Math 1650 Project: "A Piece of π"

This project will introduce to various techniques that mathematicians used to calculate π before the advent of computers. All of these techniques involve the creative use of trig identities.

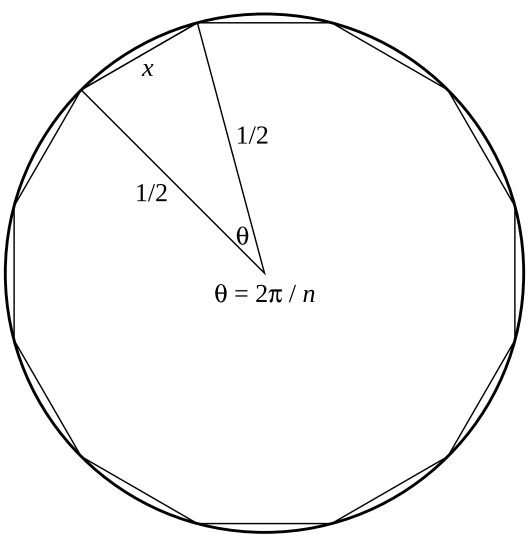
Although you have a month to do this project, I *strongly* urge you not to put this off until the last minute, as the project does require some creative thinking.

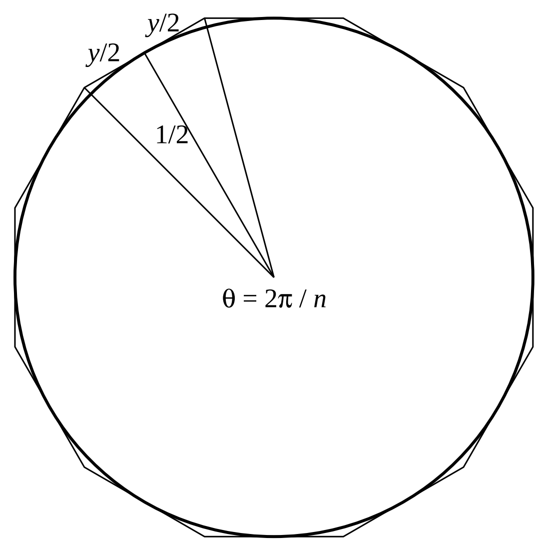
The grading for the project will be 80% correctness, 20% clarity of presentation. Collaboration with your classmates is permitted and encouraged; however, you may *not* discuss the project with anyone else (e.g., the seniors). On the front page on your write-up, you must describe all such collaboration.

1. Archimedes (c. 250 B.C.).

Consider regular n-sided polygons (called n-gons for short) which are either inscribed in or circumscribed about a circle with diameter 1. Then the perimeters of the inscribed n-gons must be less than the circumference of the circle, which is π. Likewise, the perimeters of the circumscribed n-gons must be greater than π. As n increases, then the perimeters of the n-gons will approach π.

Notice that the perimeter p_n of an inscribed regular n-gon is equal to nx, where x is the third side of a triangle with two sides of length $1/2$ and an included (central) angle of $2\pi/n$. Likewise, the perimeter P_n of a circumscribed n-gon is equal to ny, where y is the base of an isoceles triangle with altitude is equal to $1/2$ and central angle $2\pi/n$.





(a) Use the law of cosines and a double-angle identity to show that

$$x^2 = \sin^2\left(\frac{\pi}{n}\right),$$

and hence conclude that the perimeter of an inscribed n-gon is given by

$$p_n = n\sin\left(\frac{\pi}{n}\right).$$

(b) Show that $y = n\tan(\pi/n)$, and hence conclude that the perimeter of a circumscribed n-gon is

$$P_n = n\tan\left(\frac{\pi}{n}\right).$$

(c) Without using a calculator, fill in the second and third columns of the following table. You will need to use the half-angle trig identities to obtain P_{12} and P_{24}. Copy the following table, with the values filled in, onto your write-up.

n	p_n (exact)	P_n (exact)	p_n (approximate)	P_n (approximate)
4				
6				
8				
12				
24				

(c) Using a calculator, evaluate each of these approximations to π. Not surprisingly, observe that

$$p_n = n \sin\left(\frac{\pi}{n}\right) < \pi < n \tan\left(\frac{\pi}{n}\right) = P_n. \tag{1}$$

Note: Archimedes used this method with inscribed and circumscribed 96−gons to evaluate π to three decimal places. However, this method converges quite slowly, as you can check on your calculator: p_{1000} only gives π to five decimal places, while p_{10000} is accurate to seven.

