Life On Long Street

Purpose: In this project you will investigate compositions of functions.

The Problem: Tom lives on Long street. Long street is straight and goes from El Paso to Texarkana, in other words it is infinite in both directions. The addresses along the street are given by real numbers. (Tom lives at $\sqrt{2}$, while his friend Mary lives at π.) There are only two moving companies who will move people along Long street. The companies are company f and company g. When someone at address x moves using company f then that person is moved to the address $f(x)$. When someone at address x moves using company g then that person moves to $g(x)$. Now, the problem is to determine all the possible places Tom could move by using the companies f and g. He can use each company as many times as he wishes and he can use them in any order he wishes.

Procedure: You are to follow the steps below. You do not need to do the steps in exactly the order indicated as long as you cover everything you are asked to do. After completing the outline below you should write your results and calculations **neatly**.

Project Outline:

1. Consider the functions $f(x) = 1 - x$ and $g(x) = \frac{1}{x}$.
   a. Find all possible functions you can obtain by taking repeated compositions of f and g.
   b. Prove that you get the functions you say and no more.
   c. Based on these computations determine the possible addresses Tom could have if he moves.
   d. If company g attempts to move someone at address 0 then that person moves to Arkansas. What are the possible addresses one could have so that moving with f and g could make that person move to Arkansas?
2. Now consider the functions $f(x) = 2x$ and $g(x) = \frac{1}{x}$. There are an infinite number of different functions you can get by repeatedly composing these functions. Find a general form for all the functions you can get by taking repeated compositions of these two functions and prove that you get exactly the functions you claim. Based on these computations determine where Tom can move.
3. Notice that both functions f and g in Part 1 could be written in the form $\frac{ax+b}{cx+d}$ and both have the property that when you compose the function with itself you get the identity function $i(x) = x$.
   a. Determine all values for a, b, c and d which make $f(x) = \frac{ax+b}{cx+d} = x$. (That is, find equations which these numbers satisfy exactly when $f(x) = x$ as long as $x \neq -\frac{d}{c}$.) Prove your answer.
   b Find all functions of the form $f(x) = \frac{ax+b}{cx+d}$ with the property that $f \circ f(x) = x$ for all values of x in the domain of $f \circ f$. (Be careful, consider all the cases.)