Dynamics of the homeomorphism group of the Lelek fan

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joint work with Dana Bartošová

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PART 1

The Lelek fan
Lelek fan

- $C$ – the Cantor set
Lelek fan

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- continuum - compact and connected metric space
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- **Cantor fan** $F$ is the cone over the Cantor set:
  
  $C \times [0, 1]/C \times \{0\}$
Lelek fan

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Lelek fan

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- **Lelek fan** $L$ is a subfan of the Cantor fan with a dense set of endpoints in $L$
Lelek fan

Universal minimal flow of $H(L)$
More applications to the dynamics of $H(L)$
Let $\mathcal{F}$ be the family of all finite reflexive fans (graphs $(A, R^A)$ as below)
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The family $\mathcal{F}$ has the following two properties:

1. (F1) (joint projection property: JPP) for any $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ and epimorphisms from $C$ onto $A$ and from $C$ onto $B$;

2. (F2) (amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and any epimorphisms $\phi_1 : B_1 \to A$ and $\phi_2 : B_2 \to A$, there exist $C$, $\phi_3 : C \to B_1$ and $\phi_4 : C \to B_2$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$. 
Construction of the Lelek fan, part 3

1. $\mathcal{F}$ is an example of a projective Fraïssé family
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2. By a theorem of Irwin and Solecki there is a unique limit $\mathbb{L}$ of $\mathcal{F}$, the projective Fraïssé limit.
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2. By a theorem of Irwin and Solecki there is a unique limit $\mathbb{L}$ of $\mathcal{F}$, the projective Fraïssé limit.
3. $\mathbb{L}$ is the Cantor set equipped with a closed binary relation $R^\mathbb{L}$, and it satisfies:
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3. $\mathbb{L}$ is the Cantor set equipped with a closed binary relation $R^\mathbb{L}$, and it satisfies:

4. (L1) (projective universality) for any $A \in \mathcal{F}$ there is an epimorphism from $\mathbb{L}$ onto $A$

5. (L2) (projective ultrahomogeneity) for any $A \in \mathcal{F}$ and any epimorphisms $\phi_1 : \mathbb{L} \to A$ and $\phi_2 : \mathbb{L} \to A$ there exists an isomorphism $h : \mathbb{L} \to \mathbb{L}$ such that $\phi_2 = \phi_1 \circ h$
Construction of the Lelek fan, part 4

Proposition

The relation $R^L_S$, where $R^L_S(x, y)$ iff $R^L(x, y)$ or $R^L(y, x)$, is an equivalence relation such that each equivalence class has at most two elements.
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Theorem

\( L/R^L_S \) is the Lelek fan.
Properties of the Lelek fan

$H(L)$ – the homeomorphism group of the Lelek fan

1. $H(L)$ is totally disconnected
2. $H(L)$ is generated by every neighbourhood of the identity
3. $H(L)$ has a dense conjugacy class
4. $H(L)$ is simple
PART 2

The universal minimal flow of $H(L)$
Universal minimal flows

$G$ – topological group

**Definition**

A *$G$-flow* is a continuous action of $G$ on a compact Hausdorff space.
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Universal minimal flows

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A **G-flow** is a continuous action of \( G \) on a compact Hausdorff space.

**Definition**

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**Definition**

The **universal minimal flow** of \( G \) is the unique minimal \( G \)-flow that has all other minimal \( G \)-flows as its homomorphic images.
**Extreme amenability**

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- Therefore every extremely amenable group is amenable.
Extreme amenability

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We say that $G$ is **extremely amenable** if every $G$-flow has a fixed point.

- A topological group $G$ is **amenable** if every continuous action of $G$ on a compact Hausdorff space has an invariant Borel probability measure.
- Therefore every extremely amenable group is amenable.
- If $G$ is non-trivial and locally compact then $G$ is **not** extremely amenable. (Veech)
Examples of extremely amenable groups

- the isometry group of the Urysohn metric space (Pestov)
- the group of all linear isometries of the Gurarij Banach space (Bartosova, Lopez-Abad, Mbombo)
- the group of all measure preserving transformations of $([0, 1], \lambda)$ (Giordano-Pestov)
Examples of extremely amenable groups

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- many automorphism groups of countable structures (ex. of rationals, of the random ordered graph,...) (Kechris-Pestov-Todorcevic)
Let $\mathcal{G}$ be a projective Fraïssé family with the projective Fraïssé limit $\mathbb{G}$. 
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Let $G = \text{Aut}(\mathcal{G})$ be the automorphism group of $\mathcal{G}$.

