ON FUZZY PRIME IDEALS OF LATTICES

by

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BLAST2015, June 12th, 2015
Definition 1.1[2, B.A. Davey and H.A. Priestley] A non-empty set $J$ of a lattice $L$ is called an ideal of $L$ if for all $x, y \in L$

1. $\forall x, y \in J$, imply $x \lor y \in J$

2. if $x \in J$ with $y \leq x$ then $y \in J$
Definition 1.2 [2, B.A. Davey and H.A. Priestley] Let $J$ be a proper ideal of $L$ then $J$ is said to be prime if $a, b \in L$ and $a \land b \in J$ imply $a \in J$ or $b \in J$.

Definition 1.3 A fuzzy subset of $L$ is a function $\mu : L \rightarrow [0, 1]$. Let $\mu$ be a fuzzy subset of $L$, for $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in L / \mu(x) \geq \alpha\}$ is called a $\alpha$-level subset of $\mu$ or $\alpha$-cut set of $\mu$.

Definition 1.4 [1, M. Attallah] A fuzzy subset $\mu$ of $L$ is called a fuzzy sublattice of $L$ if $\mu(x \land y) \land \mu(x \lor y) \geq \mu(x) \land \mu(y)$ for all $x, y \in L$. 

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Definition 1.5 [1, M. Attallah] Let $\mu$ be a fuzzy sublattice of $L$ then $\mu$ is a fuzzy ideal of $L$ if $\mu(x \lor y) = \mu(x) \land \mu(y)$ for all $x, y \in L$.

Proposition 1.1 [1, M. Attallah] Let $\mu$ be a fuzzy sublattice of $L$ then $\mu$ is a fuzzy ideal of $L$ if and only if $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ for all $x, y \in L$. 

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Remark 1.1 [3, B.B. N. Koguep, C. Nkuimi, C. Lele] If $\mu$ is a fuzzy ideal of $L$, then $\mu(0) \geq \mu(x) \geq \mu(1)$ for all $x, y \in L$.

Theorem 1.1 [1, M. Attallah] Let $\mu$ be a fuzzy subset of $L$. Then $\mu$ is a fuzzy ideal of $L$ if and only if for any $\alpha \in [0, 1]$ such that $\mu_\alpha \neq \phi$, $\mu_\alpha$ is an ideal of $L$.

Definition 1.6 [6, V.V. Swavy, D. Viswanadha Raju] A proper fuzzy ideal $\mu$ of $L$ is called a prime fuzzy ideal, if for any two fuzzy ideal $\eta, \upsilon$ of $L$, $\eta \land \upsilon \leq \mu$ implies $\eta \leq \mu$ or $\upsilon \leq \mu$.

Definition 1.7 [1, M. Attallah] A proper fuzzy ideal $\mu$ of $L$ is called a fuzzy prime ideal if $\mu(x \land y) \leq \mu(x) \lor \mu(y)$ for all $x, y \in L$. 
We will show that every prime fuzzy ideal is always fuzzy prime ideal. Now we will give an example of a fuzzy prime ideal which is not a prime fuzzy ideal.
Example 1.1 Let $L$ be the following lattice,

\[\begin{array}{c}
& 1 \\
\downarrow & & \downarrow \\
a & & b \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}\]

if $\mu(0) = 0.7$, $\mu(a) = 0.3$, $\mu(b) = 0.7$, $\mu(1) = 0.3$ then $\mu$ is a fuzzy prime ideal. As every proper level set is a prime ideal of $L$ by Theorem 2.15 [3 B.B.N.Koguep, C. Nkuimi, C.Lee].

Now consider $\eta(0) = 1$, $\eta(a) = 0.2$, $\eta(b) = 0.8$, $\eta(1) = 0.2$, $\gamma(0) = 0.7$, $\gamma(a) = 0.7$, $\gamma(b) = 0.6$, $\gamma(1) = 0.6$. Then $(\eta \land \gamma) \leq \mu$. But $\eta(0) > \mu(0)$ and $\gamma(a) > \mu(a)$. So $\mu$ is not a prime fuzzy ideal of $L$.

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Theorem 1.2 If $\mu$ is a non-constant prime fuzzy ideal in $L$, then $|Im \mu| = 2$, where $Im(\mu)$ is the image of $\mu$.

