Uniform properties of metric spaces

Michael Hrušák

joint with Ondřej Zindulka and Wolfgang Wohofsky

Instituto de matemáticas
Universidad Nacional Autónoma de México
michael@matmor.unam.mx

UNT Denton
June 2015
Contents

1 Uniform properties of metric spaces

2 Monotone spaces

3 Strong measure zero in Polish groups

4 Uniformity of $\text{Smz}$

5 Transitive covering in Polish groups

6 The small ball property
Definition

A set $A$ in a metric space $(X, d)$ is uniformly nowhere dense if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $x \in X$ there is a $y \in X$ such that $B(y, \delta) \subseteq B(x, \varepsilon) \setminus A$, and $A \subseteq X$ is uniformly meager if it is a countable union of uniformly nowhere dense sets.

Definition (Denjoy 1920)

A set $A$ in a metric space $(X, d)$ is (strongly) porous if for every $x \in X$ there is a $p > 0$ and $r_0 > 0$ such that for any $r \leq r_0$ there is $y \in X$ such that $B(y, pr) \subseteq B(x, r) \setminus A$, and $A \subseteq X$ is $\sigma$-porous if it is a countable union of porous sets.
The small ball property and (σ-) monotonicity

Definition (Behrends and Kadets - 2001)

A metric space \((X, d)\) has the small ball property (sbp) if for any \(\varepsilon > 0\) there is a sequence \(\langle B(x_n, \varepsilon_n) : n \in \omega \rangle\) of balls with \(\varepsilon \geq \varepsilon_n\) and \(\lim_{n \to \infty} \varepsilon_n = 0\) that cover \(X\).

Definition (Zindulka - 2005)

A metric space \((X, d)\) is said to be monotone if there is a linear order \(<\) on \(X\) and a \(c > 0\) such that if \(x < y < z\) then \(d(x, y) \leq c \cdot d(x, z)\). The space is \(\sigma\)-monotone if it is a countable union of monotone subspaces.
Motivating questions

Question (Zindulka - 2009)
Is there a metric space of size $\aleph_1$ which is not $\sigma$-monotone?

Question (Miller-Steprāns - 2006)
How many translates of a meager set are needed to cover a Polish group $G$?
Cardinal invariants of $\sigma$-ideals

**Definition**

Let $I$ be an ideal on a set $X$ containing all singletons. Then

- $\text{add}(I) = \min \{|A| : A \subseteq I : \bigcup A \notin I\}$
- $\text{cov}(I) = \min \{|C| : C \subseteq I : \bigcup C = X\}$
- $\text{non}(I) = \min \{|Y| : Y \subseteq X : Y \notin I\}$
- $\text{cof}(I) = \min \{|C| : C \subseteq I : \forall I \in I \exists J \in C \ I \subseteq J\}$. 
Contents

1 Uniform properties of metric spaces

2 Monotone spaces

3 Strong measure zero in Polish groups

4 Uniformity of Smz

5 Transitive covering in Polish groups

6 The small ball property
The ideal of all $\sigma$-monotone sets in a metric space $X$ is denoted $\text{Mon}(X)$. The ideal $\text{Mon}(\mathbb{R}^2)$ of all $\sigma$-monotone sets in the plane is denoted $\text{Mon}$.

**Theorem. (HZ)**

- $\text{add}(\text{Mon}) = \omega_1$,
- $\text{cof}(\text{Mon}) = c$,

**Proof.**

Let $\mathcal{L}$ be a family of lines in $\mathbb{R}^2$. Then $\bigcup \mathcal{L}$ is $\sigma$-monotone if and only if $\mathcal{L}$ is countable.
The ideal of all $\sigma$-porous sets in a metric space $X$ is denoted $\text{SP}(X)$, and $\text{SP}(2^\omega) = \text{SP}$.

**Theorem (HZ)**

- $\text{cov}(\text{Mon}) = \text{cov}(\text{SP})$ and $\text{non}(\text{Mon}) = \text{non}(\text{SP})$.

