Math 1720 Midterm 1 Review Problems
(Section 7.4 not included in this review, but it is examinable for the midterm.)

0.
(a) Find

$$\log_{27}(9).$$

Solution.
Since $9 > 0$, $\log_{27}(9)$ is defined. We have

$$\log_{27}(9) = y$$

iff

$$27^y = 9$$

iff

$$(3^3)^y = 3^2$$

iff

$$3^{3y} = 3^2.$$

Since $f(x) = 3^x$ is 1-1 (as a^x is a 1-1 function whenever $0 < a < 1$ or $1 < a$), this is equivalent to

$$3y = 2$$

iff

$$y = 2/3.$$

So

$$\log_{27}(9) = 2/3.$$

(b) Find all solutions to the equation

$$\ln(2 - x) + \ln(5 - x) = 2\ln(5).$$

*Solution.*First note that $\ln(2 - x)$ is defined just when $2 - x > 0$, i.e. when $x < 2$. And $\ln(5 - x)$ is defined just when $5 - x > 0$, i.e. when $x < 5$. Putting both these conditions together, since $2 < 5$, any valid solution must be in the interval $x < 2$.

So, for $x < 2$, we have $2 - x > 0$ and $5 - x > 0$, so we can apply the rule $\ln(a) + \ln(b) = \ln(ab)$ (which applies just when $a, b > 0$), and the equation becomes:

$$\ln((2 - x)(5 - x)) = 2\ln(5), \quad x < 2$$

This is equivalent to

$$\ln((2 - x)(5 - x)) = \ln(5^2) = \ln(25), \quad x < 2.$$

Since ln is a 1-1 function, this is equivalent to

$$(2 - x)(5 - x) = 25, \quad x < 2$$

iff

$$10 - 7x + x^2 = 25, \quad x < 2$$

iff

$$x^2 - 7x - 15 = 0, \quad x < 2$$

iff

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-15)}}{2(1)}, \quad x < 2$$

iff

$$x = \frac{7 \pm \sqrt{49 + 60}}{2}, \quad x < 2$$

iff

$$x = \frac{7}{2} \pm \frac{\sqrt{109}}{2}, \quad x < 2.$$

So we have two potential solutions, but for each, it is in fact a solution iff it is < 2. So we check this for each potential solution.

Now $\frac{7}{2} = 3.5 > 2$, so

$$\frac{7}{2} + \frac{\sqrt{109}}{2} > 2,$$

so this is not a solution.

But $109 > 100$, so $\sqrt{109} > \sqrt{100} = 10$, so $\frac{1}{2}\sqrt{109} > 5$, so

$$\frac{7}{2} - \frac{\sqrt{109}}{2} < 3.5 - 5 = -1.5 < 2.$$

So

$$x = \frac{7}{2} - \frac{\sqrt{109}}{2} < 2,$$

and so this is a solution, and in fact is the unique solution.

(c) Find all solutions to the equation

$$e^{x^3} = 6^x.$$

Solution.

Note that both sides of the equation are defined for all values of x. The equation is equivalent to

$$e^{x^3} = (e^{\ln(6)})^x = e^{x\ln(6)}.$$

Since $f(x) = e^x$ is a 1-1 function (it's the inverse of $g(x) = \ln(x)$, so is 1-1), this is equivalent to

$$x^3 = x\ln(6)$$

iff

$$x^3 - x\ln(6) = 0$$

iff

$$x(x^2 - \ln(6)) = 0$$

iff

$$x(x + \sqrt{\ln(6)})(x - \sqrt{\ln(6)}) = 0.$$

Note that $0 < \ln(6)$ since $1 < 6$, and since ln is an increasing function, $\ln(1) < \ln(6)$ so $0 < \ln(6)$. Therefore $\sqrt{\ln(6)}$ makes sense.

So we have 3 solutions: $x = 0$ and $x = \pm\sqrt{\ln(6)}$.

1.

