Math 1720 Midterm 2 Review Problems

1. Compute

$$\frac{d}{dx}(\operatorname{arcsec}(\ln(x))).$$

Solution. Using the chain rule,

$$= \operatorname{arcsec}'(\ln(x))\ln'(x)$$

$$= \frac{1}{|\ln(x)|\sqrt{\ln(x)^2 - 1}}\frac{1}{x}.$$

2. (a) Find

$$\arctan(-1/\sqrt{3}).$$

(b) Fully simplify the expression

$$\sec(\arctan(x)).$$

(Your answer should not involve any trig or inverse trig functions.)

(c) Find

$$\int_{-\sqrt{3}}^{-\sqrt{2}} \frac{1}{\sqrt{4-x^2}}dx$$

Solution. (a) (Method 1) $\arctan(-1/\sqrt{3})$ is the unique angle θ such that $-\pi/2 < \theta < \pi/2$ (i.e. in quadrants 1 or 4) and $\tan(\theta) = -1/\sqrt{3}$. (Since tan is 1-1 over the interval $-\pi/2 < \theta < \pi/2$, and its range over this interval is $(-\infty, \infty)$, there's exactly one θ satisfying these requirements.) Looking at the standard angles, $\tan(\theta) = -1/\sqrt{3}$ when $\sin(\theta) = \frac{1}{2}$ and $\cos(\theta) = -\sqrt{3}/2$, or when $\sin(\theta) = -\frac{1}{2}$ and $\cos(\theta) = \sqrt{3}/2$: in these cases,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = -\frac{\frac{1}{2}}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}.$$

But we need θ in quadrant 1 or 4, so $\cos(\theta) \geq 0$, so it's the latter option: $\sin(\theta) = -\frac{1}{2}$ and $\cos(\theta) = \sqrt{3}/2$. This occurs exactly when $\theta = -\pi/6$. So $\arctan(-1/\sqrt{3}) = -\pi/6$.

(a) (Method 2) $\arctan(-1/\sqrt{3}) = \alpha$ where $\tan(\alpha) = -1/\sqrt{3}$ and $-\pi/2 < \alpha < \pi/2$. Use a reference triangle or the equation

$$\tan^2(\alpha) + 1 = \sec^2(\alpha)$$

to find that

$$(-1/\sqrt{3})^2 + 1 = \sec^2(\alpha)$$

$$4/3 = \sec^2(\alpha)$$

$$2/\sqrt{3} = |\sec(\alpha)|$$

$$|\cos(\alpha)| = \sqrt{3}/2.$$

Since $-\pi/2 < \alpha < \pi/2$, we have $\cos(\alpha) = \sqrt{3}/2$, so either $\alpha = \pi/6$ or $\alpha = -\pi/6$. Since $\tan(\alpha) < 0$, we have $\alpha = -\pi/6$. So

$$\arctan(-1/\sqrt{3}) = -\pi/6.$$

(b) Let $\alpha = \arctan(x)$; so $\tan(\alpha) = x$. We need to find $\sec(\alpha)$ in terms of $x = \tan(\alpha)$. As in part (a), use a reference triangle (see sketches document) or the equation

$$\tan^2 + 1 = \sec^2(\alpha)$$

to see that

$$x^2 + 1 = \sec^2(\alpha)$$

$$|\sec(\alpha)| = \sqrt{x^2 + 1}.$$

Since $\arctan(x) = \alpha$ is in the interval $-\pi/2 < \alpha < \pi/2$, we have $\cos(\alpha) > 0$, so $\sec(\alpha) > 0$. So

$$\sec(\alpha) = \sqrt{x^2 + 1}$$

So

$$\sec(\arctan(x)) = \sqrt{x^2 + 1},$$

as required.

