Point realizations of Boolean actions

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Outline of Topics

1. Main problem
2. Some answers
3. Groups of isometries and unifying results
4. The borderline case: $C(M, \mathbb{T})$
All metric and topological spaces are assumed to be second countable.
Main problem
$X$ a standard Borel space, for example, $X = [0,1]$
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$\mu$ a Borel probability measure on $X$
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$\text{Aut}(\mu) =$ all measure preserving Boolean transformations of Borel/$\mu$
Halmos–von Neumann:

given $f \in \text{Aut}(\mu)$, $f : \text{Borel}/\mu \rightarrow \text{Borel}/\mu$, 
Halmos–von Neumann:
given \( f \in \text{Aut}(\mu) \), \( f : \text{Borel}/\mu \to \text{Borel}/\mu \),
there exists \( F : X \to X \) a **Borel bijection**
**Halmos–von Neumann:**

given $f \in \text{Aut}(\mu)$, $f : \text{Borel}/\mu \to \text{Borel}/\mu$, there exists $F : X \to X$ a **Borel bijection** such that

$$f(A/\mu) = F[A]/\mu$$

for each Borel set $A \subseteq X$. 
**Halmos–von Neumann:**

given \( f \in \text{Aut}(\mu) \), \( f : \text{Borel}/\mu \to \text{Borel}/\mu \),

there exists \( F : X \to X \) a **Borel bijection** such that

\[
f(A/\mu) = F[A]/\mu
\]

for each Borel set \( A \subseteq X \).

\( f \) has a **point realization** \( F \).
**Topology** on $\text{Aut}(\mu) =$ the weakest topology making all the functions

$$\text{Aut}(\mu) \ni f \rightarrow \mu(f(a)\triangle b) \in \mathbb{R}$$

continuous, where $a, b \in \text{Borel}/\mu$. 
**Topology** on \( \text{Aut}(\mu) \) = the weakest topology making all the functions 

\[
\text{Aut}(\mu) \ni f \rightarrow \mu(f(a) \triangle b) \in \mathbb{R}
\]

continuous, where \( a, b \in \text{Borel}/\mu \).

This is a **Polish group** (separable, completely metrizable) topology on \( \text{Aut}(\mu) \).
$G$ a Polish group
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Consider a **continuous homomorphism**

$$\phi : G \to \text{Aut}(\mu).$$
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Such a homomorphism \( \phi \) can be viewed as an action

\[ G \times (\text{Borel}/\mu) \ni (g, a) \to \phi(g)(a) \in \text{Borel}/\mu. \]
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Actions of this sort are called **Boolean actions**.
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Actions of this sort are called \textbf{Boolean actions}.

We will write $g(a)$ for $\phi(g)(a)$. 

A point realization (or a spatial model) of a continuous homomorphism $G \to \text{Aut}(\mu)$ (or a Boolean action) is a Borel action $G \times X \to X$. 
A **point realization** (or a **spatial model**) of a continuous homomorphism $G \rightarrow \text{Aut}(\mu)$ (or a Boolean action) is a Borel action $G \times X \rightarrow X$ such that

$$g(A/\mu) = g[A]/\mu$$

for all $g \in G$ and $A \in \text{Borel}$. 
A **point realization** (or a **spatial model**) of a continuous homomorphism \( G \to \text{Aut}(\mu) \) (or a Boolean action) is a Borel action \( G \times X \to X \) such that

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With notation \( \phi: G \to \text{Aut}(\mu) \) and \( \alpha: G \times X \to X \),
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for all $g \in G$ and $A \in \text{Borel}$.

