From Odometers to Rotations

University of North Texas
June 2, 2014
Plan of the talk

- The ergodic theory "big picture"
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- The classification “big picture”
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- Quick survey of older results
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- The ergodic theory “big picture”
- The classification “big picture”
- Quick survey of older results
- New Results
  - odometers
  - circular systems
Measure Preserving Transformations

- Standard measure space $X$, an invertible measure preserving transformation
- Measure preserving transformations capture the "statistical" behavior of a dynamical system.
Two Examples
Two Examples

\[
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix} : T \times T \rightarrow T \times T.
\]
[.., 0, 0, 0, 0, 0, 1, 1, 0, ...

The resulting sequence of 0’s and 1’s is generic for coin flipping a (p, 1-p) coin.
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**UPSHOT:**
A completely deterministic method generates apparently random behavior.
Opposite behavior

A rotation of the circle by an irrational $\alpha$ yields completely predictable behavior.

More generally:
translations of compact groups.
Von Neumann 1932:

Classify the ergodic measure preserving transformations.
Successes

- Ornstein: Entropy is a complete classification of Bernoulli systems
- Halmos-von Neumann: Spectral equivalence is a complete invariant for translations of compact groups
Classifications
We are given a space of objects $X$ and an equivalence relation $E$ on $X$.

We “classify” $(X, E)$ by finding another space $Y$ (the “invariants”) and an equivalence relation $F$ on $Y$ and building a function

$$r : X \rightarrow Y$$

such that $xEx'$ iff $r(x) F r(x')$
To make sense as a classification:

- $r$ must be “computable” in some fashion.
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• $r$ must be “computable” in some fashion.
• At worst $r$ is a Borel function.
• $F$ should be “concrete,” say Borel or “structural”.
• If $F$ is a Borel subset of $Y \times Y$ then $E$ is a Borel subset of $X \times X$. 
In Ergodic Theory

- $X$ is some collection of measure preserving transformations.
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- Equivalence relation: conjugacy by measure theoretic transformations

$\sim MPT$
Hjorth's Results

X is MPT

- $\sim MPT$ is not Borel
- $\sim MPT$ is not classifiable by countable structures
Hjorth's Results

\[ X \text{ is MPT} \]

\[ \sim_{MPT} \text{ is not Borel} \]
\[ \sim_{MPT} \text{ is not classifiable by countable structures} \]

Results don't address Ergodicity
We now consider the case:
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$X$ is the space of ergodic measure preserving transformations.
Anti-classification: I

- (F-Rudolph-Weiss)\n  \[ \{ T : T \cong T^{-1} \} \]

is not Borel. (It is complete analytic.)

- Corollary:\n  \[ \{(S, T) : S \cong T \} \]

is not Borel.
However:

- (F-R-W) There is a generic class (the “Rank 1” transformations) for which conjugacy IS Borel.
Anti-Classification II

Maybe there is a "natural" (structural) equivalence relation that classifies the Ergodic measure preserving transformations.
**Anti-Classification II**

- (F-Weiss) The action of MPT on MPT is turbulent.
- **Corollary:** There is no generic class that can be classified by “isomorphism of countable structures.”
Anti-Classification II

- A real puzzle: the conjugacy relation rank one transformations is Borel, but can't be classified in familiar ways...
- Recent work of Gao and Hill is shedding some light on this problem.
For the logicians

Theorem (F) There is a reduction of “graph isomorphism” to the isomorphism relation of ergodic measure preserving systems.

It follows that any $S^\infty$ action can be reduced to isomorphism of ergodic MPT’s.
The “anti-classification” game:

Find some really badly behaved subset of the space for which the isomorphism relation is not Borel.
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If \( B \subseteq X \) is Borel but \( E \upharpoonright B \times B \) is not Borel then \( E \) is not Borel.
How do we build badly behaved MPT's?
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- The FRW examples are symbolic systems
Fix a language $\Sigma$, and a sequence of collections of words $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ with the properties that:

1. for each $n$ all of the words in $\mathcal{W}_n$ have the same length $q_n$,
2. each $w \in \mathcal{W}_n$ occurs at least twice as a subword of some $w' \in \mathcal{W}_{n+1}$.