(c)

$$\int_{-\sqrt{3}}^{-\sqrt{2}} \frac{1}{\sqrt{4 - x^2}} dx$$

We will use the fact that $d/dx(\arcsin(x)) = 1/\sqrt{1 - x^2}$. For this, we first convert the $4 - x^2$ to the form $1 - u^2$. So:

$$= \int_{-\sqrt{3}}^{-\sqrt{2}} \frac{1}{\sqrt{4(1 - (x/2)^2)}} dx$$

So sub $u = x/2$. Then $du = dx/2$, so $dx = 2du$, and $\sqrt{4} = 2$, so

$$= \int_{x=-\sqrt{3}}^{-\sqrt{2}} \frac{1}{2\sqrt{1 - u^2}} 2du$$

$$= \int_{x=-\sqrt{3}}^{-\sqrt{2}} \frac{1}{\sqrt{1 - u^2}} du$$

$$= \arcsin(u) \Big| \int_{x=-\sqrt{3}}^{-\sqrt{2}}$$

$$= \arcsin(x/2) \Big| \int_{x=-\sqrt{3}}^{-\sqrt{2}}$$

$$= \arcsin(-\sqrt{2}/2) - \arcsin(-\sqrt{3}/2)$$

And $\arcsin(-\sqrt{2}/2) = \alpha$ where $\sin(\alpha) = -\sqrt{2}/2$ and $-\pi/2 \le \alpha \le \pi/2$, which is $\alpha = -\pi/4$. Similarly, $\arcsin(-\sqrt{3}/2) = \alpha$ where $\sin(\alpha) = -\sqrt{3}/2$ and $-\pi/2 \le \alpha \le \pi/2$, which is $\alpha = -\pi/3$.

$$= -\pi/4 - (-\pi/3)$$

$$= 4\pi/12 - 3\pi/12 = \pi/12.$$

3. Compute the limits

$$\lim_{x\to\infty} (1 - 3/x)^{2x}.$$

$$\lim_{x\to 0^+} \sin(x)^{\tan(x)}.$$

Solution. (a)

$$\lim_{x\to\infty} (1 - 3/x)^{2x}.$$

As $x \to \infty$, $3/x \to 0$ and $2x \to \infty$, so this limit has form 1^∞, which is indeterminate. We convert to base e:

$$= \lim_{x\to\infty} e^{\ln(1-3/x)2x}$$

$$= e^{\lim_{x\to\infty} \ln(1-3/x)2x}$$

$$= e^L$$

where

$$L = \lim_{x\to\infty} \ln(1 - 3/x)2x.$$

Now since $1 - 3/x \to 1$ as $x \to \infty$ and $\ln(1) = 0$ and ln is continuous, $\lim_{x\to\infty} \ln(1 - 3/x) = \ln(1) = 0$. And $2x \to \infty$. So this limit has form $0 \cdot \infty$. So we convert to

$$= \lim_{x\to\infty} \frac{\ln(1 - 3/x)}{1/2x}.$$

This limit has form 0/0, so L'Hopital's rule applies:

$$= \lim_{x\to\infty} \frac{\ln(1 - 3/x)'}{(1/2x)'}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1-3/x}(1 - 3/x)'}{\frac{1}{2}(-x^{-2})}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1-3/x}3x^{-2}}{\frac{1}{2}(-x^{-2})}$$

$$= \lim_{x\to\infty} -6\frac{1}{1 - 3/x}$$

And $3/x \to 0$ in the limit, so

$$= \lim_{x\to\infty} -6\frac{1}{1} = -6.$$

Thus, the final answer is

$$e^L = e^{-6}.$$

(b)

$$\lim_{x\to 0^+} \sin(x)^{\tan(x)}.$$

As $x \to 0^+$, both $\sin(x) \to 0^+$ and $\tan(x) \to 0^+$, so this limit has the form 0^0, which is indeterminate. We convert to base e:

$$= \lim_{x\to 0^+} e^{\ln(\sin(x))\tan(x)}.$$

$$= e^{\lim_{x\to 0^+} \ln(\sin(x))\tan(x)}.$$

$$= e^L$$

where

$$L = \lim_{x\to 0^+} \ln(\sin(x))\tan(x).$$

This limit has form $-\infty \cdot 0^+$, so converting,

$$= \lim_{x\to 0^+} \frac{\ln(\sin(x))}{\cot(x)}$$

This limit has form $-\infty/\infty$, so L'Hopital's rule applies:

$$= \lim_{x\to 0^+} \frac{\ln(\sin(x))'}{\cot(x)'}$$

Using the chain rule,

$$\ln(\sin(x))' = \ln'(\sin(x))(\sin(x))' = (1/\sin(x)) \cdot \cos(x) = \cos(x)/\sin(x) = \cot(x).$$

