Long Cut and Choose games and the infinite distributive law

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In this talk, we are going to discuss the infinite distributive law and Cut and Choose games on complete Boolean algebras. We will show that if $\kappa > \omega$, the Cut and Choose game $\mathcal{G}_{<\omega}^\kappa$ characterizes the $(\kappa, \infty)$-distributivity.

Throughout this talk:

- complete Boolean algebra (cBa) will be assumed to be atomless; $\forall b > 0 \exists c (b > c > 0)$.
- $\mathcal{B}$ denotes a cBa, and $\mathcal{B}^+ = \mathcal{B} \setminus \{0\}$. 
In this talk, we are going to discuss the infinite distributive law and Cut and Choose games on complete Boolean algebras. We will show that if $\kappa > \omega$, the Cut and Choose game $G^K_{<\omega}$ characterizes the $(\kappa, \infty)$-distributivity.

Throughout this talk:

- complete Boolean algebra (cBa) will be assumed to be atomless; $\forall b > 0 \exists c \ (b > c > 0)$.
- $\mathcal{B}$ denotes a cBa, and $\mathcal{B}^+ = \mathcal{B} \setminus \{0\}$. 

Definition 1

\( \kappa \): infinite cardinal.

\( \mathcal{B} \) is \((\kappa, \infty)\)-distributive

\[ \iff \text{For every cardinal } \lambda \text{ and } \{ b_{\alpha, \beta} : \alpha < \kappa, \beta < \lambda \} \subseteq \mathcal{B}^+, \]

\[ \bigwedge_{\alpha < \kappa} \bigvee_{\beta < \lambda} b_{\alpha, \beta} = \bigvee_{f: \kappa \rightarrow \lambda} \bigwedge_{\alpha < \kappa} b_{\alpha, f(\alpha)}. \]
Fact 2

T.F.A.E. for $\kappa$ and $\mathcal{B}$;

1. $\mathcal{B}$ is $(\kappa, \infty)$-distributive.

2. For every $b \in \mathcal{B}^+$ and every $\mathcal{B}^+$-name $\dot{f}$ for a function from $\kappa$ to $V$, there is $c \leq b$ and $g : \kappa \to V$ such that $c \models_{\mathcal{B}^+} \text{“} \dot{f} = g \text{”}$.  

3. For every $b \in \mathcal{B}^+$ and every partitions $\langle l_\alpha : \alpha < \kappa \rangle$ of $b$, there is $b_\alpha \in l_\alpha (\alpha < \kappa)$ such that $\bigwedge_{\alpha < \kappa} b_\alpha > 0$.

- $I \subseteq \mathcal{B}^+$ is an antichain of $\mathcal{B}$ if $b \land c = 0$ for every distinct $b, c \in I$.

- For $b \in \mathcal{B}^+$, a partition of $b$ is an antichain $I$ with $\bigvee I = b$. 

Fact 2

T.F.A.E. for $\kappa$ and $B$;

1. $B$ is $(\kappa, \infty)$-distributive.

2. For every $b \in B^+$ and every $B^+$-name $\dot{f}$ for a function from $\kappa$ to $V$, there is $c \leq b$ and $g : \kappa \to V$ such that $c \Vdash_{B^+} \langle \dot{f} = g \rangle$.

3. For every $b \in B^+$ and every partitions $\langle l_\alpha : \alpha < \kappa \rangle$ of $b$, there is $b_\alpha \in l_\alpha$ ($\alpha < \kappa$) such that $\bigwedge_{\alpha < \kappa} b_\alpha > 0$.

- $l \subseteq B^+$ is an antichain of $B$ if $b \land c = 0$ for every distinct $b, c \in l$.
- For $b \in B^+$, a partition of $b$ is an antichain $l$ with $\lor l = b$. 
Cut and Choose Game

Definition 3 (Jech)

\(\kappa\): infinite cardinal

Let \(G^K\) be the following two player game on cBA \(\mathcal{B}\): First, ONE fixes \(b^* \in \mathcal{B}^+\). At each stage,

1. ONE chooses a partition \(l_\alpha\) of \(b^*\).
2. TWO takes \(b_\alpha \in l_\alpha\).

\[
\begin{array}{c|cccc}
\text{ONE} & b^* & l_0 & l_1 & \cdots & l_\alpha & \cdots \\
\hline
\text{TWO} & b_0 & b_1 & \cdots & b_\alpha & \cdots
\end{array}
\]

For a play \(\langle l_\alpha, b_\alpha : \alpha < \kappa \rangle\),

- TWO wins if \(\bigwedge_{\alpha < \kappa} b_\alpha > 0\).
- Otherwise ONE wins.
Definition 3 (Jech)