With notation $\phi: G \rightarrow \text{Aut}(\mu)$ and $\alpha: G \times X \rightarrow X$, the above equality says

$$\phi(g)(A/\mu) = \{\alpha(g, x): x \in A\}/\mu.$$
Main question:
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For what Polish groups $G$ the following holds each continuous homomorphism $G \to \text{Aut}(\mu)$ (Boolean action) has a point realization?
On non point realizability, whirlyness:
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On non point realizability, whirlyness:

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A continuous homomorphism $G \to \text{Aut}(\mu)$ (a Boolean action) is called **whirly** if for each open $1 \in U \subseteq G$ and each $a \in \text{Borel}/\mu$ with $\mu(a) > 0$,

$$\mu(Ua) = 1.$$

Glasner–Weiss: A whirly Boolean action is not point realizable.

Glasner–Weiss: Ergodic Boolean actions of groups with "concentration of measure" are whirly, so they are not point realizable.
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Some answers
The bad side
Some Polish groups have Boolean actions without point realizations.
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**Vershik ’87**
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**Becker** '02:
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**Becker** ’02:
\[ G = \text{measure classes of measurable subsets of } [0, 1] \text{ with symmetric difference as group operation} \]

**Glasner–Tsirelson–Weiss** ’05:
\[ G = \text{measure classes of measurable functions } [0, 1] \to \mathbb{T} \text{ with pointwise addition as group operation and with convergence in measure} \]
Becker’s example:
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$G$ acts on $\text{Borel}/\mu$, where the underlying standard Borel space is

$$[0, 1] \times \{1, 2\}$$

and $\mu$ is the sum of Lebesgue measures on the two copies of $[0, 1]$. 
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and $\mu$ is the sum of Lebesgue measures on the two copies of $[0, 1]$.

The action:

$$a.(d_1, d_2) =$$

where $a \in G$ and $d_i \in \text{Borel}([0, 1] \times \{i\})/\mu$, $i = 1, 2$. 
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$G$ acts on $\text{Borel}/\mu$, where the underlying standard Borel space is

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and $\mu$ is the sum of Lebesgue measures on the two copies of $[0, 1]$.

The action:

$$a.(d_1, d_2) = ((d_1 \setminus a) \cup (d_2 \cap a), (d_2 \setminus a) \cup (d_1 \cap a)),$$

where $a \in G$ and $d_i \in \text{Borel}([0, 1] \times \{i\})/\mu$, $i = 1, 2$. 
The good side
Recall that $S_\infty$ is the group of all permutations of $\mathbb{N}$ with composition and the topology of pointwise convergence.
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**Mackey ’62:**
If $G$ is locally compact, then point realizations exist.
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If $G$ is a closed subgroup of $S_\infty$ (i.e., is non-archimedean), then point realizations exist.
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**Mackey ’62:**
If $G$ is locally compact, then point realizations exist.

**Glasner–Weiss ’05:**
If $G$ is a closed subgroup of $S_\infty$ (i.e., is non-archimedean), then point realizations exist.

The proofs of these two results were very different.
Groups of isometries and unifying results
Groups of isometries
$X$ a metric space
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\( \text{Iso}(X) \) the group of all isometries of $X$ with composition and pointwise convergence
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$\text{Iso}(X)$ the group of all isometries of $X$ with composition and pointwise convergence

$G$ a Polish group
$G$ is a **Polish group of isometries of** $X$ if $G$ is a subgroup of $\text{Iso}(X)$ as a topological group.
**Uspensky, Gao–Kečhris:** Polish groups = groups of isometries of Polish spaces = groups of the form $\text{Iso}(X)$ for a Polish metric space $X$
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Malicki–S.: locally compact groups = groups of isometries of proper metric spaces = groups of the form \( \text{Iso}(X) \) for a proper metric space \( X \)
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Gao–Kečrís: groups of isometries of locally compact metric spaces $=$ groups of the form $\text{Iso}(X)$ for a locally compact metric space $X$

Examples: locally compact groups, closed subgroups of $S_\infty$, closed subgroups of countable products of locally compact groups

Malicki–S.: locally compact groups $=$ groups of isometries of proper metric spaces $=$ groups of the form $\text{Iso}(X)$ for a proper metric space $X$

Melleray: compact groups $=$ groups of isometries of compact metric spaces $=$ groups of the form $\text{Iso}(X)$ for a compact metric space $X$
The unifying result
Let $G$ be a Polish group of isometries of a locally compact metric space. Then each continuous homomorphism $G \to \text{Aut}(\mu)$ has a point realization.
**Theorem (Kwiatkowska–S.)**

Let $G$ be a Polish group of isometries of a locally compact metric space. Then each continuous homomorphism $G \to \text{Aut}(\mu)$ has a point realization.