And by the quotient rule,

$$\cot'(x) = (\frac{\cos(x)}{\sin(x)})' = (\frac{\cos'(x)\sin(x) - \cos(x)\sin'(x)}{\sin^2(x)})$$

$$= \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)} = -\frac{1}{\sin^2(x)} = -\csc^2(x).$$

To the limit is

$$= \lim_{x\to 0^+} \frac{\cot(x)}{-\csc^2(x)} = \frac{\cos(x)/\sin(x)}{-1/\sin^2(x)}$$

$$= \lim_{x\to 0^+} \frac{\cos(x)}{-1/\sin(x)}$$

$$= \lim_{x\to 0^+} -\cos(x)\sin(x) = 1 \cdot 0 = 0.$$

So the final answer is

$$e^L = e^0 = 1.$$

4. Compare the growth rates of the functions

$$f(x) = 2^x,$$

$$g(x) = \ln(5^x),$$

$$h(x) = x^{\ln(x)}.$$

Hint: for comparing f with h, convert to base e. Use the fact that if functions $k(x) \to \infty$ and $j(x) \to \infty$ as $x \to \infty$, and if $k(x) >> j(x)$, then $k(x) - j(x) \to \infty$ also.

Comparing f vs g, note that $g(x) = \ln(5^x) = x\ln(5)$, and $\ln(5)$ is a positive constant. So $g(x) = kx$ for some $k > 0$. But $f(x) = 2^x$. It's a standard comparison fact that $2^x >> x$ (as the base 2 is > 1 and x is just a power of x). And the positive constant k does not change this, since

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{kx}{2^x} = \lim_{x\to\infty} k(\frac{x}{2^x}) = k \lim_{x\to\infty} \frac{x}{2^x}.$$

The limit on the right is 0 since $x << 2^x$, so we get

$$= k0 = 0.$$

So $kx << 2^x$.

Alternatively, you could compute the limit directly:

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{\ln(5^x)}{2^x}$$

$$= \lim_{x\to\infty} \frac{x\ln(5)}{2^x} = \ln(5) \lim_{x\to\infty} \frac{x}{2^x};$$

the limit on the right has form $\infty \cdot \infty$, so applying L'Hopital's rule,

$$= \ln(5) \lim_{x\to\infty} \frac{(x)'}{(2^x)'} = \ln(5) \lim_{x\to\infty} \frac{1}{\ln(2)2^x};$$

This limit has form $1/\infty$, which gives 0 so

$$= \ln(5)0 = 0.$$

So $g << f$.

Now consider h. Note that it is $h(x) = x^{\ln(x)} = e^{\ln(x)^2}$. Compare with f: write $f(x) = 2^x = e^{\ln(2)x}$. So,

$$\lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \frac{e^{\ln(2)x}}{e^{\ln(x)^2}}$$

$$= \lim_{x\to\infty} e^{\ln(2)x - \ln(x)^2}.$$

$$= e^{\lim_{x\to\infty} \ln(2)x - \ln(x)^2}.$$

$$= e^L$$

where

$$L = \lim_{x\to\infty} \ln(2)x - \ln(x)^2.$$

Now using the hint with $k(x) = \ln(2)x$ and $j(x) = \ln(x)^2$, we have that $\ln(2)x - \ln(x)^2 \to \infty$ as $x \to \infty$, since it's a standard comparison fact that $x >> \ln(x)^2$, and therefore $\ln(2)x >> \ln(x)^2$, since $\ln(2)$ is just a positive constant. Alternatively, computing this,

$$\lim_{x\to\infty} \frac{\ln(2)x}{\ln(x)^2} = \ln(2) \lim_{x\to\infty} \frac{x}{\ln(x)^2},$$

