Introduction

The \( \text{FIN}_k \) theorem was formulated and proved by Gowers [Gow92] to obtain a stabilization theorem for \( c_0 \), the space of sequences converging to zero with the supremum norm. The theorem is a generalization of Hindman’s theorem.
Let $PS_{c_0} = \{(a_n) \in S_{c_0} : a_n > 0 \text{ for all } n \in \mathbb{N}\}$. 
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The structure $FIN_k$ is a discretization of the positive unit sphere of $c_0$. 
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The structure $FIN_k$ is a discretization of the positive unit sphere of $c_0$.

The structure $FIN_k^\pm$ is a discretization of the unit sphere of $c_0$. 

The structure $\text{FIN}_k$

For a fixed $k \in \mathbb{N}$, $\text{FIN}_k$ is the set of all finitely supported functions $f : \mathbb{N} \to \{0, 1, \ldots, k\}$ that attain the maximum value $k$. 
The structure $\text{FIN}_k$

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1. **Sum**: $(f + g)(x) = f(x) + g(x)$ for $f < g$.
2. **Tetris**: $T(f)(x) = \max\{0, f(x) - 1\}$.

The space generated by a block sequence $(f_i)_{i \in I}$ is $\langle f_i \rangle_{i \in \mathbb{N}} = \{T_l^1(f_{i_1}) + \ldots + T_l^n(f_{i_n}) : i_1 < \ldots < i_n, \min\{l_1, \ldots, l_n\} = 0\}$. 

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Finite forms of Gowers' $\text{FIN}_k$ Theorem
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(i) Sum: $(f + g)(x) = f(x) + g(x)$ for $f < g$, 

(ii) Tetris: $T : \text{FIN}_k \rightarrow \text{FIN}_{k-1}$.

For $f \in \text{FIN}_k$, $(Tf)(x) = \max\{0, f(x) - 1\}$. The space generated by a block sequence $(f_i)_{i \in \mathbb{N}}$ is $\langle f_i \rangle_{i \in \mathbb{N}} = \{Tl_1(f_{i_1}) + \ldots + Tl_n(f_{i_n}) : i_1 < \ldots < i_n, \min\{l_1, \ldots, l_n\} = 0\}$. 

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The structure $\text{FIN}_k$
Theorem

[Gow92] For any finite coloring $c : \text{FIN}_k \rightarrow r$ there exists a block sequence $(f_i)_{i \in \mathbb{N}}$ that generates a monochromatic subspace.
The structure $\text{FIN}^\pm_k$

For a fixed $k \in \mathbb{N}$, $\text{FIN}^\pm_k$ is the set of all functions $f : \mathbb{N} \to \{-k, \ldots, -1, 0, 1, \ldots, k\}$ that attain one of the values $k$ or $-k$ at least once and whose support is finite.
The structure $\text{FIN}^\pm_k$

For a fixed $k \in \mathbb{N}$, $\text{FIN}^\pm_k$ is the set of all functions $f : \mathbb{N} \rightarrow \{-k, \ldots, -1, 0, 1, \ldots, k\}$ that attain one of the values $k$ or $-k$ at least once and whose support is finite. We consider two operations in $\text{FIN}^\pm_k$ defined pointwise as follows:

(i) \textit{Sum:} $(f + g)(n) = f(n) + g(n)$ for $f < g$,

(ii) \textit{Tetris:} $T : \text{FIN}^\pm_k \rightarrow \text{FIN}^\pm_{k-1}$. For $f \in \text{FIN}^\pm_k$,

$$(Tf)(n) = \begin{cases} f(n) - 1 & \text{if } f(n) > 0 \\ f(n) + 1 & \text{if } f(n) < 0 \\ 0 & \text{otherwise.} \end{cases}$$
The structure $\text{FIN}_k^\pm$

For a fixed $k \in \mathbb{N}$, $\text{FIN}_k^\pm$ is the set of all functions $f : \mathbb{N} \rightarrow \{-k, \ldots, -1, 0, 1, \ldots, k\}$ that attain one of the values $k$ or $-k$ at least once and whose support is finite. We consider two operations in $\text{FIN}_k^\pm$ defined pointwise as follows:

(i) Sum: $(f + g)(n) = f(n) + g(n)$ for $f < g$,
(ii) Tetris: $T : \text{FIN}_k^\pm \rightarrow \text{FIN}_{k-1}^\pm$. For $f \in \text{FIN}_k^\pm$,

$$(Tf)(n) = \begin{cases} f(n) - 1 & \text{if } f(n) > 0 \\ f(n) + 1 & \text{if } f(n) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

The space generated by a block sequence $(f_i)_{i \in I}$ is

$$\langle f_i \rangle_{i \in I}^\pm = \{ \delta_0 T^{l_0}(f_{i_0}) + \ldots + \delta_n T^{l_n}(f_{i_n}) : i_0 < \ldots < i_n, \min\{l_0, \ldots, l_n\} = 0, \delta_i = \pm 1, n \in \mathbb{N} \}.$$
The $\text{FIN}_k^\pm$ theorem

Theorem

[ Gow92 ] For any finite coloring $c : \text{FIN}_k^\pm \to \{0, \ldots, r-1\}$ there exists an infinite block sequence $(f_i)_{i \in \mathbb{N}}$ such that the space $\langle f_i \rangle_{i \in \mathbb{N}}^\pm$ generated by the sequence $(f_i)_{i \in \mathbb{N}}$ is almost monochromatic; in the sense that there exists $i < r$ such that $\langle f_i \rangle_{i \in \mathbb{N}}^\pm \subseteq (c^{-1}(i))_1$. 

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The \( \text{FIN}^\pm_k \) theorem

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In the structure \( \text{FIN}^\pm_k \) we consider the metric defined by
\[
\| f - g \|_\infty = \max\{ |f(n) - g(n)| : n \in \mathbb{N} \}.
\]
For \( A \subset \text{FIN}^\pm_k \) let
\[
(A)_1 = \{ g : \| f - g \|_\infty \leq 1 \text{ for some } f \in A \}. 
\]
Theorem

[Hin74] For every finite coloring of $\text{FIN}$ there exists an infinite sequence $x_0 < x_1 < \ldots$ such that all the finite unions of elements of the sequence have the same color.
Hindman’s theorem, Folkman’s theorem

Theorem

\[\text{[Hin74]}\] For every finite coloring of \(\text{FIN}\) there exists an infinite sequence \(x_0 < x_1 < \ldots\) such that all the finite unions of elements of the sequence have the same color.

Theorem

\[\text{[NR83]}\] For every \(n \in \mathbb{N}\) there exists \(M = M(n)\) such that for any coloring \(c : \text{FIN}(M) \rightarrow 2\) there exists a sequence \(x_0 < \ldots < x_{n-1}\) of finite subsets of \(M\) such that the set of finite unions of the sets \(x_0, \ldots, x_{n-1}\) is monochromatic.
In [OA] we obtain direct combinatorial proofs of the finite versions of the $\text{FIN}_k$ Theorem and of the $\text{FIN}_k^\pm$ Theorem.
Theorem

For any $m, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every 2-coloring of $\text{FIN}_k(n)$, the functions in $\text{FIN}_k$ supported below $n$, there exists a block sequence in $\text{FIN}_k(n)$ of length $m$ that generates a monochromatic subspace.
Theorem

For any $m, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every 2-coloring of $\text{FIN}_k(n)$, the functions in $\text{FIN}_k$ supported below $n$, there exists a block sequence in $\text{FIN}_k(n)$ of length $m$ that generates a monochromatic subspace.

Let $g_k(m)$ be the minimal $n$ given by the theorem. The bounds found in [OA] for $k > 1$ in are

$$g_k(n) \leq f_{4+2(k-1)} \circ f_4(6m),$$

where for $i \in \mathbb{N}$, $f_i$ denotes the $i$-th function in the Ackermann Hierarchy.
The proof goes by induction on $k$. The strategy is to code an element of $\text{FIN}_{k+1}$ in a sequence of elements of $\text{FIN}_k$ and apply the result for $k$ and its higher dimensional versions.
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For $k, n, d \in \mathbb{N}$, the $d$-dimensional version of $\text{FIN}_k(n)$ is

$$\text{FIN}_k(n)[d] = \{(f_i)_{i < d} \mid f_i \in \text{FIN}_k(n) \text{ and } f_i < f_j \text{ for } i < j < d\}.$$
The proof goes by induction on $k$. The strategy is to code an element of $\text{FIN}_{k+1}$ in a sequence of elements of $\text{FIN}_k$ and apply the result for $k$ and its higher dimensional versions.