$\kappa$: infinite cardinal

Let $G^\kappa$ be the following two player game on cBA $\mathcal{B}$: First, ONE fixes $b^* \in \mathcal{B}^+$. At each stage,

1. ONE chooses a partition $l_\alpha$ of $b^*$.
2. TWO takes $b_\alpha \in l_\alpha$.

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Definition 3 (Jech)

\( \kappa \): infinite cardinal

Let \( G^\kappa \) be the following two player game on cBA \( B \): First, ONE fixes \( b^* \in B^+ \). At each stage,

1. ONE chooses a partition \( l_\alpha \) of \( b^* \).
2. TWO takes \( b_\alpha \in l_\alpha \).

<table>
<thead>
<tr>
<th>ONE</th>
<th>( b^* )</th>
<th>( l_0 )</th>
<th>( l_1 )</th>
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<td>TWO</td>
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For a play \( \langle l_\alpha, b_\alpha : \alpha < \kappa \rangle \),

- TWO wins if \( \land_{\alpha < \kappa} b_\alpha > 0 \).
- Otherwise ONE wins.
Fact 4 (Jech)

1. If ONE does not have a winning strategy in the $G^\kappa$ on $B$, then $B$ is $(\kappa, \infty)$-distributive.
2. The converse also holds.
3. That is, $B$ is $(\kappa, \infty)$-distributive if and only if ONE does not have a winning strategy in $G^\kappa$ on $B$. 
Weak infinite distributivity

Definition 5

\( \kappa \): infinite cardinal, \( \mu \): (finite or infinite) cardinal. \( \mathcal{B} \) is \((\kappa, \infty, < \mu)\)-distributive if

- For every \( b \in \mathcal{B}^+ \) and every partitions \( \langle I_\alpha : \alpha < \kappa \rangle \) of \( b \), there is \( J_\alpha \subseteq I_\alpha \ (\alpha < \kappa) \) such that \( |J_\alpha| < \mu \) and \( \bigwedge_{\alpha<\kappa} \bigvee J_\alpha > 0 \).

- \( \iff \) For every \( b \in \mathcal{B}^+ \) and every \( \mathcal{B}^+ \)-name \( \dot{f} \) for a function from \( \kappa \) to \( V \), there is \( c \leq b \) and \( g : \kappa \to V \) such that \( |g(\alpha)| < \mu \) and \( c \models_{\mathcal{B}^+} \forall \alpha < \kappa \ (\dot{f}(\alpha) \in g(\alpha)) \).

So \((\kappa, \infty)\)-distributive \( \iff \) \((\kappa, \infty, < 2)\)-distributive.
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So \((\kappa, \infty)\)-distributive \( \iff \) \((\kappa, \infty, < 2)\)-distributive.
Weak infinite distributivity

**Definition 5**

κ: infinite cardinal, μ: (finite or infinite) cardinal. B is (κ, ∞, < μ)-distributive if

- For every \( b \in B^+ \) and every partitions \( \langle l_\alpha : \alpha < \kappa \rangle \) of \( b \), there is \( J_\alpha \subseteq l_\alpha \) (\( \alpha < \kappa \)) such that \( |J_\alpha| < \mu \) and \( \bigwedge_{\alpha < \kappa} \bigvee J_\alpha > 0 \).
- \( \iff \) For every \( b \in B^+ \) and every \( B^+ \)-name \( \dot{f} \) for a function from \( \kappa \) to \( V \), there is \( c \leq b \) and \( g : \kappa \to V \) such that \( |g(\alpha)| < \mu \) and \( c \models_{B^+} \forall \alpha < \kappa (\dot{f}(\alpha) \in g(\alpha)) \).

So (κ, ∞)-distributive \( \iff \) (κ, ∞, < 2)-distributive.
Remark 6

If $\mathcal{B}$ is $(\kappa, \infty, < n)$-distributive for some natural number $n \geq 2$, then $\mathcal{B}$ is $(\kappa, \infty)$-distributive. Hence one may assume $\mu$ is 2 or an infinite cardinal.

Definition 7

$\mathcal{B}$ is weakly $(\kappa, \infty)$-distributive if $\mathcal{B}$ is $(\kappa, \infty, < \omega)$-distributive.