Proof Since $\mu$ is not constant, we have $|Im(\mu)| \geq 2$. Assume that $|Im(\mu)| \geq 3$. Let $\mu(0) = s$ and $k = glb\{\mu(x)/x \in L\}$. Then there exist $t, m \in Im(\mu)$ such that $k \leq t < m < s$. Let $\gamma$ and $\rho$ be fuzzy sets in $L$ defined by $\gamma(x) = \frac{t+m}{2}$ and $\rho(x) = \begin{cases} s & \text{if } x \leq a \\ k & \text{otherwise} \end{cases}$.

Let $a, b \in L$ such that $\mu(a) = m$, $\mu(b) = t$. Clearly $\gamma$ is a fuzzy ideal of $L$. Now we claim that $\rho$ is a fuzzy ideal of $L$. First to prove that $\rho(x \land y) \geq \rho(x) \land \rho(y)$.

Case 1: Let $x \leq a$ and $y \leq a$. By definition of lattice $x \land y \leq x \leq a$. It follows that $\rho(x \land y) = s$ and hence $\rho(x \land y) \geq \rho(x) \land \rho(y)$.

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Case 2: Let $x \leq a$ and $y \not\leq a$. It follows that $\rho(x) = s$ and $\rho(y) = k$ and hence $\rho(x) \land \rho(y) = s \land k = k$. Therefore $\rho(x \land y) \geq \rho(x) \land \rho(y)$.

Case 3: Let $x \not\leq a$ and $y \not\leq a$. It follows that $\rho(x) = k$ and $\rho(y) = k$ and hence $\rho(x) \land \rho(y) = k \land k = k$. Therefore $\rho(x \land y) \geq \rho(x) \land \rho(y)$. The other case, $x \not\leq a$, $y \leq a$ is similar to case 2.

Similarly we can prove that $\rho(x \lor y) \geq \rho(x) \land \rho(y)$. Hence $\rho(x \land y) \land \rho(x \lor y) \geq \rho(x) \land \rho(y)$ for all $x, y \in L$. This implies that $\rho$ is a fuzzy sublattice of $L$. Suppose $x \leq y$, If $\rho(y) = k$ and since $\rho(x) \geq k$ for all $x$, $\rho(x) \geq \rho(y)$. If $\rho(y) = s$ then $y \leq a$. Since $x \leq y$ implies $x \leq a$. Hence $\rho(x) = s$ and $\rho(x) \geq \rho(y)$. Therefore
\( \rho \) is a fuzzy ideal of \( L \). Since \( \rho(a) = s \) and \( \mu(a) = m \) we have \( \rho \not\leq \mu \). Since \( \gamma(b) = \frac{t+m}{2} \) and \( \mu(b) = t \) we have \( \gamma \not\leq \mu \). Now we claim that \( \gamma \land \rho \leq \mu \).

**Case 1:** Suppose \( x \leq a \). Since \( \mu \) is a fuzzy ideal of \( L \), \( \mu(x) \geq \mu(a) = m \). Consider \( (\gamma \land \rho)(x) = \frac{t+m}{2} \land s = \frac{t+m}{2} < \mu(x) \).

**Case 2:** Suppose \( x > a \), \( \rho(x) = k \), \( \gamma(x) = \frac{t+m}{2} \). Now \( (\rho \land \gamma)(x) = k \land \frac{t+m}{2} = k \leq \mu(x) \). Therefore \( (\gamma \land \rho) \leq \mu \) with \( \gamma \not\leq \mu \), \( \rho \not\leq \mu \) which is a contradiction to \( \mu \) is a prime fuzzy ideal of \( L \). Hence \( |Im\mu| = 2 \).

**Lemma 1.1** If \( A \) and \( B \) are the ideals of a lattice \( L \), then \( A \cap B = A \land B \).

**Proof** If \( x \in A \land B \), then \( x = a \land b \) for some \( a \in A \), \( b \in B \). Then \( a \land b \leq a \) and \( a \in A \) implies \( a \land b \in A \). Similarly \( a \land b \in B \).
Then \( a \land b \in A \cap B \). On the other hand, if \( x \in A \cap B \), then
\( x = x \land x \in A \land B \). Therefore \( A \cap B = A \land B \).