2. Francois Vieta (1593).

(a) Use a double-angle identity repeatedly to show that

$$\begin{aligned}
\sin x &= 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) \\
&= 4\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\sin\left(\frac{x}{4}\right) \\
&= 8\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\sin\left(\frac{x}{8}\right) \\
&= 2^n\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\ldots\cos\left(\frac{x}{2^{n-1}}\right)\cos\left(\frac{x}{2^n}\right)\sin\left(\frac{x}{2^n}\right)
\end{aligned}$$

(b) Recall that, when θ is measured in radians, $\sin\theta \approx \theta$ for small θ. Let $\theta = x/2^n$, and conclude that

$$\frac{\sin x}{x} \approx \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\ldots\cos\left(\frac{x}{2^{n-1}}\right)\cos\left(\frac{x}{2^n}\right).$$

(c) Substitute $x = \pi/2$ into (b) to obtain the following approximations:

$$\begin{aligned}
\frac{2}{\pi} &\approx \cos\frac{\pi}{4}\cos\frac{\pi}{8} \\
\frac{2}{\pi} &\approx \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16} \\
\frac{2}{\pi} &\approx \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cos\frac{\pi}{32} \qquad (2)
\end{aligned}$$

These approximations become quite accurate as n, the number of terms, increases.

(d) Use $\cos(\pi/4) = \sqrt{2}/2$ and a half-angle identity to show that

$$\begin{aligned}
\cos\frac{\pi}{8} &= \frac{\sqrt{2+\sqrt{2}}}{2} \\
\cos\frac{\pi}{16} &= \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \\
\cos\frac{\pi}{32} &= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}
\end{aligned}$$

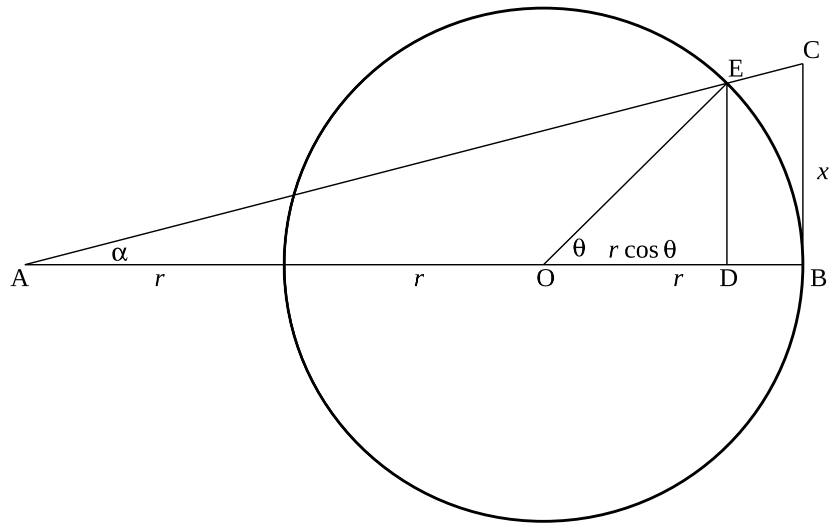
(e) Use your calculator and the answers to (c) and (d) to obtain three successive approximations of π.

Note: As you can check on your calculator, π can be obtained to four decimal places by going up to $\pi/256$. Vieta used this method with *thirty terms* to evaluate π to seventeen decimal places:

$$\pi \approx 3.14159265358979323.$$

This was a fairly amazing accomplishment since he obtained all of those square roots *by hand*.

3. Willebrod Snell (1621) and Christian Huygens (1654).



(a) In the above figure, show that

$$\tan\alpha = \frac{\sin\theta}{2+\cos\theta}.$$

(b) Since $x = 3r\tan\alpha$, show that

$$x = \frac{3r\sin\theta}{2+\cos\theta}.$$

(c) We now make the approximation that $x \approx \widehat{BE}$. Using the formula $\widehat{BE} = r\theta$, show that

$$\theta \approx \frac{3\sin\theta}{2+\cos\theta}.$$

(d) Let $\theta = \pi/n$ to obtain the following approximation to π:

$$\pi \approx \frac{3n\sin(\pi/n)}{2+\cos(\pi/n)}.$$

(e) In the following figure, use simple angle formulas from geometry (e.g., isoceles triangles, exterior angles) to show that $\theta = 3\beta$.