The result unifies the theorems of Mackey and Glasner–Weiss.
Theorem (Kwiatkowska–S.)

Let $G$ be a Polish group of isometries of a locally compact metric space. Then each continuous homomorphism $G \to \text{Aut}(\mu)$ has a point realization.

The result unifies the theorems of Mackey and Glasner–Weiss.

New cases: closed subgroups of countable products of locally compact groups.
We need a new characterization of groups of isometries of locally compact metric spaces.
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Recall that for $H$ a subgroup of $G$
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Recall that for $H$ a subgroup of $G$

$$N(H) = \{ g \in G : gHg^{-1} = H \}.$$
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**Theorem (Kwiatkowska–S.)**

Let $G$ be a Polish group. Then $G$ is a group of isometries of a locally compact metric space if and only if
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Recall that for $H$ a subgroup of $G$

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**Theorem (Kwiatkowska–S.)**

Let $G$ be a Polish group. Then $G$ is a group of isometries of a locally compact metric space if and only if for each $U \ni 1$ open there exists $H \subseteq U$ a closed subgroup of $G$ such that $G/H$ is a locally compact space and
We need a new characterization of groups of isometries of locally compact metric spaces.

Recall that for $H$ a subgroup of $G$

$$N(H) = \{g \in G : gHg^{-1} = H\}.$$

**Theorem (Kwiatkowska–S.)**

Let $G$ be a Polish group. Then $G$ is a group of isometries of a locally compact metric space if and only if for each $U \ni 1$ open there exists $H \subseteq U$ a closed subgroup of $G$ such that

- $G/H$ is a locally compact space and
- $N(H)$ is open.
Proofs
An outline of the proof of the second theorem
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Gao–Kechris:

$G$ a Polish group.
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An outline of the proof of the second theorem

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$G$ is an isometry group of a locally compact metric space if and only if $G$ is a **closed subgroup** of a **countable product** of groups of the form $S^\infty \rtimes H^N$, where $H$ is locally compact and $S^\infty$ acts by homomorphisms on $H^N$ by permuting coordinates.
An outline of the proof of the second theorem

Gao–Kechris:

$G$ a Polish group.

$G$ is an isometry group of a locally compact metric space if and only if $G$ is a closed subgroup of a countable product of groups of the form

$$S_\infty \ltimes H^\mathbb{N},$$

where $H$ is locally compact and $S_\infty$ acts by homomorphisms on $H^\mathbb{N}$ by permuting coordinates.
Condition (✳) on \( G \):
**Condition (\(\ast\)) on \(G\):**

\[
\forall U \ni 1 \text{ open } \exists H \subseteq U \text{ a closed subgroup of } G \text{ such that}
\]
\[
G/H \text{ is locally compact;}
\]
\[
N(H) \text{ is open.}
\]
**Condition** ($\ast$) on $G$:

$\forall U \ni 1$ open $\exists H \subseteq U$ a closed subgroup of $G$ such that

- $G/H$ is locally compact;
- $N(H)$ is open.

**Lemma**

(i) $S_\infty \times H^\mathbb{N}$, where $H$ is locally compact, has ($\ast$).
**Condition** (\(\ast\)) on \(G\):
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\[
G/H \text{ is locally compact;}
\]
\[
N(H) \text{ is open.}
\]

**Lemma**

(i) \(S_\infty \ltimes H^\mathbb{N}\), where \(H\) is locally compact, has \((\ast)\).