Koguep, Nkuimi and Lele [3, Theorem 2.15], have given a
characterization for a fuzzy ideal to be fuzzy prime ideal. Now we
give a another characterization for an ideal to be fuzzy prime.

**Theorem 1.3** Let \( \mu \) be a fuzzy ideal of a lattice \( L \). Then \( \mu \) is a
fuzzy prime ideal of \( L \) if and only if for any fuzzy ideal \( \sigma, \rho \) of \( L \),
\( \sigma_{t_1} \land \rho_{t_2} \subseteq \mu_{t_3} \) implies \( \sigma_{t_1} \subseteq \mu_{t_3} \) or \( \rho_{t_2} \subseteq \mu_{t_3} \) for all
\( t_1, t_2, t_3 \in [0, 1] \).

**Proof** Let \( \mu \) be a fuzzy prime ideal of a lattice \( L \). If there exists
the fuzzy ideals \( \sigma \) and \( \rho \) of \( L \) such that \( \sigma_{t_1} \land \rho_{t_2} \subseteq \mu_{t_3} \) but
\( \sigma_{t_1} \not\subseteq \mu_{t_3} \) and \( \rho_{t_2} \not\subseteq \mu_{t_3} \), then there exists \( y, z \in L \) such that
\( y \in \sigma_{t_1} \) but \( y \not\in \mu_{t_3} \) and \( z \in \rho_{t_2} \) but \( z \not\in \mu_{t_3} \). Then \( \mu(y) < t_3 \) and
\( \mu(z) < t_3 \). Therefore \( \max\{\mu(y), \mu(z)\} < t_3 \). Since \( \sigma_{t_1} \land \rho_{t_2} \subset \mu_{t_3} \),
\( y \land z \in \mu_{t_3} \).

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hence $\mu(y \land z) \geq t_3$. Hence $\mu(y \land z) \nleq \mu(y) \lor \mu(z)$. Thus $\mu$ is not a fuzzy prime ideal, contradiction.

Sufficient part: If there exists $a, b \in L$ such that $\mu(a \land b) = t_1 > \mu(a)$ and $\mu(a \land b) > \mu(b)$, then for any $x \in X$, let

$$\sigma(x) = \begin{cases} t_1 & : \text{if } x \leq a \\ 0 & : \text{otherwise} \end{cases},$$

$$\rho(x) = \begin{cases} t_1 & : \text{if } x \leq b \\ 0 & : \text{otherwise} \end{cases}.$$ 

Clearly $\sigma, \rho$ are fuzzy ideals of $L$. If $x \in \sigma_{t_1} \land \rho_{t_1} = \sigma_{t_1} \cap \rho_{t_1}$ then $\sigma(x) \geq t_1$ and $\rho(x) \geq t_1$. Thus $x \leq a \land b$. Since $\mu$ is a fuzzy ideal of $L$, $\mu(x) \geq \mu(a \land b) = t_1$. Then $x \in \mu_{t_1}$. Thus $\sigma_{t_1} \land \rho_{t_1} \subseteq \mu_{t_1}$. Now $\sigma(a) = t_1$ implies $a \in \sigma_{t_1}$. Since $\mu(a) < t_1$ we have $a \notin \mu_{t_1}$. Therefore $\sigma_{t_1} \nsubseteq \mu_{t_1}$. Similarly $\rho_{t_1} \nsubseteq \mu_{t_1}$. This is a contradiction to our assumption. Therefore $\mu$ is a fuzzy prime ideal of a lattice $L$. 

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Theorem 1.4 If $\mu$ is a prime fuzzy ideal of $L$, then $M_\mu = \{x \in M/\mu(x) = \mu(0)\}$ is a prime ideal of $L$.

Proof Let $\alpha = \mu(0)$. Then by Theorem 1.1, $M_\mu$ is an ideal of $L$. Now to show that $M_\mu$ is a prime ideal of $L$, let $a, b \in L$ such that $a \land b \in M_\mu$.

Define the fuzzy subsets $\sigma$ and $\tau$ as $\sigma(x) = \begin{cases} \mu(0) & : \text{if } x \leq a \\ 0 & : \text{otherwise} \end{cases}$, $\tau(x) = \begin{cases} \mu(0) & : \text{if } x \leq b \\ 0 & : \text{otherwise} \end{cases}$. Clearly $\sigma$ and $\tau$ are fuzzy ideal of $L$.