There are many interesting cBas which are weakly $(\omega, \infty)$-distributive but not $(\omega, \infty)$-distributive;

- Random algebra, (completion of) Sacks forcing,...
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There are many interesting cBas which are weakly $\omega$-distributive but not $(\omega, \infty)$-distributive;

- Random algebra, (completion of) Sacks forcing,...
The game corresponding the weak \((\kappa, \infty)\)-distributivity is \(G^\kappa<\omega\):

**Definition 8 (Jech)**

\(\kappa\): infinite cardinal, \(\mu\): (finite or infinite) cardinal

Let \(G^\kappa<\mu\) be the following two player game on cBA \(B\):

First, ONE fixes \(b^* \in B^+\). At each stage,

1. ONE chooses a partition \(l_\alpha\) of \(b^*\).
2. TWO takes \(J_\alpha \subseteq l_\alpha\) with \(|J_\alpha| < \mu\).

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For a play \(\langle l_\alpha, J_\alpha : \alpha < \kappa \rangle\),

- TWO wins if \(\bigwedge_{\alpha < \kappa} (\bigvee J_\alpha) > 0\).
- Otherwise ONE wins.
The game corresponding the weak \((\kappa, \infty)\)-distributivity is \(G_{<\omega}^\kappa\):

**Definition 8 (Jech)**

\(\kappa\): infinite cardinal, \(\mu\): (finite or infinite) cardinal
Let \(G_{<\mu}^\kappa\) be the following two player game on cBA \(B\): First, ONE fixes \(b^* \in B^+\). At each stage,

1. ONE chooses a partition \(I_\alpha\) of \(b^*\).
2. TWO takes \(J_\alpha \subseteq I_\alpha\) with \(|J_\alpha| < \mu\).

\[
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  \text{ONE} & b^* & I_0 & I_1 & \cdots & I_\alpha & \cdots \\
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\end{array}
\]

For a play \(\langle I_\alpha, J_\alpha : \alpha < \kappa \rangle\),

- TWO wins if \(\land_{\alpha < \kappa} (\lor J_\alpha) > 0\).
- Otherwise ONE wins.
Fact 9 (Jech)

If ONE does not have a winning strategy in $G^{\kappa}_{<\mu}$ on $\mathcal{B}$, then $\mathcal{B}$ is $(\kappa, \infty, < \mu)$-distributive.

Hence the following are equivalent for $\kappa$, $\mathcal{B}$, and a natural number $n \geq 2$:

1. $\mathcal{B}$ is $(\kappa, \infty, < n)$-distributive.
2. ONE does not have a winning strategy in $G^{\kappa}_{<n}$ on $\mathcal{B}$.

Question 10 (Jech)

If $\mathcal{B}$ is weakly $(\kappa, \infty)$-distributive, does ONE have no winning strategy in $G^{\kappa}_{<\omega}$ on $\mathcal{B}$?
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If ONE does not have a winning strategy in $G_{<\mu}^\kappa$ on $\mathcal{B}$, then $\mathcal{B}$ is $(\kappa, \infty, < \mu)$-distributive.

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Fact 11 (Kamburelis)

There is a cBa $\mathcal{B}$ which is weakly $(\omega, \infty)$-distributive but ONE has a winning strategy in $\mathcal{G}^\omega_{<\omega}$ on $\mathcal{B}$.

So the game $\mathcal{G}^\omega_{<\omega}$ does not characterize the weak $(\omega, \infty)$-distributivity.

Question 12

How about the game $\mathcal{G}^\kappa_{<\omega}$ and the weak $(\kappa, \infty)$-distributivity for $\kappa > \omega$?
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How about the game $G^\kappa_{<\omega}$ and the weak $(\kappa, \infty)$-distributivity for $\kappa > \omega$?
Main result

Theorem 13

Let $\kappa$ be an uncountable cardinal. If ONE does not have a winning strategy in $G^\kappa_{<\omega}$ on $\mathcal{B}$, then $\mathcal{B}$ is $(\kappa, \infty)$-distributive.

Corollary 14

For an uncountable cardinal $\kappa$ and a cBa $\mathcal{B}$, the following are equivalent:

1. ONE does not have a winning strategy in $G^\kappa_{<\omega}$ on $\mathcal{B}$.
2. $\mathcal{B}$ is $(\kappa, \infty)$-distributive.
3. ONE does not have a winning strategy in $G^\kappa$ on $\mathcal{B}$.

So when $\kappa > \omega$, the game $G^\kappa_{<\omega}$ characterizes the $(\kappa, \infty)$-distributivity.
### Main result

#### Theorem 13

Let $\kappa$ be an uncountable cardinal. If ONE does not have a winning strategy in $G^\kappa_{<\omega}$ on $\mathcal{B}$, then $\mathcal{B}$ is $(\kappa, \infty)$-distributive.