For $k, n, d \in \mathbb{N}$, the $d$-dimensional version of $\text{FIN}_k(n)$ is

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Similarly, if $(f_i)_{i<l}$ is a block sequence, we define $(\langle f_i \rangle_{i<l})[d]$ to be the collection of block subsequences of $(f_i)_{i<l}$ of length $d$. 
Higher-dimensional version

The formulation of the theorem, including dimensions is

**Theorem**

For every $k, m, d \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every coloring $c : \text{FIN}_k(n)^[d] \to 2$ there exists $(f_i)_{i < m} \in \text{FIN}_k(n)^[m]$ such that $c \upharpoonright (\langle f_i \rangle_{i < m})^[[d]]$ is constant.
The following definition is important when coding an element of $\text{FIN}_{k+1}$ in a sequence of elements of $\text{FIN}_k$. 
The following definition is important when coding an element of $\text{FIN}_{k+1}$ in a sequence of elements of $\text{FIN}_k$.

Given $f = (f_i)_{i < m} \in \text{FIN}^{[m]}_{k+1}$, for

$$g = \sum_{i < m} T^{n_i} f_i,$$

we define $\text{supp}^f_{k+1}(g)$ to be the set of all $i < m$ such that $n_i = 0$. 
The $\text{FIN}_k$ Theorem
Precursors in Ramsey theory
The finite versions
Closing remarks

The positive case
Working with signs

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Finite forms of Gowers' $\text{FIN}_k$ Theorem
Lemma

[OA] For every $N \in \mathbb{N}$ there exists $\bar{N}$ such that for every $c : \text{FIN}_{k+1}(\bar{N}) \rightarrow 2$ there exists $h = (h_i)_{i < N} \in \text{FIN}_{k+1}(\bar{N})^{[N]}$ such that for

$$f = \sum_{i < N} T^{s_i}(h_i)$$

$$g = \sum_{i < N} T^{t_i}(h_i),$$

$c(f) = c(g)$ whenever $\text{supp}^h_{k+1}(f) = \text{supp}^h_{k+1}(g)$, that is, whenever for all $i < N$, $s_i = 0$ if and only if $t_i = 0$. 
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The positive case
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Assume the result for $k$. Given a coloring of $\text{FIN}_{k+1}$, use the result for $k$ and its higher dimensional versions and reduce the coloring to a coloring of finite sets. Finally apply Folkman’s Theorem.
Theorem

For any $m, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every 2-coloring $c$ of $\text{FIN}_k^\pm(n)$, the functions in $\text{FIN}_k^\pm$ supported below $n$, there exist $i < 2$ and a block sequence $(f_i)_{i < m}$ in $\text{FIN}_k^\pm(n)$ such that $\langle f_i \rangle_{i < m}^\pm \subseteq (c^{-1}(i))_1$. 

Let $g_k^\pm(n)$ be the minimal $n$ given by the theorem. The bounds we find in [OA] for $k > 1$ are $g_k(n) \leq f_{4+2((k-1)\circ f_4)(12m)}$, where for $i \in \mathbb{N}$, $f_i$ denotes the $i$-th function in the Ackermann Hierarchy.
**Theorem**

*For any $m, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every 2-coloring $c$ of FIN\(\pm\)\(_k\)(n), the functions in FIN\(\pm\)\(_k\) supported below $n$, there exist $i < 2$ and a block sequence $(f_i)_{i \leq m}$ in FIN\(\pm\)\(_k\)(n) such that $\langle f_i \rangle_{i < m} \subseteq (c^{-1}(i))_1$.  

