Henry and the Fan

Purpose: This project will give you an introduction to maximizing and minimizing functions. Although you will not learn a general technique, you will use what you know about graphing functions and solving equations to find the maximum of a family of functions.

Procedure: You are to follow the steps below. Do the steps in order because what you learn in one step will often be needed in the next step. After completing the outline below you should write your results and calculations **neatly**. Be sure to include graphs in what you turn in. Your grade will be determined in part by your presentation.

Project Outline:

1. Henry has a fan connected to a power supply as indicated in the simple circuit shown in Figure 1. Note that the fan acts as a simple resistor. Henry wishes to add a variable resistor to the circuit as show in in Figure 2. His problem is that the variable resistor (which is located in the wall) may dissipate energy so fast that a fire may start. In fact, if the power dissipated through the variable resistor is at least $\frac{1}{3}$ the power used by the fan in Figure 1, then there is danger of starting a fire. Any less than $\frac{1}{3}$ is fine. Of course, if the variable resistor is set at a high resistance then not much current flows and little power is dissipated. If the resistance is set very low, it is essentially as if the variable resistor is not there and the power dissipated is essentially zero. You want to be sure that at any intermediate setting the power dissipated is no more than $\frac{1}{3}$ the power consumed by the fan in Figure 1. Is the circuit in Figure 2 safe?
   a. Use Kirchhoff's law and the formula for power to relate the constants V_0, R_0, I_0, and P_0 which represent voltage, resistance, current, and power respectively in Figure 1.
   b. Use Kirchhoff's law and the formula for power on Figure 2 to relate the constants V_0 and R_0 and the variables R, I, and P where R is resistance of the variable resistor, I is the current in the circuit, and P is the power dissipated by the variable resistor.
   c. Combine the formulas of part a to write P as a function of V_0, R, and R_0.
   d. Let $x = \frac{R}{R_0}$ and rewrite the formula in part c so P is written as $P_0 f(x)$ where f is some function of x. (The formula for f should involve only x, not R or R_0.)
   e. Use Mathematica to plot the function $y = f(x)$ for a wide range of values for x. Looking at the graph, guess approximately the value of x which makes y as large as possible. Then make another plot using a smaller domain for the values of x. Continue doing this until you feel you can determine the value of x which makes $f(x)$ as large as possible.
   f. Now prove that your answer in part e is correct. One way to do this is to assume there is some x which makes $f(x)$ larger than the value you obtained in part e and arrive at a contradiction. (Answer the original question concerning the circuits.)
2. Now you are to use the techniques you learned in part 1 to maximize (or minimize) two other functions. Do parts e and f above using the functions $f(x) = \frac{x+2}{(x+3)^2}$ and $f(x) = \frac{x+1}{(2x+1)^2}$. Note that on one the asymptote goes up instead of down, so try to find the smallest value for y instead of the largest for that functions.

3. In this step you are to generalize the techniques you learned in parts 1 and 2. Consider the function $f(x) = \frac{(ax+b)}{(cx+d)^2}$ where $ad \neq bc$, $c \neq 0$, and $a \neq 0$.
   a. Find conditions on the numbers a, b, c, and d which determine if the vertical asymptote goes up or goes down.
   b. Find a formula for x in terms of a, b, c, d which makes $f(x)$ as large as possible in the case of the asymptote going down and as small as possible in the case of the asymptote going up. (Try to use what you learned in part 2. It may be useful to use Mathematica for solving certain equations.)

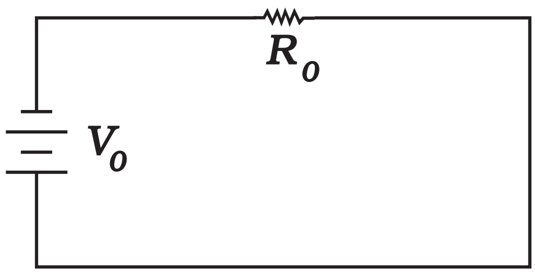

Figure 1.

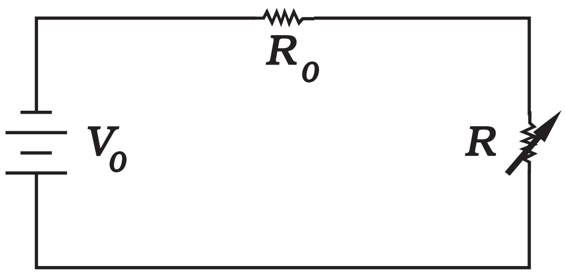

Figure 2.