#### Corollary 14

For an uncountable cardinal $\kappa$ and a cBa $\mathcal{B}$, the following are equivalent:

1. ONE does not have a winning strategy in $G^\kappa_{<\omega}$ on $\mathcal{B}$.
2. $\mathcal{B}$ is $(\kappa, \infty)$-distributive.
3. ONE does not have a winning strategy in $G^\kappa$ on $\mathcal{B}$.

So when $\kappa > \omega$, the game $G^\kappa_{<\omega}$ characterizes the $(\kappa, \infty)$-distributivity.
However the difference between the $(\omega_1, \infty)$-distributivity and the weak $(\omega_1, \infty)$-distributivity is sensitive:

**Lemma 15 (folklore?)**

If $\kappa \geq 2^\omega$ and $B$ is weakly $(\kappa, \infty)$-distributive, then $B$ is in fact $(\kappa, \infty)$-distributive.

**Corollary 16**

Suppose the Continuum Hypothesis ($2^\omega = \omega_1$). For an uncountable cardinal $\kappa$ and a cBa $B$, the following are equivalent:

1. ONE does not have a winning strategy in $G^\kappa_{<\omega}$ on $B$.
2. $B$ is weakly $(\kappa, \infty)$-distributive.
3. $B$ is $(\kappa, \infty)$-distributive.
4. ONE does not have a winning strategy in $G^\kappa$ on $B$. 


However the difference between the \((\omega_1, \infty)\)-distributivity and the weak \((\omega_1, \infty)\)-distributivity is sensitive:

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1. ONE does not have a winning strategy in \(G^\kappa_{<\omega}\) on \(B\).
2. \(B\) is weakly \((\kappa, \infty)\)-distributive.
3. \(B\) is \((\kappa, \infty)\)-distributive.
4. ONE does not have a winning strategy in \(G^\kappa\) on \(B\).
Fact 17 (Folklore)

Let $\mathcal{B}$ be the random algebra.

1. $\mathcal{B}$ is not $(\omega, \infty)$-distributive.

2. Suppose Martin’s Axiom. For every $\kappa < 2^{\omega}$, $\mathcal{B}$ is weakly $(\kappa, \infty)$-distributive.

Corollary 18

The statement that:

For every $\mathcal{B}$ and uncountable $\kappa$, $\mathcal{B}$ is weakly $(\kappa, \infty)$-distributive if and only if ONE does not have a winning strategy in $\mathcal{G}_{<\omega}^\kappa$ on $\mathcal{B}$,

is independent from ZFC.
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Corollary 18

The statement that:

For every $\mathcal{B}$ and uncountable $\kappa$, $\mathcal{B}$ is weakly $(\kappa, \infty)$-distributive if and only if ONE does not have a winning strategy in $\mathcal{G}_{\leq \omega}^\kappa$ on $\mathcal{B}$,

is independent from ZFC.
Proposition 19

Let $\mu$ be a regular cardinal, and $\kappa > \mu$. Suppose $B$ is $(\mu, \infty)$-distributive. Then the following are equivalent:

1. ONE does not have a winning strategy in $G^\kappa_{<\mu}$ on $B$, and for every $\mu$-Suslin tree, $\Vdash_{B^+} \text{"}T \text{ does not have a cofinal branch".}$

2. $B$ is $(\kappa, \infty)$-distributive.
When $\kappa = \mu = \omega_1$

**Proposition 20**

Suppose Proper Forcing Axiom. Then for every cBa $\mathcal{B}$, if $\mathcal{B}$ is $(\omega, \infty)$-distributive and ONE does not have a winning strategy in $\mathcal{G}_{<\omega_1}^{\omega_1}$ on $\mathcal{B}$, then $\mathcal{B}$ is $(\omega_1, \infty)$-distributive.

In particular, if $\mathcal{B}$ is $(\omega, \infty)$-distributive, then the following are equivalent for $\kappa \geq \omega_1$:

1. $\mathcal{B}$ is $(\kappa, \infty)$-distributive.
2. ONE does not have a winning strategy in $\mathcal{G}_{<\omega_1}^{\kappa}$ on $\mathcal{B}$. 
Question 21

1. How about the weak \((\kappa, \lambda)\)-distributivity and the game \(G^\kappa_{<\omega}(\lambda)\)?

2. More generally, how about the \((\kappa, \lambda, < \mu)\)-distributivity and \(G^\kappa_{<\mu}(\lambda)\)?

Thank you for your attention!
Question 21

1. How about the weak $(\kappa, \lambda)$-distributivity and the game $G^\kappa_{<\omega}(\lambda)$?

2. More generally, how about the $(\kappa, \lambda, < \mu)$-distributivity and $G^\kappa_{<\mu}(\lambda)$?

Thank you for your attention!