(a) Find and simplify

$$\int_{-1}^{1} 6^{2x} dx$$

*Solution.*Since $6^{2x} = (6^2)^x = 36^x$, this is

$$\int_{-1}^{1} (36)^x dx$$

$$= \frac{1}{\ln(36)} 36^x\Big|_{-1}^{1}$$

$$= \frac{1}{\ln(6^2)}(36^1 - 36^{-1})$$

$$= \frac{1}{2\ln(6)}(36 - \frac{1}{36})$$

$$= \frac{1}{2\ln(6)}(\frac{36^2 - 1}{36})$$

$$= \frac{6^4 - 1}{72\ln(6)}.$$

(I would consider $6^4 - 1$ to be simpler than 1295.)

(b) Find and simplify

$$\int_{3}^{10} \frac{5}{2-x} dx$$

*Solution.*Since $\frac{d}{dx}(\ln(|2-x|)) = \frac{1}{2-x}(2-x)' = -\frac{1}{2-x}$, we have

$$\int_{3}^{10} \frac{5}{2-x} dx = -5\ln(|2-x|)\Big|_{3}^{10}$$

(Alternatively, you might also have done a substitution of $u = 2 - x$ to get started.)

$$= -5\ln(|2-10|) - (-5\ln(|2-3|))$$

$$= -5\ln(|-8|) + 5\ln(|-1|)$$

$$= -5\ln(8) + 5\ln(1)$$

$$= -5\ln(8) + 5(0)$$

$$= -5\ln(8) = -5\ln(2^3) = -15\ln(2).$$

(Remark: the solution here has been corrected from an earlier version. In the earlier version I swapped over the locations of plugging in $x = 3$ and $x = 10$, so ended up with an answer of $+15\ln(2)$ instead. Note that $-15\ln(2)$ is negative, since $\ln(2) > 0$ since $\ln(x) > 0$ for $x > 1$. You can verify independently that the final answer *should* be negative, because the function we're integrating, $f(x) = 5/(2-x)$, has $f(x) < 0$ for all x in the interval of integration, $3 \le x \le 10$. And $3 \le 10$, i.e. the bounds of integration in $\int_3^{10}$ are in the usual order. So the integral $\int_3^{10} f(x)dx$ must come out negative. (I'm also using the fact that f is continuous over $3 \le x \le 10$.))

(c) Find

$$\int_0^{\pi/2} e^{3\sin(x)} \cos(x)dx$$

*Solution.*Since $(3\sin(x))' = 3\cos(x)$, we have

$$\frac{d}{dx}(e^{3\sin(x)}) = e^{3\sin(x)}(3\sin(x))' = 3e^{3\sin(x)}\cos(x).$$

So

$$\int_0^{\pi/2} e^{3\sin(x)} \cos(x)dx$$

$$= \frac{1}{3}e^{3\sin(x)}|_0^{\pi/2}$$

(You might alternatively have done a substitution of $u = 3\sin(x)$.)

$$= \frac{1}{3}(e^{3\sin(\pi/2)} - e^{3\sin(0)})$$

$$= \frac{1}{3}(e^3 - e^0)$$

$$= \frac{1}{3}(e^3 - 1).$$

(d) Find

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

*Solution.*Since $(e^x + e^{-x})' = (e^x - e^{-x})$, and that's the numerator, subbing $u = e^x + e^{-x}$ will be useful. Then $du = (e^x - e^{-x})dx$, so

$$\int = \int \frac{du}{u} = \ln(|u|) + c$$

$$= \ln(|e^x + e^{-x}|) + c$$

$$= \ln(e^x + e^{-x}) + c,$$

the latter since $e^x + e^{-x} > 0$, since both $e^x > 0$ and $e^{-x} > 0$.

2.

(a) Find

$$\frac{d}{dx}(\frac{x^3}{\ln(x^3)}).$$

*Solution.*Note that $\ln(x^3)$ is only defined when $x^3 > 0$, i.e. when $x > 0$, so we have $\ln(x^3) = 3\ln(x)$ for over its domain. So the derivative is equivalent to:

$$\frac{d}{dx}(\frac{x^3}{3\ln(x)})$$

$$= \frac{1}{3}\frac{d}{dx}(\frac{x^3}{\ln(x)}).$$

Appying the quotient rule,

$$= \frac{1}{3}\frac{(x^3)'\ln(x) - x^3(\ln(x))'}{(\ln(x))^2}$$

$$= \frac{1}{3}\frac{3x^2\ln(x) - x^3(1/x)}{\ln(x)^2}$$

$$= \frac{1}{3}\frac{3x^2\ln(x) - x^2}{(\ln(x))^2}$$

$$= \frac{1}{3}\frac{x^2(3\ln(x) - 1)}{(\ln(x))^2}.$$

The domain of the original function, $x^3/\ln(x^3)$, is the union of the intervals $(0, 1)$ and $(1, \infty)$. For $x^3/\ln(x^3)$ is defined iff both numerator and denominator are defined, and denominator is non-0. The numerator x^3 is defined for all x. The denominator $\ln(x^3)$ is defined iff $x^3 > 0$ iff $x > 0$, and is non-0 iff $x^3 \neq 1$ iff $x \neq 1$. Therefore the function is defined just for $0 < x < 1$ and $1 < x$. The formula we found for the derivative gives the same domain (its numerator works for $x > 0$ and its denominator $(\ln(x))^2$ works for $x > 0$ and is non-0 for $x \neq 1$), so it is correct as written and needn't be restricted.

(b) Find

$$\frac{d}{dx}(\sqrt{x}^{\sqrt{x}}).$$

*Solution.*The function to be differentiated has the form $f(x)^{g(x)}$, where $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x}$. Here the domain is $\sqrt{x} > 0$, i.e. $x > 0$, since we're using $\sqrt{x}$ in the base. We first convert the function to be differentiated to base e:

$$= \frac{d}{dx}(e^{\ln(\sqrt{x})})^{\sqrt{x}}$$

$$= \frac{d}{dx}(e^{\sqrt{x}\cdot\ln(x^{\frac{1}{2}})})$$

$$= \frac{d}{dx}(e^{(\frac{1}{2}\sqrt{x}\ln(x))}).$$

Now applying the chain rule and that $(e^x)' = e^x$:

$$= e^{(\frac{1}{2}\sqrt{x}\ln(x))}(\frac{1}{2}\sqrt{x}\ln(x))'$$

Pulling the $\frac{1}{2}$ through and then applying the product rule:

$$= \frac{1}{2} e^{(\frac{1}{2}\sqrt{x}\ln(x))}(\sqrt{x}' \ln(x) + \sqrt{x}(\ln(x))')$$

$$= \frac{1}{2} e^{(\frac{1}{2}\sqrt{x}\ln(x))}(\frac{1}{2\sqrt{x}} \ln(x) + \sqrt{x}\frac{1}{x})$$

$$= \frac{1}{2} e^{(\frac{1}{2}\sqrt{x}\ln(x))}(\frac{1}{2\sqrt{x}} \ln(x) + \frac{1}{\sqrt{x}})$$

$$= \frac{1}{2\sqrt{x}} e^{(\frac{1}{2}\sqrt{x}\ln(x))}(\frac{1}{2} \ln(x) + 1)$$

From the first few steps, we have $e^{(\frac{1}{2}\sqrt{x}\ln(x))} = \sqrt{x}^{\sqrt{x}}$ so:

$$= \frac{1}{2\sqrt{x}} \sqrt{x}^{\sqrt{x}}(\frac{1}{2} \ln(x) + 1)$$

You could also combine the powers of x into a single term: $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ and $\sqrt{x}^{\sqrt{x}} = (x^{\frac{1}{2}})^{\sqrt{x}} = x^{(\frac{1}{2}\sqrt{x})}$, so

$$= \frac{1}{2}(x^{-\frac{1}{2}})x^{(\frac{1}{2}\sqrt{x})}(\frac{1}{2} \ln(x) + 1)$$

$$= \frac{1}{2}x^{(-\frac{1}{2}+\frac{1}{2}\sqrt{x})}(\frac{1}{2} \ln(x) + 1)$$

$$= \frac{1}{2}x^{(\frac{1}{2}(-1+\sqrt{x}))}(\frac{1}{2} \ln(x) + 1)$$

(c) Let $a = (e^{13} + 6/\pi)$. Find

$$\frac{d}{dx}(a^x x^{\ln(a)}).$$

We must differentiate the product of the functions a^x and $x^{\ln(a)}$, so by the product rule:

$$= (a^x)' x^{\ln(a)} + a^x (x^{\ln(a)})'$$

For the first term: Since a is a constant, a^x has a constant base and variable exponent, so it's an exponential function, so we use the exponential function rule:

$$= \ln(a) a^x x^{\ln(a)} + a^x (x^{\ln(a)})'$$

For the second term: since a is a constant, so is $\ln(a)$, so $x^{\ln(a)}$ has variable base and constant exponent, so we can just use the power rule:

$$= \ln(a) a^x x^{\ln(a)} + a^x \ln(a) x^{\ln(a)-1}$$

$$= \ln(a) a^x (x^{\ln(a)} + x^{\ln(a)-1})$$

3. Let $f(x) = \sqrt{x^6 - 4}$. Find the longest intervals over which f has an inverse, and find the formula for the inverse over each such interval.

*Solution.*We have $f(x)$ is defined just when $x^6 - 4 \geq 0$, i.e. when

$$x^6 \geq 4$$

iff

$$(x^6)^{1/6} \geq 4^{1/6}$$

(since $f(x) = x^{1/6}$ is an increasing function) iff

$$|x| \geq 4^{1/6}$$

iff

$$x \geq 4^{1/6} \quad \text{or} \quad x \leq -4^{1/6}.$$

We look at f' to determine where f is increasing/decreasing, in order to determine where it is 1-1. Now

$$f(x) = (x^6 - 4)^{\frac{1}{2}}$$

so by the chain and power rules,

$$\begin{aligned} f'(x) &= \frac{1}{2}(x^6 - 4)^{-\frac{1}{2}}(x^6 - 4)' \\ &= \frac{1}{2\sqrt{x^6 - 4}}(6x^5) \\ &= \frac{3x^5}{\sqrt{x^6 - 4}}. \end{aligned}$$

Now $f'(x) = 0$ iff the numerator is 0 and the denominator non-0. The denominator is 0 just when $x^6 - 4 = 0$, i.e. $x = \pm 4^{1/6}$ (so the derivative is not defined at these points). And so calculating like above, $f'(x)$ is defined for $x > 4^{1/6}$ and for $x < -4^{1/6}$ (not at $x = \pm 4^{1/6}$).

The numerator is 0 just when $3x^5 = 0$, which is iff $x = 0$. But $x = 0$ is not in the domain of f anyway.

The denominator $\sqrt{x^6 - 4} > 0$ for all x in the domain of f (and therefore for all x in the domain of f'). Therefore the sign of $f'(x)$ is the same as that of its numerator, $3x^5$.

For $x > 4^{1/6}$, we have $x > 0$, so the numerator $3x^5 > 0$. Therefore $f'(x) > 0$, so f is increasing over the interval $(4^{1/6}, \infty)$. Since f is also continuous, it is therefore in fact increasing over $[4^1/6, \infty)$.

For $x < -4^{1/6}$, we have $x < 0$, so the numerator $3x^5 < 0$. Therefore $f'(x) < 0$, so f is decreasing over the interval $(-\infty, -4^{1/6})$. Since f is also continuous, it is therefore in fact decreasing over $(-\infty, 4^{1/6}]$.

Since the domain of f is $(-\infty, -4^{1/6}] \cup [4^{1/6}, \infty)$, these are the longest intervals over which f is increasing/decreasing: If we wanted to enlarge $D_1 = (-\infty, -4^{1/6}]$ to a larger interval, and actually include more points in the domain of f, we would need to include $+4^{1/6}$ (since nothing in the interval $(-4^{1/6}, 4^{1/6})$ is included in f's domain). But $f(-4^{1/6}) = 0 = f(4^{1/6})$, so f is not decreasing over $(-\infty, 4^{1/6}]$, and therefore is neither decreasing over any interval larger than that.

(We can ignore points in the interval $(-4^{1/6}, 4^{1/6})$ since none of these are included in the domain of f.)

Since f is decreasing over $D_1 = (-\infty, -4^{1/6}]$, it is 1-1 over this interval. Since f is increasing over $D_2 = [4^{1/6}, \infty)$, it is 1-1 over this interval. And f is not 1-1 over any intervals which are larger in any useful way: as remarked above, if we extend $(-\infty, -4^{1/6}]$ to the larger interval $(-\infty, 4^{1/6}]$, then since $f(-4^{1/6}) = 0 = f(4^{1/6})$, we have f is not 1-1 over the latter interval. So D_1, D_2 are the largest intervals over which f is 1-1.