Let $g_k^\pm(m)$ be the minimal $n$ given by the theorem. The bounds we find in [OA] for $k > 1$ are

$$g_k(n) \leq f_{4+2(k-1)} \circ f_4(12m),$$

where for $i \in \mathbb{N}$, $f_i$ denotes the $i$-th function in the Ackermann Hierarchy.
We prove the $\text{FIN}_k^\pm$ Theorem using the following lemma:

**Lemma**

[Kan04] For all $m, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every 2-coloring of $\text{FIN}_k^\pm(n)$, the functions in $\text{FIN}_k^\pm$ supported below $n$, there exists a block sequence $(f_i)_{i < m}$ of elements of $\text{FIN}_k^\pm(n)$ such that the set

$$\langle f_i \rangle_{i < m} = \left\{ (-T)^{l_0} f_{i_0} + \ldots + (-T)^{l_s} f_{i_s} : i_0 < \ldots < i_s \right\}$$

$$\min\{l_0, \ldots, l_s\} = 0, s \leq m$$

is monochromatic.
Let $\text{FIN}_k^-(n)$ be the set of functions $f : \mathbb{N} \to \{(-1)^j(k-j) : j = 0, 1, \ldots, k\}$ supported below $n$. 
Let $\text{FIN}^{-}_k(n)$ be the set of functions $f : \mathbb{N} \to \{(-1)^j(k - j) : j = 0, 1, \ldots, k\}$ supported below $n$.

The structures $(\text{FIN}_k(n), +, T)$ and $(\text{FIN}^{-}_k(n), +, -T)$ are isomorphic as witnessed by the map $I : \text{FIN}^{-}_k(n) \to \text{FIN}_k(n)$ that sends each $f \in \text{FIN}^{-}_k(n)$ to its pointwise absolute value.
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This follows from the fact that $I^{-1} \circ T(g) = -T \circ I^{-1}(g)$ for all $g \in \text{FIN}_k^n$. 


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This follows from the fact that $I^{-1} \circ T(g) = -T \circ I^{-1}(g)$ for all $g \in \text{FIN}_k^n$.

Note that the inverse image of a block sequence $(g_i)_{i < m}$ in $\text{FIN}_k(n)$ is a block sequence in $\text{FIN}_k^-(n)$, and $I^{-1}(\langle g_i \rangle_{i < m}) = \langle I^{-1}(g_i) \rangle_{i < m}^{(-T)}$. 
Lemma

[OA] Let \((f_i)_{i<2m}\) be a block sequence of elements of \(\text{FIN}_k^\pm\). If we set \(h_i = f_{2i} - f_{2i+1}\) for each \(i < m\), then \(\langle h_i \rangle_{i<m}^\pm \subseteq \langle \langle f_i \rangle (-T) \rangle_1\).
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To illustrate this, suppose we have \(f_0^0 < f_0^1 < f_1^0 < f_1^1\). Let \(h_i = f_i^0 - f_i^1\) for \(i < 2\), and say we want to approximate \(\delta_0 T^3(h_0) + \delta_1 h_1\), where \(\delta_0 = -1\) and \(\delta_1 = 1\).
Lemma

[OA] Let $(f_i)_{i<2m}$ be a block sequence of elements of $\text{FIN}_k^{\pm}$. If we set $h_i = f_{2i} - f_{2i+1}$ for each $i < m$, then $\langle h_i \rangle_{i<m}^{\pm} \subseteq (\langle f_i \rangle (\mp T))_1$.

To illustrate this, suppose we have $f_0^0 < f_0^1 < f_1^0 < f_1^1$. Let $h_i = f_i^0 - f_i^1$ for $i < 2$, and say we want to approximate $\delta_0 T^3(h_0) + \delta_1 h_1$, where $\delta_0 = -1$ and $\delta_1 = 1$.

$$\delta_0 T^3(f_0^0) - \delta_0 T^3(f_0^1) + \delta_1 f_1^0 - \delta_1 f_1^1$$
Lemma

[OA] Let \((f_i)_{i < 2m}\) be a block sequence of elements of \(\text{FIN}^\pm_k\). If we set \(h_i = f_{2i} - f_{2i+1}\) for each \(i < m\), then \(\langle h_i \rangle_{i < m}^\pm \subseteq (\langle f_i \rangle(-T))^1\).