Such a sequence will be called a *construction sequence*. 
Given a construction sequence:

Define $\mathcal{K}$ to be the collection of $x \in \Sigma^\mathbb{Z}$ such that every finite contiguous subword of $x$ occurs inside some $w \in \mathcal{W}_n$. Then $\mathcal{K}$ is a closed shift-invariant subset of $\Sigma^\mathbb{Z}$ that is compact if $\Sigma$ is finite.
Odometer based systems:

If there is a sequence $\langle r_n : n \in \mathbb{N} \rangle$ such that

$$\mathcal{W}_{n+1} \subseteq (\mathcal{W}_n)^{r_n}$$

then we call our system an **odometer based system**.
Let $\langle r_n : n \in \mathbb{N} \rangle$ be a sequence of natural numbers greater than or equal to 2. Let

$$\mathcal{O} = \prod_{0}^{\infty} \mathbb{Z}/r_n \mathbb{Z}$$
Example:

<table>
<thead>
<tr>
<th></th>
<th>r_0</th>
<th>r_1</th>
<th>r_2</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td></td>
<td>...</td>
</tr>
</tbody>
</table>
Example:

<table>
<thead>
<tr>
<th>( r_0 )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>...</td>
</tr>
</tbody>
</table>

Apply T once:

<table>
<thead>
<tr>
<th>3</th>
<th>1</th>
<th>1</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td>...</td>
</tr>
</tbody>
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Example:

<table>
<thead>
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<th>r_0</th>
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<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>...</td>
</tr>
</tbody>
</table>

Apply T once:

3  1  1  ...  \rightarrow  4  1  1  ...

Apply T a second time

4  1  1  ...  \rightarrow  0  0  2  ...
Odometer Based Systems have canonical odometer factors:

\[ W_2 \quad W_3 \quad W_4 \quad \text{location of 0} \]
What about classifying “concrete” transformations?
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Measure Preserving diffeomorphisms of the disk or the torus.
Obstacle:

- The transformations we built in FRW were odometer based.
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- There is NO known example of an odometer based diffeomorphism.
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- There is NO known example of an odometer based diffeomorphism
- Famous Problem: Realize the dyadic odometer as a diffeomorphism
We need a different construction

Anosov-Katok

diffeomorphisms
Some annoying numerology

Fix $\langle k_n, l_n : n \in \mathbb{N} \rangle$ such that $\sum \frac{1}{k_n}, \sum \frac{1}{l_n} < \infty$.

Let

$$p_{n+1} = p_n q_n k_n l_n + 1 \quad q_{n+1} = k_n l_n q_n^2.$$ 

$$i_i = (p_n)^{-1} i \mod q_n.$$ 

$$\alpha_n = \frac{p_n}{q_n}$$
Then $\alpha_n \to \alpha$ rapidly.
The briefest description of A-K diffeos. At stage $n$:

- We view the torus as $[0,1] \times [0,1]$.
- We rotate in the $x$-direction by $\alpha_n$.
- In the $y$-direction we permute subrectangles of $[0, 1/q_{n-1}] \times [0,1]$.
- Copy this permutation over equivariantly.
- Approximate by a diffeo.
Translating by $\alpha_n$

Geometric ordering of $1/q_n$ intervals

width $= 1/q_n$  translated here
Translating by $\alpha_n$

Geometric ordering of $1/q_n$ intervals

\[
\begin{array}{cccc}
1 & 3 & \text{translated here} & 2 & 4 \\
\end{array}
\]

width=$1/q_n$

Dynamical ordering of $1/q_n$ intervals

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

...
Translating by $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}}$

Width = $1/q_{n+1}$

translated here

1  2  3

...
Translating by \[
\alpha_{n+1} = \frac{pn+1}{qn+1}
\]
Translating by

\[ \alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \]