(Again we ignore points in the interval $(-4^{1/6}, 4^{1/6})$.)

Since f has an inverse over an interval iff it is 1-1 over that interval, we know that the longest intervals over which f has an inverse are D_1 and D_2. Note that the range of f over $[4^{1/6}, \infty)$ is $[0, \infty)$. This is because $f(4^{1/6}) = 0$, and f is increasing over the interval $[4^{1/6}, \infty)$ (established earlier), and $f(x) \geq 0$ for all x in $[4^{1/6}, \infty)$. And f is continuous over $[4^{1/6}, \infty)$,

$$\lim_{x\to\infty} (x^6 - 4) = \infty,$$

and since also

$$\lim_{z\to\infty} \sqrt{z} = \infty,$$

we get

$$\lim_{x\to\infty} \sqrt{x^6 - 4} = \infty.$$

Therefore by the intermediate value theorem, for every $a \geq 0$ there is c in the interval $[4^{1/6}, \infty)$ such that $f(c) = a$. So the range of f over $D_1 = [4^{1/6}, \infty)$ is $R_1 = [0, \infty)$.

Since f is an even function, and $D_2 = (-\infty, -4^{1/6}]$ is the reflection of $D_1 = [4^{1/6}, \infty)$ across the y-axis, we get that the range of f over D_2 is $R_2 = R_1 = [0, \infty)$. (That is, if x is in $(-\infty, -4^{1/6}]$ then $f(x) = f(-x)$, and $-x \geq 4^{1/6}$, so $f(x) = f(-x) \geq 0$. And if a is in $[0, \infty)$, then there's c in $[4^{1/6}, \infty)$ such that $f(c) = a$, so $f(-c) = a$ also, and $-c$ is in $(-\infty, -4^{1/6}]$. Thus the range of f over $(-\infty, -4^{1/6}]$ is $[0, \infty)$.)

Now for x in D_1 or in D_2, and for any number y, we have

$$f(x) = y$$

iff

$$\sqrt{x^6 - 4} = y$$

iff (note the above line implies $y \geq 0$)

$$\sqrt{x^6 - 4} = y, \quad y \geq 0$$

iff

$$x^6 - 4 = y^2, \quad y \geq 0$$

iff

$$x^6 = y^2 + 4, \quad y \geq 0$$

iff

$$x = \pm(y^2 + 4)^{1/6}, \quad y \geq 0.$$

Now for f over the interval $D_1 = [4^{1/6}, \infty)$, we have $x \geq 4^{1/6}$, and $(y^2+4)^{1/6} \geq 0$, so in this case $f(x) = y$ iff

$$x = +(y^2+4)^{1/6}, \quad y \geq 0.$$

So letting

$$g(y) = (y^2+4)^{1/6}, \quad y \geq 0,$$

we have that g is the inverse of f over $D_1 = [4^{1/6}, \infty)$. And we swap over the domains and ranges, so the domain of g is $R_1 = [0, \infty)$ (as also indicated above by "$y \geq 0$"), and the range of g is $D_1 = [4^{1/6}, \infty)$.

Now for f over the interval $D_2 = (-\infty, -4^{1/6}]$, we have $x \leq -4^{1/6}$, and $(y^2+4)^{1/6} \geq 0$, so in this case $f(x) = y$ iff

$$x = -(y^2+4)^{1/6}, \quad y \geq 0.$$

So letting

$$h(y) = -(y^2+4)^{1/6}, \quad y \geq 0,$$

we have that h is the inverse of f over $D_2 = (-\infty, -4^{1/6}]$. And we swap over the domains and ranges, so the domain of h is $R_2 = [0, \infty)$ (as also indicated above by "$y \geq 0$"), and the range of h is $D_2 = (-\infty, -4^{1/6}]$.

4. Let $f(x) = e^{-x^2}$. (a) Does f have an inverse over the interval $(-2, 5)$? (b) Find the formula for the inverse of f over the interval $D = [5, 10]$, and find the domain and range of the inverse.

Solution.

(a) No, because f is not 1-1 over the interval $(-2, 5)$, because e.g. $f(-1) = e^{-(-1)^2} = e^{-1}$, and $f(1) = e^{-(1)^2} = e^{-1}$, so $f(-1) = f(1)$, but $-1 \neq 1$, and -1 and 1 both lie in the interval $(-2, 5)$.