**Proof.**

- $\text{cov}(\text{SP}(\mathbb{R})) = \text{cov}(\text{SP}(\mathbb{R}^2)) = \text{cov}(\text{SP}(2^\omega))$ (and likewise for non).
- Every monotone set $X \subseteq \mathbb{R}^2$ is strongly porous.
- If $A, B \subseteq \mathbb{R}$ are $\sigma$-porous, then $A \times B \subseteq \mathbb{R}^2$ is $\sigma$-monotone.
Covering of \textbf{Mon}

**Theorem (HZ)**

Each of the following is relatively consistent with ZFC that
- $\text{cov}(\text{Mon}) = \text{cov}(\text{SP}) < \mathfrak{c}$, and
- $\text{cof}(\mathcal{N}) < \text{cov}(\text{SP}) = \text{cov}(\text{Mon})$.

**Proof (Hint).**

A set $A \subseteq 2^\omega$ is strongly porous if and only if
\[
\exists n \forall p \in 2^{<\omega} \, \exists q \supseteq p \, |q| = |p| + n \land A \cap \langle q \rangle = \emptyset.
\]

A tree $T \subseteq 2^{<\omega}$ is \textbf{hyper-perfect} if
\[
\forall s \in T \, \forall n \exists t \supseteq s \, \forall r \in 2^n \, t \upharpoonright r \in T.
\]

Force with hyper-perfect trees.
Uniform properties of metric spaces

Monotone spaces
Strong measure zero in Polish groups
Uniformity of Smz
Transitive covering in Polish groups
The small ball property

Uniformity of Mon

Theorem (HZ)
Each of the following

- \( m_{\sigma-linked} \leq \text{non}(SP) = \text{non}(Mon) \), while
- it is relatively consistent with ZFC that \( m_{\sigma-centered} = c > \omega_1 \) and \( \text{non}(SP) = \text{non}(Mon) = \omega_1 \).

In fact:

Theorem (HZ)  \( MA_{\sigma-linked} + \neg CH \)
Every separable metric space of size \( \aleph_1 \) is \( \sigma \)-monotone.

Question (Zindulka - 2009)
Is there a metric space of size \( \aleph_1 \) which is not \( \sigma \)-monotone?
Uniform properties of metric spaces

Monotone spaces

Strong measure zero in Polish groups

Uniformity of SMZ

Transitive covering in Polish groups

The small ball property

Uniformity of Mon

Theorem (HZ)

Each of the following

- $m_{\sigma-linked} \leq \non(SP) = \non(Mon)$, while
- it is relatively consistent with ZFC that $m_{\sigma-centered} = c > \omega_1$ and $\non(SP) = \non(Mon) = \omega_1$.

In fact:

Theorem (HZ) $\text{MA}_{\sigma-linked} + \neg CH$

Every separable metric space of size $\aleph_1$ is $\sigma$-monotone.

Question (Zindulka - 2009)

Is there a metric space of size $\aleph_1$ which is not $\sigma$-monotone?
Contents

1 Uniform properties of metric spaces
2 Monotone spaces
3 Strong measure zero in Polish groups
4 Uniformity of $\text{Smz}$
5 Transitive covering in Polish groups
6 The small ball property
Miller-Steprāns question

Question (Miller-Steprāns - 2006)
How many translates of a meager set are needed to cover a Polish group $G$?
Strong measure zero

Definition (Borel - 1919)

A metric space $X$ has strong measure zero ($\text{Smz}$) if for any sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive numbers there is a cover $\{U_n : n \in \omega\}$ of $X$ such that $\text{diam } U_n \leq \varepsilon_n$ for all $n$.

- No perfect set is $\text{Smz}$.
- (Sierpiński - 1928) Every Luzin set has $\text{Smz}$.
- (Laver - 1976) Consistently, every $\text{Smz}$ set of reals (equivalently, of separable metric space) is countable.
**Theorem**

- *(Prikry - 197?)* Let $X \subseteq \mathbb{R}$ be such that $X + M \neq \mathbb{R}$ for all $M \in \mathcal{M}$. Then $X$ has strong measure zero.

- *(Galvin-Mycielski-Solovay - 1976)* A set $X \subseteq \mathbb{R}$ has Smz if and only if $X + M \neq \mathbb{R}$ for all $M \in \mathcal{M}$.

- *(Fremlin and Kysiak - 200?)* Let $G$ be a separable locally compact metric group. A set $S \subseteq G$ has strong measure zero if and only if $S \cdot M \neq G$ for all $M \in \mathcal{M}(G)$. 

---

**Smz and translates of meager sets**
Our proof of the Kysiak-Fremlin result

**Proposition (HWZ)**

Let $G$ be a Polish group which is either locally compact, or equipped with a complete invariant metric $d$, and let $S \subseteq G$ be Smz with respect to $d$. Then $S \cdot M \neq G$ for every uniformly meager $M \subseteq G$.