Translated here
Translating by

\[ \alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \]

width = \frac{1}{q_n}

translated here

\[ \cdots \]
Translating by

\[ \alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \]

width = 1/q \_n

translated here
Translating by

\[ \alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \]

width = 1/q \_n

translated here

\[ \cdots \]
Translating by

\[ \alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \]

width = 1/q_n

translated here

...
Translating by

\[ \alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \]

width = \frac{1}{q_n}

translated here

\[ \ldots \]
Translating by

\[ \alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \]

width = \frac{1}{q_n}

translated here

\[ \cdots \]
Translating by

\[ \alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \]

width = $1/q_{-n}$

translated here
Symbolic representation of AK systems

We define an operation $C$ on sequences $w_0, \ldots w_{k-1}$ of words in a language $\Sigma \cup \{b, e\}$ (where we assume that neither $b$ nor $e$ belong to $\Sigma$) by setting:

$$C(w_0, w_1, w_2, \ldots w_{k-1}) = \prod_{i=0}^{q-1} \prod_{j=0}^{k-1} (b^{q-j_i} w_j^{l-1} e^{j_i}).$$
A circular system is a symbolic system built from a sequence of collections of words

\[ \langle W_n^c : n \in \mathbb{N} \rangle \]

where

\[ W_{n+1}^c \subseteq \{ \mathcal{C}(w_1, \ldots w_{k_n}) : w_1, \ldots w_{k_n} \in W_n^c \} \]
Example

It follows from the summability of \( \langle 1/k_n, 1/l_n \rangle \) that \( \alpha_n \to \alpha \) for some irrational \( \alpha \).

If we take \( \Sigma = \{\ast\} \), the one element language, the resulting *Circular System* is a symbolic representation of the rotation of the circle by \( \alpha \).
It follows that every circular system has the rotation of the circle as a factor.

With a minimal amount of “randomness” the rotation of the circle IS the Kronecker factor.
The Theorems:

**Theorem 1** Fix a sequence \( \langle k_n, l_n \rangle \) such that \( \langle 1/k_n, 1/l_n \rangle \) are summable. If \( T \in \text{Diff}^\infty(\mathbb{T}^2) \) is built from the AK construction then \( T \) is conjugate to a circular system.

**Theorem 2** If \((\mathbb{K}, sh)\) is a circular system, then there is an AK \( T \in \text{Diff}^\infty_\lambda(\mathbb{T}^2) \) conjugate to \( \mathbb{K} \).
Thus the circular systems are exactly the symbolic representations of the AK diffeomorphisms
The functor Theorem

- There is a functor $K \rightarrow AK$
- mapping odometer systems to circular systems
- and taking joinings of $K$ with $L^{\pm 1}$
- to joinings of $AK$ with $AL^{\pm 1}$
- This functor takes graph joinings to graph joinings and invertible joinings to invertible joinings
This result says that all of the complexity of the conjugacy relation on odometer transformations is canonically realized as complexity of the conjugacy relation on smooth measure preserving transformations of the torus.
Anti-Classification

III

Theorem (F-Weiss) The set of ergodic measure preserving diffeomorphisms of the torus (or the disk) that are measure conjugate to their inverses is complete analytic (NOT a Borel set).
Anti-Classification

III

As a corollary, the set of pairs \((S,T)\) that are conjugate is NOT Borel.
Anti-Classification

III

The existence of the functor also yields the fact that every $S^\infty$-action can be reduced to isomorphism of measure preserving diffeomorphisms.
Is there a turbulent action that can be reduced to isomorphism of ergodic diffeomorphisms of the torus?
Open questions

Can this be repeated for diffeomorphisms of the torus up to conjugacy by Homeomorphisms?
$\text{Diff}^\infty (\mathbb{T}^2)$

Structurally Stable
$\text{Diff}^\infty (\mathbb{T}^2)$

Structurally Stable

Newhouse Transformations
$\text{Diff}^\infty(T^2)$

Structurally Stable

Newhouse Transformations

measure preserving
Structurally Stable

\[ \text{Diff}^\infty (T^2) \]

Newhouse Transformations

Measure preserving

Anosov-Katok Transformations
Some technical comments on this problem.

- In the FRW reduction from trees to ergodic mpt's, the difficult part was showing that if there is NO branch then $T$ is NOT conjugate to its inverse.
- Because the AK transformations are uniquely ergodic this direction follows automatically for homeo’s.
Thank You!