(ii) Condition \(\ast\) is preserved under taking countable products.
Lemma

Condition (*) is preserved under taking closed subgroups.
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Proof uses
Lemma

Condition (*) is preserved under taking closed subgroups.

Proof uses Yamabe’s theorem connecting locally compact groups with Lie groups (Hilbert’s 5-th problem)
Lemma

Condition (\(\ast\)) is preserved under taking closed subgroups.

Proof uses Yamabe’s theorem connecting locally compact groups with Lie groups (Hilbert’s 5-th problem) and well behaved dimension on Lie groups.
The borderline case: \( C(M, \mathbb{T}) \)
Let $M$ be a compact metric space.
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Let $C(M, \mathbb{T})$ be the group of all continuous functions from $M$ to $\mathbb{T}$ with pointwise multiplication and with the uniform convergence topology.
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Let $C(M, \mathbb{T})$ be the group of all continuous functions from $M$ to $\mathbb{T}$ with pointwise multiplication and with the uniform convergence topology.

$C([0, 1], \mathbb{T})$ lies exactly between
\{ $f : [0, 1] \rightarrow \mathbb{T}$ measurable $\}$ (which has whirly actions)
and
groups with property (\ast) (which have point realizations).
Let $M$ be a compact metric space. Let $C(M, \mathbb{T})$ be the group of all continuous functions from $M$ to $\mathbb{T}$ with pointwise multiplication and with the uniform convergence topology.

$C([0, 1], \mathbb{T})$ lies exactly between 
\{ $f: [0, 1] \to \mathbb{T}$ measurable \} (which has whirly actions) 
and 
groups with property (*) (which have point realizations).

It does not have “concentration of measure.”
Theorem (Moore–S.)

Let $M$ be a compact uncountable metric space. The group $C(M, \mathbb{T})$ has a whirly Boolean action.
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Let $M$ be a compact uncountable metric space. The group $C(M, \mathbb{T})$ has a whirly Boolean action (so a Boolean action without a point realization).
An outline of proof in the case $M = 2^\mathbb{N}$
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Identify $\mathbb{C}$ with $\mathbb{R}^2$. 
The borderline case: $C(M, T)$

An outline of proof in the case $M = 2^\mathbb{N}$

Identify $\mathbb{C}$ with $\mathbb{R}^2$.

Let $\gamma$ be the standard Gaussian measure on $\mathbb{C}$ with density

$$
\frac{1}{2\pi} e^{-\frac{1}{2}(x_0^2 + x_1^2)}
$$
The borderline case: $C(M, T)$

Note

$\gamma$ is preserved under rotations of $C$ by elements of $T$, $\pi$:

$C \times C \ni (z_1, z_2) \mapsto z_1 + z_2 \sqrt{2} \in C$ is measure preserving if $C \times C$ is taken with $\gamma \times \gamma$ and $C$ with $\gamma$,

$\iota: T \ni z \mapsto (z, z) \in T \times T$ is a continuous embedding.
Note $\gamma$ is **preserved** under rotations of $\mathbb{C}$ by elements of $\mathbb{T}$. 
Note $\gamma$ is **preserved** under rotations of $\mathbb{C}$ by elements of $\mathbb{T}$,

$$
\pi : \mathbb{C} \times \mathbb{C} \ni (z_1, z_2) \rightarrow \frac{z_1 + z_2}{\sqrt{2}} \in \mathbb{C}
$$