(f) Show that

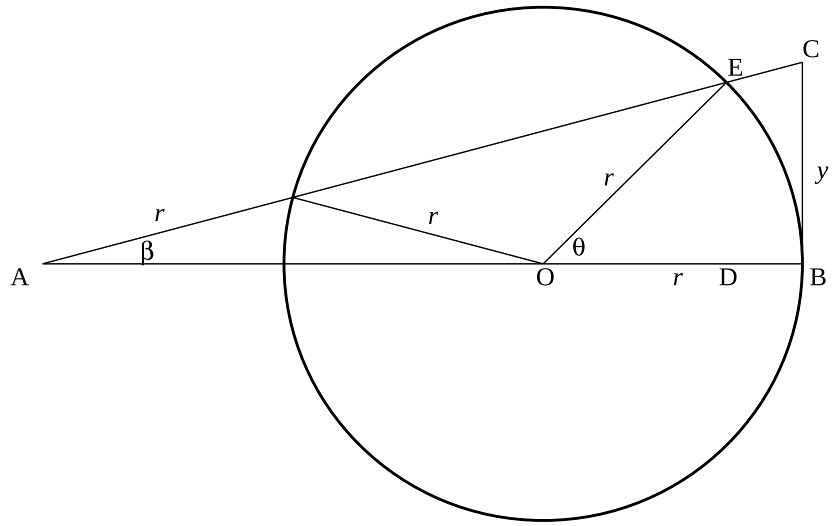
$$y = r\left(1 + 2\cos\frac{\theta}{3}\right)\tan\frac{\theta}{3}.$$

(g) Again using the approximation $y \approx \widehat{BE}$ and the formula for the length of a circular arc, show that

$$\theta = \left(1 + 2\cos\frac{\theta}{3}\right)\tan\frac{\theta}{3}.$$

(h) Let $\theta = \pi/n$ to obtain the following approximation to π:

$$\pi \approx n\left(1 + 2\cos\frac{\pi}{3n}\right)\tan\frac{\pi}{3n}.$$



Although we do not prove this here, the approximations (d) and (h) are in fact *bounds* on π:

$$\frac{3n\sin(\pi/n)}{2+\cos(\pi/n)} < \pi < n\left(1+2\cos\frac{\pi}{3n}\right)\tan\frac{\pi}{3n}. \tag{3}$$

We will now show that these bounds are *tighter* than the bounds (1); that is,

$$n\sin\left(\frac{\pi}{n}\right) < \frac{3n\sin(\pi/n)}{2+\cos(\pi/n)} < \pi < n\left(1+2\cos\frac{\pi}{3n}\right)\tan\frac{\pi}{3n} < n\tan\left(\frac{\pi}{n}\right)$$

(i) Show that

$$n\sin\left(\frac{\pi}{n}\right) < \frac{3n\sin(\pi/n)}{2+\cos(\pi/n)}.$$

Hint: $\cos(\pi/n) < 1$ for all positive integers $n \geq 2$.

(j) To show that the Snell-Huygens upper bound is less than Archimedes' upper bound, first prove the trig identity

$$\begin{aligned}\tan 3x &= \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}\\ &= 3\tan x + \frac{8\tan^3 x}{1-3\tan^2 x}\end{aligned}$$

(k) Substitute $x = \pi/(3n)$ into the above identity, and show that the second term is positive for $n \geq 3$. Conclude that

$$n\left(1+2\cos\frac{\pi}{3n}\right)\tan\frac{\pi}{3n} < 3n\tan\frac{\pi}{3n} < n\tan\left(\frac{\pi}{n}\right)$$

(l) **Beauty contest.** Evaluate the approximations (1), (2) and (3) up to terms involving $\pi/128$. That is, substitute $n = 128$ into (1) and (3), and evaluate a product similar to (2) up to and including $\cos(\pi/128)$. Which method provides the best estimate of π?

4. John Machin (1706).

(a) Use a double-angle identity twice to prove the following identity:

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

(b) Use (a) to show that

$$\tan\left[4\tan^{-1}\left(\frac{1}{5}\right)\right] = \frac{240}{238}.$$

(c) Prove the following identity:

$$\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}.$$

(d) Use (c) to find the solution of the equation

$$\tan\left(\frac{\pi}{4} + x\right) = \frac{240}{238}.$$

(e) Use (b) and (d) to show that

$$\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right).$$

(f) In calculus, we will show that, for small x, $\tan^{-1} x$ can be accurately approximated by

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} \cdots$$

We will use this result to approximate π. Use the first two terms of the above expression to approximate $\tan^{-1}(\frac{1}{239})$, and all six terms to approximate $\tan^{-1}(\frac{1}{5})$. You may use a calculator if you wish, but observe that this calculation could be done without a calculator with sufficient patience.

(g) Use (e) and (f) to approximate π to seven decimal places.

With sufficient creativity, other trig identities can be exploited to obtain approximations to π. Before the advent of computers, the record was held by William Shanks in 1829, who used the series in (f), expanded to *over two hundred* terms, to evaluate π (or so he thought) to 707 decimal places. Unfortunately, it turned out there was an error in the 528th decimal place, so that all subsequent digits were wrong. At present and with the help of supercomputers, π is known to several billion decimal places. The first two million places were published about fifteen years ago in what must be the world's most boring 800-page book.