- This gives the Fremlin-Kysiak result as: *In a locally compact space every meager set is uniformly meager.*
- Is the proposition true for all Polish groups equipped with a complete left-invariant metric?
The Baer-Specker group $\mathbb{Z}^\omega$

**Theorem (CH)**

There is a set $X \subseteq \mathbb{Z}^\omega$ such that $X + M \neq \mathbb{Z}^\omega$ for every uniformly meager $M$, and yet $X \notin \text{Smz}(\mathbb{Z}^\omega)$.

**Theorem (CH)**

There is a Smz set $X \subseteq \mathbb{Z}^\omega$ such that $X + M \neq \mathbb{Z}^\omega$ for every $M \in \mathcal{M}(\mathbb{Z}^\omega)$.

**Question**

Is the Galvin-Mycielski-Solovay/Fremlin-Kysiak result (in ZFC) true only for locally compact Polish groups?
The Baer-Specker group $\mathbb{Z}^\omega$

**Theorem (CH)**

There is a set $X \subseteq \mathbb{Z}^\omega$ such that $X + M \neq \mathbb{Z}^\omega$ for every uniformly meager $M$, and yet $X \notin \text{Smz}(\mathbb{Z}^\omega)$.

**Theorem (CH)**

There is a $\text{Smz}$ set $X \subseteq \mathbb{Z}^\omega$ such that $X + M \neq \mathbb{Z}^\omega$ for every $M \in \mathcal{M}(\mathbb{Z}^\omega)$.

**Question**

Is the Galvin-Mycielski-Solovay/Fremlin-Kysiak result (in ZFC) true only for locally compact Polish groups?
Uniform properties of metric spaces
Monotone spaces
Strong measure zero in Polish groups
Uniformity of Smz
Transitive covering in Polish groups
The small ball property
**Definition**

\[ \varepsilon_q = \min \{|F| : F \text{ is a bounded family, } \forall g \in \omega^\omega \exists f \in F \forall n \in \omega f(n) \neq g(n) \} \].

\[ \varepsilon_\omega = \min \{|F| : F \text{ is a bounded family of infinite partial functions, } \forall g \in \omega^\omega \exists f \in F \forall n \in \text{dom } f f(n) \neq g(n) \} \].

- (Bartoszyński - 1987) omitting “bounded” in the definition of \( \varepsilon_q \) gives \( \text{cov}(\mathcal{M}) \).
Related cardinal invariants

- \( \text{add}(\mathcal{M}) = \min\{b, \text{eq}\} = \min\{b, \text{ed}\} \)
- \( \text{cov}(\mathcal{M}) = \min\{d, \text{ed}\} \)

\[ \begin{align*}
\vdash & \rightarrow d \\
add(\mathcal{M}) & \rightarrow \text{cov}(\mathcal{M}) & \rightarrow \text{ed} & \rightarrow \text{eq} & \rightarrow \text{non} (\mathcal{N})
\end{align*} \]

- (Goldstern-Judah-Shelah - 1993) \( \text{Con} (\text{cov}(\mathcal{M}) < \min\{d, \text{eq}\}) \)
- (Fremlin-Miller - 1988) \( \text{non}(\text{Smz}(\omega^\omega)) = \text{cov}(\mathcal{M}) \)
- (Bartoszyński-Judah - 1995) \( \text{non}(\text{Smz}(2^\omega)) = \text{eq} \)
Theorem (HWZ)

Let $X$ be a separable metric space. Then:

1. If $X$ is $\sigma$-totally bounded then $\varepsilon_\mathfrak{q} \leq \text{non}(\text{Smz}(X))$,
2. if $X$ is sbp then $\varepsilon_\mathfrak{d} \leq \text{non}(\text{Smz}(X))$, and
3. if $X$ is not sbp then $\text{cov}(\mathcal{M}) \leq \text{non}(\text{Smz}(X)) \leq \mathfrak{d}$.

