/2}(n-1)!}{(2\pi)^n} \int_{S^{n-1}} \int_{-\infty}^{\infty} R_g(\alpha, w) \, d\alpha \, d\sigma(w), \quad n = 2m
\]

and substitute \( R_g(\alpha, w) = e^{-\alpha^2/2} G_f(\alpha, w) \) to arrive at

\[
f(x) = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{m+1}} \frac{(-1)^{n/2}(n-1)!}{(2\pi)^n} \int_{S^{n-1}} \int_{-\infty}^{\infty} e^{-\alpha^2/2} G_f(\alpha, w) \, d\alpha \, d\sigma(w), \quad n = 2m
\]

which yields

\[
f(x) = \frac{(-1)^{n/2}(n-1)!}{(2\pi)^{m+1}} e^{\frac{|x|^2}{2}} \int_{S^{n-1}} \int_{-\infty}^{\infty} G_f(\alpha, w) \, d\alpha \, d\sigma(w).
\]
The equation in (4.3) is proved in a similar way to that of (4.1). We begin by using Theorem 2.6 with \( g \) as defined above to get

\[
g(x) = \frac{(-1)^{n/2+1}}{(2\pi)^n} \int_{S^{n-1}} \int_{-\infty}^{\infty} \frac{R^{(n-1)}(w,x)}{\langle w, x \rangle - \alpha} \, d\alpha \, d\sigma(w), \quad \text{where } n = 2m
\]

We then substitute (4.5) into the above to arrive at

\[
g(x) = \frac{(-1)^{n/2+1}}{(2\pi)^n} \int_{S^{n-1}} \int_{-\infty}^{\infty} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k e^{-\alpha^2/2} h_k(\alpha) \, G_f^{n-1-k}(\alpha, w) \, d\alpha \, d\sigma(w).
\]

which yields

\[
f(x) = \frac{e^{-x^2/2}}{(2\pi)^{n+1/2}} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{k+m+1} \int_{S^{n-1}} \int_{-\infty}^{\infty} e^{-\alpha^2/2} h_k(\alpha) \, G_f^{n-1-k}(\alpha, w) \, d\alpha \, d\sigma(w).
\]

As we have done previously the above theorem can be transformed to an inversion formula for a random variable in terms of conditional expectations by using Proposition 4.3. We forgo explicitly displaying the lengthy corollary as its construction is similar to that of Corollary 4.7 and Corollary 4.5.

5. A NEW INVERSION FORMULA FOR THE RADON TRANSFORM

5.1. Segal-Bargmann Transform.

**Definition 5.1.** For a function \( f \in L^2(\mathbb{R}^n) \) the Segal-Bargmann transform of \( f \) is given by

\[
S_f(z) = \int_{\mathbb{R}^n} f(x) e^{\langle x, z \rangle - \langle z, z \rangle^2/2} e^{-|x|^2/2} (2\pi)^{n/2} \, dx \quad \text{for } z \in \mathbb{C}^n
\]

where the pairing \( \langle z, x \rangle \) is complex bilinear (not sesquilinear), i.e. if \( z = a + bi \), then \( \langle z, x \rangle = \langle a, x \rangle + i\langle b, x \rangle \).

**Theorem 5.2.** The Segal-Bargmann transform is a unitary isomorphism between \( L^2(\mathbb{R}^n) \) and \( \mathcal{H}(\mathbb{C}^n) \) of holomorphic functions on \( \mathbb{C}^n \)

**Theorem 5.3.** If \( f \) is in the Sobolev space \( H^d(\mathbb{R}^n) \) and \( d > n/2 \), then for all \( x \in \mathbb{R}^n \) we have

\[
f(x) = \int_{\mathbb{R}^n} S_f(x + iy) e^{-y^2/2} (2\pi)^{n/2} \, dy.
\]

with absolute convergence of the interval for all \( x \in \mathbb{R}^n \).

**Theorem 5.4.** Suppose \( f \in L^2(\mathbb{R}^n) \). For each unit vector \( w \in \mathbb{R}^n \), let \( F_w \) be the function on \( \mathbb{R} \) given by \( F_w(\alpha) = G_f(\alpha w + w^\perp) \). Then

\[
S_{F_w}(\alpha) = S_f(\alpha w).
\]
Proof. Starting on the left-hand side we have that
\[
S_{F_w}(\alpha) = \int F_w(\beta) e^{\alpha \beta - \alpha^2/2} e^{-\beta^2/2} d\beta
\]
\[
= \int G_f(\beta w + w^\perp) e^{\alpha \beta - \alpha^2/2} e^{-\beta^2/2} d\beta
\]
\[
= \int \int_{\beta w + w^\perp} f(x) e^{-|x|^2/2 + \alpha(x,w) - \alpha^2/2} dx \frac{d\alpha}{(2\pi)^{n/2}} e^{\alpha \beta - \alpha^2/2} e^{-\beta^2/2} d\beta
\]
\[
= \int \int_{\beta w + w^\perp} f(x) e^{-|x|^2/2 + \alpha \beta - \alpha^2/2} dx \frac{d\beta}{(2\pi)^{n/2}}
\]
\[
= S_f(\alpha w)/2
\]
Since \( x \in \beta w + w^\perp \) we have \( \beta = \langle x, w \rangle \), combining this with the fact that \( w \) is a unit vector and the above becomes
\[
= \int \int_{\beta w + w^\perp} f(x) e^{-|x|^2/2 + \alpha \langle x,w \rangle - \alpha^2/2} dx \frac{d\beta}{(2\pi)^{n/2}}
\]
\[
= \int f(y) e^{-|y|^2/2} e^{-\langle y, \alpha w \rangle - |\alpha w|^2/2} dy \frac{d\beta}{(2\pi)^{n/2}}
\]
\[
= S_f(\alpha w)
\]
\]
Notice that in inversion formula we need \( S_f \) to be defined for values \( z \in \mathbb{C}^n \).
\[
f(x) = \int_{\mathbb{R}^n} S_f(x + iy) e^{-y^2/2} \frac{d\beta}{(2\pi)^{n/2}}
\]
We could like to replace \( S_f \) in the interior of the integral with the Gauss Radon transform. However, (5.1) really only makes sense for values in \( \mathbb{R}^n \). For instance, for each nonzero \( x \) we have
\[
S_f(x) = S_{F_{x/|x|}}(|x|) = \int G_f(|x| \frac{x}{|x|} + x^\perp) d|x|.
\]
\[
S_f(\alpha w) = S_{F_{\alpha w/|\alpha w|}}(|x|) = \int G_f(\alpha w + w^\perp) d\alpha.
\]
However for values \( z \in \mathbb{C}^n \) is unclear how to interpret the above expression. Luckily by Theorem 5.2 \( S_f \) is holomorphic function on \( \mathbb{C}^n \). Thus knowing its real values is enough to completely determine the function (IS THIS TRUE?...REFERENCE?). Hence we can use the Segal Bargmann transform to find in inversion formula

References


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