Next we show that $(\sigma \land \tau) \subseteq \mu$. Let $x \in L$. If $x \leq a$ and $x \leq b$ then $x \leq a \land b \in M_\mu$. By definition $\mu(x) = \mu(0_L)$, hence $(\sigma \land \tau)(x) \leq \mu(x)$. If either $x \not\leq a$ or $x \not\leq b$ then $(\sigma \land \tau)(x) = 0$. Hence $(\sigma \land \tau) \leq \mu$. Since $\mu$ is a prime fuzzy, it follows that $\sigma \leq \mu$ or $\tau \leq \mu$. Suppose $\sigma \leq \mu$. If $a \not\in M_\mu$, then means $\mu(a) \neq \mu(0)$.  

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Now $\mu(0) = \mu(0 \land a) \geq \mu(a)$ since $\mu$ is a fuzzy ideal. Therefore $\mu(a) < \mu(0)$. Hence $\sigma(a) = \mu(0) > \mu(a)$ which is a contradiction to the fact that $\sigma \leq \mu$. Thus we have proved that if $\sigma \leq \mu$, then $a \in M_\mu$. Similarly, if $\tau \leq \mu$, we can show that $b \in M_\mu$. Hence $M_\mu$ is a prime ideal of $L$.

**Theorem 1.5** If $\mu$ is a prime fuzzy ideal of $L$ then $\mu(0) = 1$.

**Proof** By Theorem 1.2 we have $|Im(\mu)| = 2$. Let $|Im(\mu)| = \{s, t\}$ with $s < t$. Then $\mu(0) = t$. Let $\mu(a) = s$. We can claim that $t = 1$. Assume that $t \neq 1$. Let $\gamma$ and $\rho$ be fuzzy sets in $L$ defined by $\gamma(x) = \frac{1}{2}(s + t)$ and $\rho(x) = \begin{cases} 1 & : \text{if } x \in \mu_t \\ s & : \text{otherwise} \end{cases}$.

It is clear that $\gamma$ and $\rho$ are fuzzy ideals of $L$. We claim that $(\gamma \land \rho) \leq \mu$.
Case 1: If $x \notin \mu_t$ then $(\gamma \land \rho)(x) = \min\{\frac{s+t}{2}, s\} = s \leq \mu(x)$.

Case 2: If $x \in \mu_t$ then $(\gamma \land \rho)(x) = \min\{\frac{s+t}{2}, 1\} = \frac{s+t}{2} \leq t = \mu(x)$. Thus $(\gamma \land \rho) \leq \mu$.

From $\mu(0) = t < 1 = \rho(0)$, it follows that $\rho \not\subseteq \mu$. Note that $M_{\mu} \neq L$ since $|Im(\mu)| = 2$. Hence there exists $a \in L$ such that $\mu(a) = s < \frac{1}{2}(s + t) = \gamma(a)$. This means that $\gamma \not\subseteq \mu$. This is contradiction and so $\mu(0) = t = 1$.

**Theorem 1.6** Let $\mu$ be a fuzzy set in $L$ such that $|Im(\mu)| = 2$ and $\mu(0_L) = 1$. If the set $M_\mu = \{x \in L/\mu(x) = \mu(0)\}$ is a prime ideal of $L$, then $\mu$ is a prime fuzzy ideal of $L$.

**Proof** Since $|Im(\mu)| = 2$ and $\mu(0_L) = 1$, we may denote $Im(\mu) = \{t, 1\}$, where $t < 1$. If $x, y \in M_{\mu}$, then $x \land y \in M_{\mu}$ and $\mu(x \land y) = \mu(0) = 1 = \mu(x) \land \mu(y)$. Now $x \in M_{\mu}$ and $y \notin M_{\mu}$. By definition $x \land y \leq x$. Since $x \in M_{\mu}$ implies $x \land y \in M_{\mu}$, $\mu(x \land y) = 1$. Therefore $\mu(x \land y) \geq \mu(x) \land \mu(y)$. The other case $x \notin M_{\mu}$.
and \( y \in M_\mu \) is similar to the above proof. If \( x \notin M_\mu \) and \( y \notin M_\mu \), then \( \mu(x) = \mu(y) = t \) and so \( \mu(x \land y) \geq t = \mu(x) \land \mu(y) \). Hence \( \mu(x \land y) \geq \mu(x) \land \mu(y) \), for all \( x, y \in L \). Similarly we can show that \( \mu(x \lor y) \geq \mu(x) \land \mu(y) \) for all \( x, y \in L \). Then \( \mu \) is a fuzzy sublattice of \( L \).