Theorem (HWZ)

Let $X$ be an uncountable analytic separable metric space. Then:

1. If $X$ is $\sigma$-totally bounded then $\text{non}(\text{Smz}(X)) = \varepsilon_\mathfrak{q}$,
2. if $X$ is sbp then $\varepsilon_\mathfrak{d} \leq \text{non}(\text{Smz}(X)) \leq \varepsilon_\mathfrak{q}$, and
3. if $X$ is not sbp then $\text{cov}(\mathcal{M}) \leq \text{non}(\text{Smz}(X)) \leq \min\{\varepsilon_\mathfrak{q}, \mathfrak{d}\}$. 
Uniform properties of \textit{S}mz in \textit{P}olish groups

There is a Hurewicz type dichotomy:

\textbf{Proposition (HWZ)}

\textit{A Polish group equipped with a complete, left-invariant metric is either locally compact, or else contains a uniform copy of }\omega^\omega\textit{ and, in particular, is not sbp.}

\textbf{Corollary (HWZ)}

\textit{Let }\mathcal{G}\textit{ be a Polish group equipped with a complete, left-invariant metric.}

1. \textit{If }\mathcal{G}\textit{ is locally compact, then }\text{non}(\text{Smz}(\mathcal{G})) = \text{eq},
2. \textit{if }\mathcal{G}\textit{ is not locally compact, then }\text{non}(\text{Smz}(\mathcal{G})) = \text{cov}(\mathcal{M}).

\textbf{Question}

Is the corollary true for all Polish groups?
Consistency results

Theorem (Szpilrajn - 1935)

*If* $X$ *is a separable metric which is not of universal measure zero, then*

$$\text{non}(\text{Smz}(X)) \leq \text{non}(\mathcal{N}).$$

Recall that a metric space $X$ is of *universal measure zero* if there is no probability Borel measure on $X$ vanishing on singletons.

Theorem (HWZ)

*It is consistent that there is a non-$\text{sbp}$ set $X \subseteq \omega^\omega$ such that*

$$\omega = \text{non}(\text{Smz}(X)) > \text{non}(\mathcal{N}) = \text{cov}(\mathcal{M}).$$

Question

Is $\text{cof}(\mathcal{N})$ an upper bound for $\text{non}(\text{Smz}(X))$ for any non-$\text{Smz}$ separable space $X$?
Theorem (HWZ)

*It is consistent that there is an sbp set \( X \subseteq \omega^\omega \) such that*

\[
\text{non}(\text{Smz}(X)) = \omega < \text{eq}.
\]

**Question**

- Is \( \text{non}(\text{Smz}(X)) = \text{eq} \) for all sbp analytic (Borel, Polish) spaces \( X \)?
- Is \( \text{non}(\text{Smz}(X)) = \text{cov}(\mathcal{M}) \) for all non-sbp analytic (Borel, Polish) spaces \( X \)?
Contents

1 Uniform properties of metric spaces
2 Monotone spaces
3 Strong measure zero in Polish groups
4 Uniformity of Smz
5 Transitive covering in Polish groups
6 The small ball property
**Definition (Miller-Steprāns - 2006)**

Given a Polish group \( G \), let \( \text{cov}^*_G \) be the minimal cardinality of a set \( A \subseteq G \) such that \( A \cdot M = G \) for some meager set \( M \subseteq G \).

- Related to \( \text{Smz} \) via the Prikry/Galvin-Mycielski-Solovay/Fremlin-Kysiak results.

**Questions (Miller-Steprāns - 2006)**

1. Is it consistent to have a compact Polish group \( G \) such that \( \text{cov}^*_G > \varepsilon q \)?
2. Is it true that \( \text{cov}^*_G \geq \varepsilon q \) for any infinite compact Polish group \( G \)?
3. Is it true that for every non-discrete Polish group \( G \) either \( \text{cov}^*_G = \varepsilon q \) or \( \text{cov}^*_G = \text{cov}(M) \)?
Transitive covering

Questions (Miller-Steprāns - 2006)

- Is it consistent to have a compact Polish group $G$ such that $\text{cov}^*_G > \varepsilon_q$?
- Is it true that $\text{cov}^*_G \geq \varepsilon_q$ for any infinite compact Polish group $G$?
- Is it true that for every non-discrete Polish group $G$ either $\text{cov}^*_G = \varepsilon_q$ or $\text{cov}^*_G = \text{cov}(M)$?

Theorem (HWZ)

- If $G$ is a Polish group, then $\text{cov}(M) \leq \text{cov}^*_G \leq \varepsilon_q$.
- If $G$ is a non-discrete, locally compact Polish group, then $\text{cov}^*_G = \varepsilon_q$.
- Let $G$ be a Polish group endowed with a complete left-invariant metric. If $G$ is not locally compact, then $\text{cov}^*_G = \text{cov}(M)$. 
Question

Is there consistently a non-locally compact Polish group $G$ such that $\text{cov}^*_G > \text{cov}(\mathcal{M})$?