**Theorem**

The following are equivalent

1. *The group $G$ is extremely amenable.*
2. *The family $\mathcal{G}$ is a projective Ramsey class.*
Main question

Question

What is the universal minimal flow of $H(L)$?
Chains

- $K$ – compact Hausdorff topological space
Chains

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- A **chain** $C$ on $K$ is a family of closed subsets of $K$ such that for every $C_1, C_2 \in C$, either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. 
Chains

- $K$ – compact Hausdorff topological space
- A chain $\mathcal{C}$ on $K$ is a family of closed subsets of $K$ such that for every $C_1, C_2 \in \mathcal{C}$, either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.
- A chain $\mathcal{C}$ is maximal if for every closed set $D \subseteq K$, if $\{D\} \cup \mathcal{C}$ is a chain then $D \in \mathcal{C}$.  

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Dynamics of the homeomorphism group of the Lelek fan
If $K$ is a fan (ex: a finite fan from $\mathcal{F}$ or the Lelek fan $L$), say that a chain is \textit{compatible} if it is compatible with respect to the linear order on each segment/branch of the fan.
Main theorem

Let

\[ H = \{ h \in H(L) : h(C^L) = C^L \}, \]

where \( C^L \) is a “generic” chain on \( L \).
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Theorem

The universal minimal flow of \( H(L) \) is equal to \( H(L) \bowtie \widehat{H(L)}/H \), the completion of \( H(L)/H \).
The generic chain on $L$

- Let $\mathcal{F}_c$ be the family of all finite reflexive fans, each equipped with a maximal compatible chain.
The generic chain on $L$

- Let $\mathcal{F}_c$ be the family of all finite reflexive fans, each equipped with a maximal compatible chain.
- We can take a limit of $\mathcal{F}_c$.
- This limit will be $\mathbb{L}_c = (\mathbb{L}, \mathcal{C}^\mathbb{L})$, where $\mathcal{C}^\mathbb{L}$ – a maximal compatible chain on $\mathbb{L}$. 
The generic chain on $L$

- Let $\mathcal{F}_c$ be the family of all finite reflexive fans, each equipped with a maximal compatible chain.
- We can take a limit of $\mathcal{F}_c$.
- This limit will be $L_c = (\mathbb{I}, C^\mathbb{I})$, where $C^\mathbb{I}$ – a maximal compatible chain on $\mathbb{I}$.
- Define the generic chain $C^L$ on $L$ to be $\pi(C^\mathbb{I})$, where $\pi: \mathbb{I} \rightarrow L$ denotes the projection.
Main steps in the proof of the Main Theorem

Finding the expansion $\mathcal{F}_c$ of $\mathcal{F}$ and showing
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**Theorem**

1. $\mathcal{F}_c$ is a projective Ramsey class (needed for the universality).
2. $\mathcal{F}_c$ has the expansion property (needed for the minimality).
3. $\mathcal{F}_c$ is a precompact expansion
Main steps in the proof of the Main Theorem

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**Theorem**

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**Theorem**

The group $\text{Aut}(\mathcal{F}_c)$ is extremely amenable.
And then:

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$$\text{Aut}(\mathbb{L}) \equiv \text{Aut}(\mathbb{L})/H_0,$$

where $H_0 = \{ f \in \text{Aut}(\mathbb{L}) : f(C^\mathbb{L}) = C^\mathbb{L} \}$. 

The universal minimal flow
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  \[ \text{Aut} (\mathbb{L}) \bowtie \widehat{\text{Aut}} (\mathbb{L}) / H_0, \]
  where $H_0 = \{ f \in \text{Aut} (\mathbb{L}) : f (C^\mathbb{L}) = C^\mathbb{L} \}$.
- Deduce that the universal minimal flow of $H (L)$ is
  \[ H (L) \bowtie \widehat{H} (L) / H, \]
  where $H = \{ h \in H (L) : h (C^L) = C^L \}$. 
PART 3

More applications to the dynamics of $H(L)$
Fans with ordered branches

\[ \mathcal{F}_{\leq} \] – the family of all finite reflexive fans with an order on the set of branches.
Fans with ordered branches

- $\mathcal{F}_\leq$ – the family of all finite reflexive fans with an order on the set of branches.
- Precisely: $A_\leq = (A, R^A, \leq^A) \in \mathcal{F}_\leq$ iff $(A, R^A) \in \mathcal{F}$ and for some ordering $a_1 < a_2 < \ldots < a_n$ of branches in $A$ we have $x \leq^A y$ if and only if there are $i \leq j$ such that $x \in a_i$ and $y \in a_j$. 

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- This is a projective Fraïssé family.