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Note $\gamma$ is **preserved** under rotations of $\mathbb{C}$ by elements of $\mathbb{T}$,

$$\pi: \mathbb{C} \times \mathbb{C} \ni (z_1, z_2) \rightarrow \frac{z_1 + z_2}{\sqrt{2}} \in \mathbb{C}$$

is **measure preserving** if $\mathbb{C} \times \mathbb{C}$ is taken with $\gamma \times \gamma$ and $\mathbb{C}$ with $\gamma$, and

$$\iota: \mathbb{T} \ni z \rightarrow (z, z) \in \mathbb{T} \times \mathbb{T}$$

is a **continuous embedding**.
The borderline case: $C(M, T)$

$$(C, \gamma)$$

$\uparrow$

$T$
The borderline case: $C(M, \mathbb{T})$

\[
\begin{array}{c}
(\mathbb{C}, \gamma) \xleftarrow{\pi} (\mathbb{C}^2, \gamma^2) \\
\uparrow \quad \quad \quad \uparrow \\
\mathbb{T} \quad \xrightarrow{\iota} \quad \mathbb{T}^2
\end{array}
\]
The borderline case: \( C(M, T) \)

\[
(\mathbb{C}, \gamma) \leftarrow \pi \rightarrow (\mathbb{C}^2, \gamma^2) \leftarrow \pi^2 \rightarrow (\mathbb{C}^4, \gamma^4)
\]

\[\uparrow \quad \uparrow \quad \uparrow \quad \uparrow\]

\[T \quad \overset{i}{\longrightarrow} \quad \mathbb{T}^2 \quad \overset{i^2}{\longrightarrow} \quad \mathbb{T}^4\]
The borderline case: $C(M, \mathbb{T})$

$$(\mathbb{C}, \gamma) \xleftarrow{\pi} (\mathbb{C}^2, \gamma^2) \xleftarrow{\pi^2} (\mathbb{C}^4, \gamma^4) \xleftarrow{\pi^3} (\mathbb{C}^8, \gamma^8)$$

$\mathbb{T} \xrightarrow{i} \mathbb{T}^2 \xrightarrow{i^2} \mathbb{T}^4 \xrightarrow{i^3} \mathbb{T}^8$$
The borderline case: $C(M, T)$

$$(C, \gamma) \leftarrow \pi \quad (C^2, \gamma^2) \leftarrow \pi^2 \quad \cdots \lim (C^{2^n}, \gamma^{2^n})$$

$$\uparrow \quad \uparrow \quad \uparrow$$

$$(\mathbb{T}, \iota) \rightarrow \mathbb{T}^2 \rightarrow \iota^2 \rightarrow \cdots \lim \mathbb{T}^{2^n}$$
The borderline case: $C(M, \mathbb{T})$

$\pi$ $\pi^2$ $\lim$ $\subseteq$

$\varphi$ $\psi$ $\varphi^2$ $\lim \varphi^{2^n} \subseteq \varphi^{\infty}$

$\mathbb{T}$ $\mathbb{T}^2$ $\mathbb{T}^{2^n} \subseteq \mathbb{C}(2^\mathbb{N}, \mathbb{T})$
The borderline case: $C(M, T)$

\[(C, \gamma) \leftarrow \pi \leftarrow (C^2, \gamma^2) \leftarrow \pi^2 \leftarrow \cdots \lim(C^{2n}, \gamma^{2n}) \rightarrow (C^\infty, \gamma^\infty)\]

\[
\begin{align*}
\uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow & \\
T & \rightarrow \quad T^2 & \rightarrow \quad T^{2^2} & \rightarrow \quad \lim T^{2^n} & \subseteq \quad C(2^N, T)
\end{align*}
\]
We get a Boolean action of $C(2^N, \mathbb{T})$ on the probability measure space $(C^\infty, \gamma^\infty)$. 
The proof of the following result is important for the proof of non-point realizability:
The proof of the following result is important for the proof of non-point realizability:

If \( a \in \mathbb{R} \) and \( B \subseteq \mathbb{R}^N \) is a Borel set of positive \( \gamma^N \)-measure, then

\[
\gamma^N(\sqrt{1 + a^2 B + ay}) > 0, \quad \text{for } \gamma^N\text{-a.e. } y \in \mathbb{R}^N.
\]
Question.
**Question.** Characterize Polish groups with the point realization property?