Question

Is there consistently a Polish group $G$ such that $\text{non}(\text{Smz}(G)) \neq \text{cov}^*_G$?
Contents

1 Uniform properties of metric spaces
2 Monotone spaces
3 Strong measure zero in Polish groups
4 Uniformity of Smz
5 Transitive covering in Polish groups
6 The small ball property
Definition (Behrends and Kadets - 2001)

A metric space \((X, d)\) has the small ball property (sbp) if for any \(\varepsilon > 0\) there is a sequence \(\langle B(x_n, \varepsilon_n) : n \in \omega \rangle\) of balls with \(\varepsilon \geq \varepsilon_n\) and \(\lim_{n \to \infty} \varepsilon_n = 0\) that cover \(X\).

- sbp is a \(\sigma\)-additive property.
- Every \((\sigma-)\) compact space has sbp.
- Every Smz space has sbp.
The small ball property

A right-continuous, nondecreasing function $g : (0, \infty) \to (0, \infty)$ with $\lim_{r \to 0} g(r) = 0$ is called a gauge. Given a gauge $g$ and $Y \subseteq X$, the $g$-dimensional Hausdorff measure of $Y$ is defined thus:

$$H^g(Y) = \sup_{\delta > 0} \inf \left\{ \sum_{E \in \mathcal{E}} g(\text{diam}(E)) : Y \subseteq \bigcup \mathcal{E} \land \forall E \in \mathcal{E} \ (\text{diam}(E) \leq \delta) \right\}.$$

**Proposition (HWZ)**

The following are equivalent.

(i) $X$ has \textbf{sbp},

(ii) $X$ admits a base $\{B_n : n \in \omega\}$ such that $\text{diam} B_n \to 0$,

(iii) for every sequence $\langle \varepsilon_m : m \in \omega \rangle$ of positive reals there is a sequence $\langle F_m : m \in \omega \rangle$ of finite sets such that $\bigcup_{m \in \omega} \{B(x, \varepsilon_m) : x \in F_m\}$ covers $X$,

(iv) there is a gauge $g$ such that $H^g(X) = 0$. 
The small ball property

Proposition (HWZ)

The following are equivalent.

(i) $X$ has sbp,

(ii) there is a gauge $g$ such that $\mathcal{H}^g(X) = 0$

Compare to:

Proposition (Besicovitch - 1933)

The following are equivalent.

(i) $X$ has Smz,

(ii) $\mathcal{H}^g(X) = 0$ for every gauge $g$. 
In a similar vein:

**Proposition (HWZ)**

A metrizable space $X$ has the Menger Property iff $X$ has \textbf{sbp} in all compatible metrics.

Compared to:

**Proposition (Fremlin-Miller - 1988)**

A metrizable space $X$ has the Rothberger Property iff $X$ has \textbf{Smz} in all compatible metrics.

A space $X$ has the \textbf{Menger Property} if for every sequence $\langle U_n : n \in \omega \rangle$ of open covers there are finite sets $F_n \subseteq U_n$ such that $\bigcup_{n \in \omega} F_n$ covers $X$, while $X$ has the \textbf{Rothberger Property} if for every sequence $\langle U_n : n \in \omega \rangle$ of open covers there are $U_n \in U_n$ such that $\{ U_n : n \in \omega \}$ covers $X$. 
Cardinal invariants of \textit{sbp}

Proposition (HWZ)

If a metric space $X$ does not have \textit{sbp}, then

(i) $\non(\textit{sbp}(X)) = \cof(\textit{sbp}(X)) = d$,

(ii) $\add(\textit{sbp}(X)) = \cov(\textit{sbp}(X)) = b$.

Proposition (HWZ)

For a subset $X \subseteq \omega^\omega$, the following are equivalent:

(i) $X$ has \textit{sbp},

(ii) no isometric copy of $X$ in $\omega^\omega$ is dominating,

(iii) no uniformly continuous image of $X$ in $\omega^\omega$ is dominating.

Question.

Is $\Borel(\omega^\omega)/\textit{sbp}$ forcing equivalent to the Laver forcing?
The end.

Thank you for your attention!