Polynomials and Trigonometry

Purpose: In this project you will investigate the feasibility of approximating the sin and cos functions using polynomials. There is a general method using calculus to determine how closely a function can be approximated using polynomials. You will investigate how well polynomials you derive approximate the sin and cos functions.

Procedure: You are to follow the steps below. You do not need to do the steps in exactly the order indicated as long as you cover everything you are asked to do. After completing the outline below you should write your results and calculations **neatly**. Be sure to include relevant graphs in what you turn in. Your grade will be determined in part by your presentation.

Project Outline: Elmer is writing an educational computer program for kids. The program contains bright graphics and cute sounds. In order to make sectors of circles appear on the screen the program needs to be able to compute $\sin x$ and $\cos x$ for random values of x. Elmer is using C++ to write the program. In his version of C++, if he includes the standard library of math functions it will increase his code size beyond what would be feasible for the market. He therefore wishes to write his own functions in C++ which would give him a good approximation for $\sin x$ and $\cos x$, so he does not need to include the whole math library. He wishes to find polynomials $p(x)$ and $q(x)$ so that $|p(x) - \sin(x)| < .0005$ and $|q(x) - \cos x| < .0005$ for any value of x between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. Elmer thinks he remembers from his calculus course several years ago that for any desired degree of accuracy it is possible to find two polynomials to approximate the sin and cos functions for every real value x. His daughter Anna, who is taking precalculus says that would be impossible. Who is right? Is it possible for Elmer to find polynomials p and q with the properties given above? If so what are the lowest degree polynomials he can find that meet the condition?

1. Plot $\sin x$ and $mx + b$ on the same axes for various values of m and b. Experiment until you find the values for m and b which give a line that seems to be the best approximation for $\sin x$ when x is **very close to 0**. Do the same for $\cos x$.
2. The next step is to prove the approximation for $\sin\theta$ found in part 1 when θ is close to 0 is a good approximation. Follow the steps below.
    a. Draw a sector of a circle of radius 1 having angle θ with one ray on the x-axis and the other in the first quadrant. Draw two right triangles using the angle of the sector as one angle. For the first triangle put the right angle at the point (1,0). For the second, drop a perpendicular from the point on the circle meeting the other side of the sector to the x-axis.
    b. Compare the area of the sector with the areas of the triangles to get inequalities.
    c. Simplify the inequalities found in c to get an approximation for $\sin\theta$. State why this proves your approximation in part 1 is a good approximation.
3. Next you will use what you learned in part 1 to find polynomial approximations for $\sin x$ and $\cos x$. The philosophy you should follow is that you have a linear approximation for $\sin x$ and $\cos x$ when x is close to zero. Use the double angle formula to

rewrite $\sin x$ in terms of trig functions of angles smaller than x. At this point you can approximate all the occurrences of sin and cos with the linear approximations you found above. Try using the double angle formula several times before making the approximations to see if you get a better polynomial approximation. Do the same for cos. (Mathematica will simplify your calculations!)

4. Plot some graphs to see how close the approximations you found in part 2 are to $\sin x$ and $\cos x$.
5. Try removing some of the highest order terms to see if you lose much accuracy in the approximation. What are the lowest degree polynomials you can find that meet Elmer's conditions for approximating the sin and cos functions? Plot graphs to show that the error is acceptable to Elmer.
6. Try to approximate π using what you found above. (Be crafty!)
7. Who is right, Elmer or Anna? Discuss why.
8. Discuss why you would expect the process you did in step 3 to work.