Your (Almost) Complete Guide to Using Calculus to Graph Functions

Follow these 8 easy steps to get enough information about your function \( f(x) \) using algebra and calculus to graph \( f(x) \).

1. Identify the domain of \( f \).
   - You should be able to write the domain in interval notation (see the Algebra Review for a refresher on this).

2. Find all \( x \) and \( y \) intercepts of \( f \).
   - To find \( x \)-intercepts: set \( y \) equal to 0 and solve for \( x \). Your answers will be ordered pairs \((x,0)\).
   - To find \( y \)-intercepts: set \( x \) equal to 0 and solve for \( y \). Your answers will be ordered pairs \((0,y)\).

3. Find all vertical, horizontal, and slant (oblique) asymptotes of \( f \). Identify the end behavior of the function if \( f \) is defined as \( x \to \pm \infty \) or if \( f \) is defined on a closed interval.
   - We’ll see vertical asymptotes come up in two different ways: from trig functions (tangent, cotangent, secant, and cosecant) and from rational functions.
     - To review the asymptotes of trig functions, you can look it up in any number of places, including the Trig Review handout.
     - To find the vertical asymptotes of a rational function: (1) simplify to write as a single fraction if necessary and then factor the numerator and denominator, (2) cancel any common factors, (3) the zeros of any factors that are left in the denominator after cancelling are vertical asymptotes of the graph.
     - Vertical asymptotes are equations of vertical lines, so they look like \( x = a \) for some real number \( a \).
   - We covered horizontal asymptotes of many different kinds of functions in Section 2.5. Remember that horizontal asymptotes are equations of horizontal lines, so they look like \( y = b \) for some real number \( b \).
   - Slant asymptotes occur if you have a rational function \( f(x) = \frac{N(x)}{D(x)} \) where \( N \) and \( D \) are polynomial functions and the degree of \( N \) is exactly one more than the degree of \( D \). If this is the case, use polynomial long division to find \( N \div D \). The quotient will be of the form \( mx + b \) and the line \( y = mx + b \) is the equation of the slant asymptote. This is the line that \( f(x) \) will approach as \( x \) goes to \( \pm \infty \).
4. Use the first and/or second derivative test to find all local extrema of $f$.
   - First find all critical values of $f$.
   - Then either use the First Derivative Test (first derivative sign chart) or Second Derivative Test (plug any $c$ such that $f'(c) = 0$ in to $f''(x)$) to identify local maxima and local minima.
   - If a local extrema occurs at $x = c$, then $f(c)$ is called the local extreme value, and you should read the question carefully to determine if your answer should be the $x$ value ($c$), the $y$ value ($f(c)$), or the ordered pair ($c, f(c)$).

5. Find all inflection points of $f$.
   - First find all places in the domain of $f$ where $f''(x) = 0$ or $f''(x)$ DNE.
   - Next use the second derivative sign chart to identify inflection points.
   - Again, read the question carefully to determine if your answer should be the $x$ value of the inflection point ($c$) or the ordered pair ($c, f(c)$). (Usually you won’t be asked just for the $y$ value of an inflection point.)

6. Identify the interval(s) where $f$ is increasing and the interval(s) where $f$ is decreasing.
   - This comes from the first derivative sign chart.

7. Identify the interval(s) where $f$ is concave up and the interval(s) where $f$ is concave down.
   - This comes from the second derivative sign chart (which you may have already made in step 5).

