Translating the Cantor set by a random

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Goal:

Determine how much of the randomness in a “random” real can be cancelled by adding (or subtracting) a member of the Cantor set.

Further, determine the range of effective dimensions of points in a random translate of the Cantor set.

Let $E \subseteq \mathbf{R}^n$. The *diameter* of E, denoted $|E|$, is the maximum distance between any two points in E. We will use card for cardinality. A *cover* $\mathcal{G}$ for a set E is a collection of sets whose union contains E, and $\mathcal{G}$ is a δ*-mesh* cover if the diameter of each member $\mathcal{G}$ is at most δ. For a number $\beta \geq 0$, the β*-dimensional Hausdorff measure* of E, written $\mathcal{H}^\beta(E)$, is given by $\lim_{\delta \to 0} \mathcal{H}^\beta_\delta(E)$ where

$$\mathcal{H}^\beta_\delta(E) = \inf \left\{ \sum_{G \in \mathcal{G}} |G|^\beta : \mathcal{G} \text{ is a countable } \delta\text{-mesh cover of } E \right\}. \tag{1}$$

The *Hausdorff dimension* of a set E, denoted $\dim_{\mathrm{H}}(E)$, is the unique number α where the α-dimensional Hausdorff measure of E transitions from being negligible to being infinitely large; if $\beta < \alpha$, then $\mathcal{H}^\beta(E) = \infty$ and if $\beta > \alpha$, then $\mathcal{H}^\beta(E) = 0$.

Let $S_\delta(E)$ denote the smallest number of sets of diameter at most δ which can cover E. The *upper box-counting dimension* of E is defined as

$$\overline{\dim}_{\mathrm{B}}(E) = \limsup_{\delta \to 0} \frac{\log S_\delta(E)}{-\log \delta}.$$

For all E we have

$$\dim_{\mathrm{H}}(E) \leq \overline{\dim}_{\mathrm{B}}(E).$$

$$\dim_{\mathrm{H}}(A \times B) \geq \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B$$

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The *effective* (or *constructive*) β-*dimensional Hausdorff measure* of a set E, $\mathrm{c}\mathcal{H}^{\beta}(E_k)$, is defined exactly in the same way as Hausdorff measure with the restriction that the covers be uniformly c.e. open sets. This yields the corresponding notion of the *effective* (or *constructive*) *Hausdorff dimension* of a set E, $\mathrm{cdim}_{\mathrm{H}}\, E$.

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Let

$$\mathcal{E}_{\leq\alpha} = \{x : \mathrm{cdim_H}\{x\} \leq \alpha\}.$$

From the above, we know that the effective Hausdorff dimension of $\mathcal{E}_{\leq\alpha}$ satisfies $\mathrm{cdim_H}\, \mathcal{E}_{\leq\alpha} = \alpha$; it turns out (Lutz) that also $\dim_\mathrm{H} \mathcal{E}_{\leq\alpha} = \alpha$.

The *Kolmogorov complexity* of a string σ, denoted $K(\sigma)$, is the length (here we will measure length in ternary units) of the shortest program (under a fixed universal machine) which outputs σ. For a real number x, $x \upharpoonright n$ denotes the first n digits in a ternary expansion of x.

All sequences s of length n have $K(s) \leq n + O(\log n)$; most of them have $K(s) \geq n - O(1)$.

From work of Levin ($\geq$) and Mayordomo ($\leq$) we have for any real number x,

$$\mathrm{cdim}_{\mathrm{H}}\{x\} = \liminf_{n\to\infty} \frac{K(x \upharpoonright n)}{n}. \tag{2}$$

We say a number is *Martin-Löf random* if it “passes” all Martin-Löf tests. A *Martin-Löf test* is a uniformly computably enumerable (c.e.) sequence of open sets $\{U_m\}_{m \in \mathbf{N}}$ with $\lambda(U_m) \leq 2^{-m}$, where λ denotes Lebesgue measure. A number x *passes* such a test if $x \notin \cap_m U_m$.

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Martin-Löf random reals have high initial segment complexity; indeed every Martin-Löf random real r satisfies $\lim_n K(r \upharpoonright n)/n = 1$.