Assume \( x \leq y \). We claim that \( \mu(x) \geq \mu(y) \). Let \( \mu(y) = \mu(0) \) this means that \( y \in M_\mu \). Therefore \( x \in M_\mu, \mu(x) = \mu(0) = \mu(y) \) and hence \( \mu(x) \geq \mu(y) \). If \( \mu(y) = t \) then \( \mu(x) \geq \mu(y) \). This shows that \( \mu \) is a fuzzy ideal of \( L \). Let \( \gamma \) and \( \rho \) be fuzzy ideal of \( L \) such that \( \gamma \land \rho \subseteq \mu \). Assume \( \gamma \nsubseteq \mu \) and \( \rho \nsubseteq \mu \). Then there exists \( x \in L \) such that \( \gamma(x) > \mu(x) \). Let \( \gamma(x) = t_1 \). Clearly \( \mu(x) = t(t_1 > t) \). Similarly there exists \( y \in M \) such that \( \rho(y) > \mu(y) \). Let \( \rho(y) = t_2 \). Clearly \( \mu(y) = t(t_2 > t) \). Hence\( x \land y \leq x \) and since \( \gamma \) is a fuzzy ideal of \( L \), \( \gamma(x \land y) \geq \gamma(x) = t_1 \). Similarly \( \rho(x \land y) \geq \rho(y) = t_2 \).
Hence $\gamma(x \wedge y) \wedge \rho(x \wedge y) \geq t_1 \wedge t_2 > t$. Therefore $(\gamma \wedge \rho)(x \wedge y) > t$. We claim that $\mu(x \wedge y) = t$. Since $\mu(x) = t$ implies $x \notin M_\mu$ and $\mu(y) = t$ implies $y \notin M_\mu$. Since $M_\mu$ is a prime ideal, $x \wedge y \notin M_\mu$. Hence $\mu(x \wedge y) \neq 1$. Thus $\mu(x \wedge y) = t$. Hence $(\gamma \wedge \rho)(x \wedge y) > \mu(x \wedge y)$, which means that $\gamma \wedge \rho \notin \mu$, is a contradiction to assumption. Thus $\gamma \subseteq \mu$ or $\rho \subseteq \mu$. Hence $\gamma \subseteq \mu$ or $\rho \subseteq \mu$. Hence $\mu$ is a fuzzy prime ideal.

**Theorem 1.7** Every prime fuzzy ideal is always fuzzy prime ideal

**Proof** Let $\mu$ be prime fuzzy ideal. Since $Im\mu = \{t, 1\}$, the level sets will be either $L$ or $\mu_0$. These two are ideals of $L$. Hence for all $\alpha \in [0, 1]$, $\mu_\alpha = L$ or $\mu_0$. By Theorem 1.4, $\mu_0$ is a prime ideal of $L$. Hence by [3, Theorem 2.15], $\mu$ is a fuzzy prime ideal.
Conclusion and further suggestions: We have shown that every prime fuzzy ideal of a lattice $L$ is always fuzzy prime ideal. If $\mu$ is a prime fuzzy ideal the image set of $\mu$ will contain only two elements $\{t, 1\}$. A generalization of prime fuzzy ideal can be defined as follows. A fuzzy ideal $\mu$ of $L$ is said to be weakly prime fuzzy ideal if $\lambda_1 \land \lambda_2 \land \lambda_3 \subseteq \mu$ implies either $\lambda_1 \land \lambda_2 \subseteq \mu$ or $\lambda_2 \land \lambda_3 \subseteq \mu$ or $\lambda_1 \land \lambda_3 \subseteq \mu$. Clearly every prime fuzzy ideal is weakly prime fuzzy ideal but the converse need not be true.
Open Problem

Question. whether the image set of weakly prime fuzzy ideal will contain three elements?

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References


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Thank you for your attention!

Any questions?

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