(b) (Note f has an inverse over $[5, 10]$ because

$$f'(x) = (e^{-x^2})' = e^{-x^2}(-x^2)' = -2xe^{-x^2},$$

and for all x, $e^{-x^2} > 0$, and for $5 \leq x \leq 10$, we have $x > 0$, so

$$f'(x) = -2xe^{-x^2} < 0$$

for $5 \leq x \leq 10$. Thus, f is decreasing over this interval, so is 1-1 over this interval, and therefore has an inverse over this interval.)

Now for $5 \leq x \leq 10$ and y any number we have

$$y = e^{-x^2}, \quad 5 \leq x \leq 10$$

iff

$$y = e^{-x^2}, \quad 5 \leq x \leq 10, \ y > 0$$

(since $e^{-x^2} > 0$ for all x) iff

$$\ln(y) = \ln(e^{-x^2}), \quad 5 \leq x \leq 10, \ y > 0$$

(since ln is 1-1 and we're assuming $y > 0$) iff

$$\ln(y) = -x^2, \quad 5 \leq x \leq 10, \ y > 0$$

iff

$$-\ln(y) = x^2, \quad 5 \le x \le 10, \; y > 0$$

iff

$$\sqrt{-\ln(y)} = |x|, \quad 5 \le x \le 10, \; y > 0$$

iff

$$\pm\sqrt{-\ln(y)} = x, \quad 5 \le x \le 10, \; y > 0$$

iff

$$\sqrt{-\ln(y)} = x, \quad 5 \le x \le 10, \; y > 0$$

(for $5 \le x \le 10$, we can't have $x = -\sqrt{-\ln(y)}$, since any $\sqrt{z} \ge 0$ for any z). So setting

$$g(y) = \sqrt{-\ln(y)}, \quad y > 0, \; 5 \le g(y) \le 10,$$

this is the formula for the inverse of f over the interval $[5, 10]$.

The range of the inverse is $[5, 10]$ (since we're computing the inverse for f over the interval $[5, 10]$). The domain of the inverse is the range of f over $[5, 10]$. Since f is decreasing over this interval and is continuous, the range of f over this interval is $[f(10), f(5)]$, by the intermediate value theorem. Thus, the domain of the inverse is $[e^{-(10)^2}, e^{-5^2}] = [e^{-100}, e^{-25}]$. So in the end, a clearer definition of the inverse is

$$g(y) = \sqrt{-\ln(y)}, \quad e^{-100} \le y \le e^{-25}.$$

5. Suppose that f is a differentiable function, and that f is one-to-one, and

- $f(2) = 4$; $f'(2) = -1$
- $f(3) = 2$; $f'(3) = -3$
- $f(4) = -2$; $f'(4) = 0$

(a) Why does f^{-1} exist?

(b) Let $g = f^{-1}$. Find $g'(2)$, if you have sufficient information. Find $g'(3)$, if you have sufficient information.

(c) Sketch a plausible graph of $y = f(x)$ over the interval $[0, 5]$. (So it should agree with all information given.) Then sketch the graph of $y = g(x)$, having the correct relationship to the graph of $y = f(x)$.

Solution.

(a) f^{-1} exists because f is one-to-one.

(b) Hint: before doing this problem it can help to do part (c) first, to guide your thinking about the relationship between f and f^{-1}.

We know $g = f^{-1}$. Consider $g'(2)$. Here $x = 2$ is an *input* to $g = f^{-1}$, and this corresponds to 2 being an *output* of f. The data given shows that $f(3) = 2$, and in this equation 2 is the output. Thus, since $g = f^{-1}$, we have $g(2) = 3$ (tho this was not asked for). The data also specifies $f'(3) = -3$. So we can find $g'(2)$:

$$g'(2) = \frac{1}{f'(3)} = \frac{1}{-3} = -\frac{1}{3}.$$

We do not have sufficient information to find $g'(3)$ because the only f-outputs specified in the data are 4, 2 and -2 (equal to $f(2)$, $f(3)$, $f(4)$ respectively); 3 is not included in this list of f-outputs.

(c) See sketches document on website.