To illustrate this, suppose we have \(f_0^0 < f_0^1 < f_1^0 < f_1^1\). Let \(h_i = f_i^0 - f_i^1\) for \(i < 2\), and say we want to approximate \(\delta_0 T^3(h_0) + \delta_1 h_1\), where \(\delta_0 = -1\) and \(\delta_1 = 1\).

\[
\begin{align*}
\delta_0 T^3(f_0^0) & - \delta_0 T^3(f_0^1) + \delta_1 f_1^0 - \delta_1 f_1^1 \\
(-T)^3(f_0^0) &
\end{align*}
\]
Lemma

[OA] Let \((f_i)_{i<2m}\) be a block sequence of elements of \(\text{FIN}_k^\pm\). If we set \(h_i = f_{2i} - f_{2i+1}\) for each \(i < m\), then \(\langle h_i \rangle_{i<m}^\pm \subseteq (\langle f_i \rangle^(-T))_1\).

To illustrate this, suppose we have \(f_0^0 < f_0^1 < f_1^0 < f_1^1\). Let \(h_i = f_i^0 - f_i^1\) for \(i < 2\), and say we want to approximate \(\delta_0 T^3(h_0) + \delta_1 h_1\), where \(\delta_0 = -1\) and \(\delta_1 = 1\).

\[
\begin{align*}
\delta_0 T^3(f_0^0) & - \delta_0 T^3(f_0^1) + \delta_1 f_1^0 - \delta_1 f_1^1 \\
(-T)^3(f_0^0) & - (-T)^4(f_0^1)
\end{align*}
\]
Lemma

[OA] Let \((f_i)_{i<2m}\) be a block sequence of elements of \(\text{FIN}_k^\pm\). If we set \(h_i = f_{2i} - f_{2i+1}\) for each \(i < m\), then \(\langle h_i \rangle_{i<m}^\pm \subseteq (\langle f_i \rangle (-T))^1\).

To illustrate this, suppose we have \(f_0^0 < f_0^1 < f_1^0 < f_1^1\). Let \(h_i = f_i^0 - f_i^1\) for \(i < 2\), and say we want to approximate \(\delta_0 T^3(h_0) + \delta_1 h_1\), where \(\delta_0 = -1\) and \(\delta_1 = 1\).

\[
\begin{align*}
\delta_0 T^3(f_0^0) & - \delta_0 T^3(f_0^1) + \delta_1 f_1^0 - \delta_1 f_1^1 \\
(-T)^3(f_0^0) & (-T)^4(f_0^1) + f_1^0
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\]
Lemma

[OA] Let $(f_i)_{i < 2m}$ be a block sequence of elements of $\text{FIN}_k^\pm$. If we set $h_i = f_{2i} - f_{2i+1}$ for each $i < m$, then $\langle h_i \rangle_{i < m}^\pm \subseteq (\langle f_i \rangle (-T))_1$.

To illustrate this, suppose we have $f_0^0 < f_0^1 < f_1^0 < f_1^1$. Let $h_i = f_i^0 - f_i^1$ for $i < 2$, and say we want to approximate $\delta_0 T^3(h_0) + \delta_1 h_1$, where $\delta_0 = -1$ and $\delta_1 = 1$.

\[
\begin{align*}
\delta_0 T^3(f_0^0) & \quad -\delta_0 T^3(f_0^1) & \quad +\delta_1 f_1^0 & \quad -\delta_1 f_1^1 \\
(-T)^3(f_0^0) & \quad (-T)^4(f_0^1) & \quad +f_1^0 & \quad (-T)f_1^1
\end{align*}
\]
Summary

We find $A \subset \text{FIN}_k^\pm(n)$ such that $(A, +, - T) \cong (\text{FIN}_k(n), +, T)$ and note that one can approximate elements of $\text{FIN}_k^\pm(n)$ by elements of $A$ using the operation $- T$. 
This method for obtaining the theorem with signs from the positive theorem also works in the infinite case.
Closing remarks

- This method for obtaining the theorem with signs from the positive theorem also works in the infinite case.
- Currently is no proof of the infinite $\text{FIN}_k$ Theorem that avoid the use of idempotent ultrafilters. The proof we present of the finite version cannot be adapted to the infinite case.
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Tyros [Tyr] obtained a different proof of the finite $\text{FIN}_k$ Theorem and of the finite $\text{FIN}_k^\pm$ theorem that provide upper bounds at the the level of $f_4$ for all $k$. 
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