Cantor set $\mathcal{C} \subseteq [0, 1]$:

$$\dim_{\mathrm{H}} \mathcal{C} = \log_3 2 \approx .6309$$

$$\mathcal{C} + \mathcal{C} = [0, 2]$$

$$\frac{1}{2}\mathcal{C} + \frac{1}{2}\mathcal{C} = [0, 1]$$

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$$\operatorname{cdim}_{\mathrm{H}} E = 1/2$$

$$B_1 = \{0, 1\}$$

$$B_2 = \{00, 02, 11\}$$

$$B_3 = \{000, 002, 021, 110, 112\}$$

$$B_4 = \{0000, 0002, 0011, 0200, 0202, 0211, 1100, 1102, 1111\}$$

$$B_5 = \{00000, 00002, 00021, 00112, 00210, 01221, 02012, 02110, 02201, 10212, 11010, 11101, 11120, 11122\}$$

Lemma 1 (Lorentz)**.** *There exists a constant* c *such that for any integer* k*, if* $A \subseteq [0, k)$ *is a set of integers with* $|A| \geq \ell \geq 2$*, then there exists a set of integers* $B \subseteq (-k, k)$ *such that* $A + B \supseteq [0, k)$ *with* $|B| \leq ck\frac{\log \ell}{\ell}$*.*

So we get E of constructive dimension $1 - \dim \mathcal{C}$ such that $E + \mathcal{C} = [0, 2]$.

Theorem 2. *Let* $1 - \dim_{\mathrm{H}} \mathcal{C} \leq \alpha \leq 1$ *and let* $r \in [0, 1]$. *Then*

$$\dim_{\mathrm{H}} \left[(\mathcal{C} + r) \cap \mathcal{E}_{\leq \alpha} \right] \geq \alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}.$$

Assume $1 - \dim_{\mathrm{H}} \mathcal{C} < \alpha < 1$. Split $\mathbf{N}$ into $A \subseteq \mathbf{N}$ and $\bar{A}$; then we can write $\mathcal{C} = \mathcal{C}_A + \mathcal{C}_{\bar{A}}$. Choose A of a suitable density $D = (1-\alpha)/\dim_{\mathrm{H}} \mathcal{C}$ so that $\dim_{\mathrm{H}} \mathcal{C}_A$ comes out to be $1 - \alpha$. Then find a closed set E such that $\operatorname{cdim}_{\mathrm{H}} E \leq \alpha$ and $\mathcal{C}_A + E = [0, 2]$. Let $F = 2 - E$, so that $F - \mathcal{C}_A = [0, 2]$. Let $S = \mathcal{C} \cap (F - r)$; it will suffice to show that $\dim_{\mathrm{H}} S \geq \alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}$.

Now for each $z \in \mathcal{C}$ there exist unique points $v \in \mathcal{C}_A$ and $w \in \mathcal{C}_{\bar{A}}$ such that $v + w = z$; let p be the projection map which takes $z \in \mathcal{C}$ to its unique counterpart $w \in \mathcal{C}_{\bar{A}}$. For each $y \in \mathcal{C}_{\bar{A}}$ we have $r + y \in [0, 2] \subseteq F - \mathcal{C}_A$, so there exists $x \in \mathcal{C}_A$ such that $r + y \in F - x$, which gives $x + y \in S$ since $\mathcal{C}_A + \mathcal{C}_{\bar{A}} = \mathcal{C}$. Thus p maps S onto $\mathcal{C}_{\bar{A}}$. Since p is Lipschitz we have

$$\dim_{\mathrm{H}} S \geq \dim_{\mathrm{H}} \mathcal{C}_{\bar{A}} \geq \alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}$$

because Lipschitz maps do not increase dimension. The theorem follows.

The set E is constructed as before, but the computation of $\operatorname{cdim}_{\mathrm{H}} E$ has an additional complication — we have not assumed that α is computable. It turns out that Kolmogorov complexity methods are useful here.

Specifically, let

$$A = \{\lfloor y/D \rfloor : y \in \mathbf{N}\}.$$

Then initial segments of A are easy to describe:

$$K(A[n]) \leq 4 \log_3 n + O(1).$$

Now a straightforward computation shows that, if $\epsilon > 0$, then for all $x \in E$ and all sufficiently large k we have

$$K(x \upharpoonright m_k) \leq m_k[\alpha + \epsilon + o(1)],$$

which is enough to give $\operatorname{cdim}_{\mathrm{H}} E \leq \alpha$.

Theorem 3. *Let* $1 - \dim_{\mathrm{H}} \mathcal{C} \leq \alpha \leq 1$. *For every Martin-Löf random real* r*,*

$$\dim_{\mathrm{H}} \left[(\mathcal{C} + r) \cap \mathcal{E}_{\leq \alpha} \right] \leq \alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}.$$

Theorem 4. *Let* $1 - \dim_{\mathrm{H}} \mathcal{C} \leq \alpha \leq 1$ *and let* $r \in [0, 1]$ *be Martin-Löf random. Then*

$$\dim_{\mathrm{H}} [(\mathcal{C} + r) \cap \mathcal{E}_{=\alpha}] = \alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}.$$

Moreover,

$$\mathcal{H}^{\alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}} [(\mathcal{C} + r) \cap \mathcal{E}_{=\alpha}] > 0.$$

Questions:

How much can the randomness of r be reduced by adding a Cantor set point if r was not completely random to begin with?

What about sets other than the Cantor set?