Chapter 4. Discrete Probability Distributions

Sections 4.9, 4.10: Moment Generating Function and Probability Generating Function

Jiaping Wang

Department of Mathematical Science

03/06/2013, Wednesday
Outline

Moment Generating Function

Probability Generating Function

More Examples

Homework #7
Part 1. Moment Generating Function
The k-th moment is defined as $E(X^k) = \sum x^k p(x)$. For example, $E(X)$ is the 1st moment, $E(X^2)$ is the 2nd moment.

The moment generating function is defined as

$$M(t) = E(e^{tX})$$

So we have $M^{(k)}(0) = E(X^k)$.

For example, $M^{(1)}(t) = \frac{dM(t)}{dt} = \frac{d}{dt} E(e^{tX}) = E \left[ \frac{d}{dt} e^{tX} \right] = E[Xe^{tX}]$

So if set $t=0$, then $M^{(1)}(0) = E(X)$.

It often is easier to evaluate $M(t)$ and its derivatives than to find the moments of the random variable directly.
Evaluate the moment generating function (mgf) for the geometric distribution and use it to find the mean and the variance of this distribution.

Answer: \( M(t) = E(e^{tX}) = \sum_{i=0}^{\infty} e^{tx}pq^x = p \sum_{x=0}^{\infty} (qet)^x = p[1 + qet + (qet)^2 + \cdots \infty] \)

\[ = p \left( \frac{1}{1-qet} \right) = \frac{p}{1-qet} \] which requires \( qet < 1 \).

\( M^{(1)}(t) = \frac{pqet}{(1-qet)^2}, \quad E(X) = M^{(1)}(0) = pq/(1-q)^2 = q/p. \)

\( M^{(2)}(t) = \frac{pqet + pq^2e^{2t}}{(1-qet)^3}, \quad E(X^2) = M^{(2)}(0) = q(1+q)/p^2. \)

Thus \( V(X) = E(X^2) - E^2(X) = p/p^2. \)
Part 2. Probability Generating Function
Definition 4.8: The probability generating function (pgf) of a random variable is denoted by \( P(t) \) and is defined as
\[
P(t) = E(t^X)
\]
For \( X \) is an integer-valued random variable,
\[
P(t) = E(t^X) = p_0 + p_1t + p_2t^2 + \ldots
\]
If we know \( P(t) \) and can expand it into a series, we can determine \( p(x) = P(X=x) \) as the coefficient of \( t^x \). Repeat differentiation of \( P(t) \) yields factorial moments for the random variable \( X \).

Definition 4.9: The \( k \)-th factorial moment for a random variable \( X \) is defined as
\[
\mu_{[k]} = E[X(X-1)(X-2)\ldots(X-k+1)]
\]
where \( k \) is a positive integer.
When a pgf exists, it can be differentiated in a neighborhood of \( t=1 \), thus with \( P(t)=E(t^X) \), we have

\[
P^{(1)}(t) = \frac{dP(t)}{dt} = \frac{d}{dt} E(t^X) = E \left( \frac{d}{dt} t^X \right) = E(Xt^X - 1)
\]

If we set \( t=1 \), then \( P^{(1)}(1)=E(X) \).
Similarly, \( P^{(2)}(t)=E[X(X-1)t^{X-2}] \) and \( P^{(2)}(1)=E[X(X-1)] \).
In general, \( P^{(k)}(t)=E[X(X-1)\cdots(X-k+1)t^{X-k}] \) and \( P^{(k)}(1)= E[X(X-1)\cdots(X-k+1)t^{X-k}]=\mu_k \).
Example 4.28

Find the probability generating function for the geometric random variable and use this function to find mean.

Answer: $P(t) = E(t^X) = \sum_{x=0}^{\infty} t^x p q^x = p \sum_{x=0}^{\infty} (qt)^x = \frac{p}{1-qt}$

So $P^{(1)}(t) = \frac{pq}{(1-qt)^2}$, with $t=1$, we have $E(X) = P^{(1)}(1) = q/p$. 
Part 3. More Examples
Additional Example 1

An actuary determines that the claim size for a certain class of accidents is a random variable, \( X \), with moment generating function

\[
M(t) = \frac{1}{(1 - 2500t)^4}
\]

Determine the standard deviation of the claim size for this class of accidents.

Answer: 
\[
M^{(1)}(t) = -4(1 - 2500t)^{-5}(-2500), \quad E(X) = M^{(1)}(0) = 10000
\]

\[
M^{(2)}(t) = (-4)(-5)(1 - 2500t)^{-6}(-2500)(-2500), \quad E(X^2) = M^{(2)}(0) = 120(2500)^2
\]

So \( V(X) = E(X^2) - E^2(X) = 120(2500)^2 - (10000)^2 \).
If the moment generating function for the random variable $X$ is $M(t) = 1/(t+1)$, find $E[(X-2)^3]$:

Answer: $M^{(1)}(t) = -1/(t+1)^2$, $M^{(2)}(t) = 2/(t+1)^3$, $M^{(3)}(t) = -6/(t+1)^4$,
So $E(X) = M^{(1)}(0) = -1$, $E(X^2) = 2$, $E(X^3) = -3/2$,
Additional Problem 1: The value of a piece of factory equipment after three years of use is $100(0.5)^X$ where $X$ is a random variable having moment generating function $M(t) = 1/(1-2t)$ for $t < \frac{1}{2}$. Calculate the expected value of this piece of equipment after three years of use.

Additional Problem 2: A typesetter, on the average, makes one error in every 500 words typeset. A typical page contains 300 words. What is the probability that there will be no more than two errors in five pages?

Additional Problem 3: Consider an urn with 7 red balls and 3 blue balls. Suppose we draw 4 balls without replacement and let $X$ be the total number of red balls we get. Compute $P(X \leq 1)$. 