Functionally Complete Sets

Purpose. The purpose of this project is to determine that the usual logical operators are sufficient to give every possible truth table. Furthermore, you will find other sets of logical operators that are sufficient to give all possible sets of truth tables.

Definitions. A set of logical operators is called **functionally complete** if every compound proposition is logically equivalent to a compound proposition involving only operations in the set. The operators in compound propositions are the usual operators of conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$) , implication ($\rightarrow$), and biconditional ($\leftrightarrow$).

Procedure. Complete the following steps to determine sets of operators that are functionally complete and to better understand what functionally complete means.

1. Prove that the operators $\vee$ and $\neg$ are functionally complete. (Hint: First, show every statement involving only one logical operator and two simple propositions, like $p \rightarrow q$, is logically equivalent to a compound propositions involving only $\vee$ and $\neg$.)
2. Next prove that the operators $\wedge$ and $\neg$ are functionally complete.
3. Consider the operator NAND, denoted by $|$. The proposition $p|q$ is defined to be equivalent with the proposition $\neg(p \wedge q)$. Show that NAND is functionally complete.
4. The operator NOR is denoted by $\downarrow$. The proposition $p \downarrow q$ is defined to be equivalent with the proposition $\neg(p \vee q)$. Show NOR is functionally complete.
5. Find a compound statement involving only NAND whose truth table is given below. You may wish to start with a smaller example (maybe involving only simple propositions) first and then figure out this one based on what you learn from the smaller example.

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

6. The definition of a functionally complete set of operators is that any compound proposition involving the usual logical operators is logically equivalent with a proposition involving only the operators in the functionally complete set. Now, suppose that you make up a "truth table" involving the propositions $p_1, p_2, \cdots, p_n$ by assigning arbitrarily T and F values to each of the 2^n rows. Is it always possible to find a compound proposition involving the statements $p_1, p_2, \cdots, p_n$ and the usual logical operators which gives the made up truth table? Determine the answer to this question. If you think the answer is no, then you should find an example of a table which is not the truth table for any compound proposition. You may want to start looking for a table with $n = 2$. If you do not find an example, then maybe try $n = 3$, and so on. On the other hand, if you think the answer is yes, then you need to prove that no

matter which table you write down, you can find a compound proposition giving this table. In this case, it may be a good idea to try an induction proof. In either case, you will need to prove your answer.