- \(\mathbb{L}_\leq\) – the projective Fraïssé limit of \(\mathcal{F}_\leq\)
The class $\mathcal{F}_\leq$ is a projective Ramsey class.
Theorem – $\mathcal{F}_{\leq}$

**The class $\mathcal{F}_{\leq}$ is a projective Ramsey class.**

To show the theorem above we need:

- The generalization of the finite Gowers’ Ramsey Theorem.
The class $\mathcal{F}_{\leq}$ is a projective Ramsey class.

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- The generalization of the finite Gowers’ Ramsey Theorem.
- The theorem about size-insensitivity.
\[ \text{FIN}_k(n) = \{ p : \{1, \ldots, n\} \to \{0, 1, \ldots, k\} : \exists l \ (p(l) = k) \} \]
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$\text{supp}(p) = \{ l \in \{1, \ldots, n\} : p(l) \neq 0 \}$
Operations on $\text{FIN}_k(n)$

**OPERATIONS**

$+ \quad p + q$ is defined to be $p + q$, whenever $\max(\text{supp}(p)) < \min(\text{supp}(q))$
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**$T_i^{(k)}$** For every $i = 1, 2, \ldots, k$ we have a function

$T_i^{(k)} : \text{FIN}_k(n) \rightarrow \text{FIN}_{k-1}(n)$

\[
T_i^{(k)}(p)(l) = \begin{cases} 
p(l) & \text{if } p(l) < i \\
p(l) - 1 & \text{if } p(l) \geq i.
\end{cases}
\]
Operations on $\text{FIN}_k(n)$

**Operations**

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$T^{(k)}_i$ For every $i = 1, 2, \ldots, k$ we have a function $T^{(k)}_i : \text{FIN}_k(n) \to \text{FIN}_{k-1}(n)$

$$T^{(k)}_i(p)(l) = \begin{cases} p(l) & \text{if } p(l) < i \\ p(l) - 1 & \text{if } p(l) \geq i. \end{cases}$$

Let also $T^{(k)}_0 = \text{id} \upharpoonright_{\text{FIN}_k}$. 

Lelek fan
Universal minimal flow of $H(L)$
More applications to the dynamics of $H(L)$
Generalization of the finite version of the Gowers’ Ramsey Theorem, part 1

A sequence $B = (b_s)_{s=1}^m$ of elements of $\text{FIN}_k(n)$ is called a block sequence if for every $s$,

$$\max(\text{supp}(b_s)) < \min(\text{supp}(b_{s+1})).$$
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Let $\langle B \rangle_k$ denote the subset of $\text{FIN}_k(n)$ of

$$\sum_{s=1}^m T_{\vec{i_s}}(b_s),$$

such that $\vec{i_s} \in \prod_{j=1}^k \{0, 1, \ldots, j\}$ and there is an $s$ with $T_{\vec{i_s}}(b_s) = b_s$. 
Generalization of the finite version of the Gowers’ Ramsey Theorem, part 2

**Theorem**

Let $k, m, r$ be natural numbers. Then there exists a natural number $n$ such that for every colouring $c : \text{FIN}_k(n) \to \{1, 2, \ldots, r\}$ there exists a block sequence $B$ of length $m$ in $\text{FIN}_k(n)$ such that $\langle B \rangle_k$ is $c$-monochromatic.
Generalization of the finite version of the Gowers’ Ramsey Theorem, part 2

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We proved and need a more general Ramsey theorem.
Let $\text{Aut}(\mathbb{L})$ and $\text{Aut}(\mathbb{L}_\leq)$ be the automorphism groups of $\mathbb{L}$ and $\mathbb{L}_\leq$.
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Theorem

The group $\text{Aut}(\mathbb{L}_{\leq})$ is extremely amenable.
Theorem – $H(L)$

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- Let $\leq_L = \pi(\leq_{\mathbb{L}})$, where $\pi: \mathbb{L} \to L$ is the quotient map.
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- Let $\leq_L = \pi(\leq_I)$, where $\pi : I \to L$ is the quotient map.

\[
H(L_{\leq}) = \\
\{ h \in H(L) : \text{for every } x, y \in L \ (x \leq_L y \implies h(x) \leq_L h(y)) \}
\]
Theorem – $H(L)$

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- Let $\leq_L = \pi(\leq_I)$, where $\pi: I \to L$ is the quotient map.

$$H(L_{\leq}) = \{ h \in H(L) : \text{for every } x, y \in L (x \leq_L y \implies h(x) \leq_L h(y)) \}$$

Theorem

*The group $H(L_{\leq})$ is extremely amenable.*