8. Sketch a graph of $f(x)$ using all of this information.
   - As long as your graph meets all the criteria of all of the above information you found, it’s a good enough sketch of the graph for our purposes. (ie, It must have the right domain, intercepts, asymptotes, end behavior, extreme values, and inflection points, and it must be increasing, decreasing, concave up, and concave down in all the right places.)
Examples

1. \( f(x) = x^3 - x^4 \)

2. \( f(x) = x + \frac{1}{x} = \frac{x^2 + 1}{x} \)

3. \( f(x) = x - 2 \cos x \) on the interval \([-\pi, \pi]\)

4. \( f(x) = \frac{3}{x-4} \)

5. \( f(x) = 3x^4 + 4x^3 - 12x^2 \)

6. \( f(x) = 3x^5 - 20x^3 \)

7. \( f(x) = \frac{4x}{x^2+1} \)

8. \( f(x) = \frac{2x-3}{x^3-5x^2+4} \)

9. \( f(x) = \frac{x^2+x-2}{2x^2-2} \)

10. \( f(x) = 3x^{2/3} - x \)

11. \( f(x) = x^{1/3}(x+4) \)

12. \( f(x) = -2(x-4)^{2/3} + 5 \)

13. \( f(x) = 2 \cos x + \sin^2 x \) on \( \left[-\frac{5\pi}{4}, \frac{\pi}{2}\right] \)

14. \( f(x) = \frac{\cos x}{2+\sin x} \) on \( \left[-\frac{2\pi}{3}, 2\pi\right] \)

15. \( f(x) = 3 \sin x - \sin^3 x \) on \([0, 3\pi]\)

16. \( f(x) = \sin(2x) - 2 \sin x \) on \([-\pi, \pi]\)

17. \( f(x) = x\sqrt{9-x^2} \)
Extra Practice and Examples

On an exam, it is possible, but unlikely, that you would be asked to complete all 8 steps of graphing a function for the same function. There is too much room for an error on a beginning or middle step that can really make the rest of the problem unnecessarily difficult. It is much more likely that you will be asked about one or two steps at a time, sometimes steps in the middle of the process. Many of your homework problems are like this, and a few other examples are below.

Problem 1. Use the following information about a function $f(x)$ to answer the questions below.

The domain of $f$ is $(-\infty, \infty)$.

<table>
<thead>
<tr>
<th>$x$-value(s)</th>
<th>$(-\infty, -3)$</th>
<th>$(-3, 0)$</th>
<th>$(0, 6)$</th>
<th>$(6, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of $f'(x)$</td>
<td>$+$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>Sign of $f''(x)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

(A) In the boxes provided, list the $x$-value(s) at which local extrema of $f(x)$ occur.

(B) In the box provided, list the $x$-value(s) at which inflection points of $f(x)$ occur.

(C) Sketch a possible graph for $f(x)$. (Please mark the axes to make your scale clear.)
Problem 2. Find the absolute maximum value and absolute minimum value of the function below on the interval \([0, 2]\) (and fill in the blanks with your answers). If the absolute extreme value does not exist, say so.

\[ f(x) = \frac{1}{3} x^3 - 2x^2 + 3x + \frac{1}{3} \]

The absolute maximum value of \(f(x)\) on \([0, 2]\) is \(\) at \(x = \)

The absolute minimum value of \(f(x)\) on \([0, 2]\) is \(\) at \(x = \)

Problem 3. A rational function and its first and second derivative are given below. Use this information to answer the questions that follow.

\[ R(x) = \frac{3x^2 - 1}{1 - x^2} \quad R'(x) = \frac{4x}{(1 - x^2)^2} \quad R''(x) = \frac{4(3x^2 + 1)}{(1 - x^2)^3} \]

(A) Write the domain of \(R(x)\).

(B) Find all asymptotes (vertical, horizontal, and/or slant) of the graph of \(R(x)\).

(C) Find all \(x\) and \(y\) intercepts of the graph of \(R(x)\).

(D) Determine on what intervals \(R(x)\) is concave up and concave down.

(E) Sketch a possible graph for \(f(x)\). (Please mark the axes to make your scale clear.)

Problem 4. Use the Second Derivative Test to find all local extrema of the function below.

\[ f(x) = 2x^3 - 3x^2 - 72x + 7 \]

Hint: \(72 = 6 \times 12\). Use this fact to help you find the critical values.

\[ \begin{array}{c|c|c}
\text{x-value(s) where local maxima occur} & \text{x-value(s) where local minima occur} \\
\end{array} \]