which has form ∞/∞, so using L'Hopital's rule,

$$= \ln(2) \lim_{x\to\infty} \frac{(x)'}{(\ln(x)^2)'} = ln(2) \lim_{x\to\infty} \frac{1}{2\ln(x)(1/x)}$$

$$= (\ln(2)/2) \lim_{x\to\infty} \frac{x}{\ln(x)},$$

which still has form ∞/∞; repeating L'Hopital's rule,

$$= (\ln(2)/2) \lim_{x\to\infty} \frac{(x)')}{(\ln(x))'}$$

$$= (\ln(2)/2) \lim_{x\to\infty} \frac{1}{1/x}$$

$$= (\ln(2)/2) \lim_{x\to\infty} x = \infty.$$

So this shows $\ln(2)x >> \ln(x)^2$. So applying the hint, we have that $\lim_{x\to\infty} \ln(2)x - \ln(x)^2 = \infty$. Therefore coming back to computing L, we have

$$L = \infty.$$

Therefore the original limit $\lim_{x\to\infty} \frac{f(x)}{h(x)} = e^L = e^\infty = \infty$, so $f >> h$.

For g vs h, we have $g(x) = \ln(5^x) = x\ln(5) = \ln(5)e^{\ln(x)}$, and $h(x) = x^{\ln(x)} = e^{\ln(x)^2}$. So

$$\lim_{x\to\infty} \frac{h(x)}{g(x)} = \lim_{x\to\infty} \frac{e^{\ln(x)^2}}{\ln(5)e^{\ln(x)}}$$

$$= (1/\ln(5)) \lim_{x\to\infty} e^{\ln(x)^2 - \ln(x)}$$

$$= (1/\ln(5)) \lim_{x\to\infty} e^{\ln(x)(\ln(x)-1)}$$

$$= (1/\ln(5)) e^{\lim_{x\to\infty}(\ln(x)(\ln(x)-1))}$$

$$= (1/\ln(5)) e^L$$

where

$$L = \lim_{x\to\infty} \ln(x)(\ln(x) - 1)).$$

And $\ln(x) \to \infty$ as $x \to \infty$, so $\ln(x) - 1 \to \infty$ also, so this limit is $\infty \cdot \infty = \infty$, so $L = \infty$, so $(1/\ln(5))e^L = \infty$, so $h >> g$.

5.(a) Integrate

$$\int_0^1 x^2 2^x dx$$

(b) Antidifferentiate

$$\int \sin(3x)\cos(x/2)dx.$$

(c) Antidifferentiate

$$\int \ln(x)^2 dx$$

(Hint: note the integrand is not $\ln(x^2)$. Use a method like that used for antidifferentiating $\int \ln(x)dx$.)

Solution.

(a) We use integration by parts. Since integrating or diffing the 2^x term leads to the form $k2^x$ for some constant k, it is reasonable to diff the x^2 term and work its power down to 0. So let $f(x) = 2^x$ and $g(x) = x^2$. Let $F(x) = \int f(x)dx = \int 2^x dx = (1/\ln(2))2^x$. We have $g'(x) = 2x$. So by integration by parts,

$$\int = Fg|_0^1 - \int_0^1 Fg'dx$$

$$= (1/\ln(2))2^x x^2|_0^1 - \int_0^1 (1/\ln(2))2^x(2x)dx$$

which is (*):

$$= \frac{1}{\ln(2)}\left[2^x x^2|_0^1 - 2\int_0^1 2^x x dx\right].$$

Looking at the remaining integral inside line (*), this is

$$\int_0^1 2^x x dx$$

Integrating by parts again, intergrating the 2^x term and diff'ing the x term (note this is in the same "direction" again, reducing the power of the x term); we get

$$= (1/\ln(2))2^x x|_0^1 - \int_0^1 (1/\ln(2))2^x(1)dx$$

which is (**):

$$= \frac{1}{\ln(2)}\left[2^x x|_0^1 - \int_0^1 2^x dx\right].$$

And the last integral inside line (**) is:

$$\int_0^1 2^x dx = \frac{1}{\ln(2)}2^x|_0^1.$$

So line (**) equals

$$= \frac{1}{\ln(2)}\left[2^x x|_0^1 - \frac{1}{\ln(2)}2^x|_0^1\right].$$

$$= \frac{1}{\ln(2)}\left[2^x(x - \frac{1}{\ln(2)})|_0^1\right].$$

So line (*) equals

$$= \frac{1}{\ln(2)}\left[2^x x^2|_0^1 - 2\left(\frac{1}{\ln(2)}\left[2^x(x - \frac{1}{\ln(2)})|_0^1\right]\right)\right].$$

$$= \frac{1}{\ln(2)}\left[2^x x^2 - \left(\frac{2}{\ln(2)}\left[2^x(x - \frac{1}{\ln(2)})\Big|_0^1\right]\right)\right].$$

$$= \frac{1}{\ln(2)}\left[2^1(1^2) - \left(\frac{2}{\ln(2)}\left[2^1(1 - \frac{1}{\ln(2)})\right]\right)\right]$$

$$-\frac{1}{\ln(2)}\left[2^0(0^2) - \left(\frac{2}{\ln(2)}\left[2^0(0 - \frac{1}{\ln(2)})\right]\right)\right]$$

$$= \frac{1}{\ln(2)}\left[2 - \left(\frac{2}{\ln(2)}\left[2(1 - \frac{1}{\ln(2)})\right]\right)\right]$$

$$-\frac{1}{\ln(2)}\left[0 - \left(\frac{2}{\ln(2)}\left[1(0 - \frac{1}{\ln(2)})\right]\right)\right]$$

$$= \frac{2}{ln(2)} + \frac{4}{(\ln(2))^3} - \frac{4}{\ln(2)^2}$$

$$-\frac{2}{\ln(2)^3}$$

$$= \frac{2}{\ln(2)} + \frac{2}{\ln(2)^3} - \frac{4}{\ln(2)^2}.$$

(b)

$$\int \sin(3x)\cos(x/2)dx.$$

We use integration by parts. Integrate the sin term and differentiate the cos. This results in:

$$= -\frac{1}{3}\cos(3x)\cos(x/2) - \int -\frac{1}{3}\cos(3x)\frac{1}{2}(-\sin(x/2))dx$$

$$= -\frac{1}{3}\cos(3x)\cos(x/2) - \frac{1}{6}\int \cos(3x)\sin(x/2)dx$$

Now with the remaining integral, again integrate by parts, going in the same direction: the term we integrated in the previous one was $\sin(3x)$, so we integrate the term this produced, i.e. $\cos(3x)$:

$$= -\frac{1}{3}\cos(3x)\cos(x/2) - \frac{1}{6}\left[\frac{1}{3}\sin(3x)\sin(x/2) - \frac{1}{3}\int \sin(3x)\frac{1}{2}\cos(x/2)dx\right]$$

$$= -\frac{1}{3}\cos(3x)\cos(x/2) - \frac{1}{18}\sin(3x)\sin(x/2) + \frac{1}{36}\int \sin(3x)\cos(x/2)dx.$$

The remaining integral is just the original one. So let $I = \int \sin(3x)\cos(x/2)dx$. Then we have the equation

$$I = -\frac{1}{3}\cos(3x)\cos(x/2) - \frac{1}{18}\sin(3x)\sin(x//2) + \frac{1}{36}I.$$

Solving for I,

$$\frac{35}{36}I = -\frac{1}{3}\cos(3x)\cos(x/2) - \frac{1}{18}\sin(3x)\sin(x//2)$$

$$I = \frac{36}{35}\left(-\frac{1}{3}\cos(3x)\cos(x/2) - \frac{1}{18}\sin(3x)\sin(x//2)\right)$$

So our final answer is

$$\int \sin(3x)\cos(x/2)dx = \frac{36}{35}\left(-\frac{1}{3}\cos(3x)\cos(x/2) - \frac{1}{18}\sin(3x)\sin(x//2)\right) + c.$$

(c)

$$\int \ln(x)^2 dx.$$

We use integration by parts. There are a couple of ways. First by the hint, using the method used for antidifferentiating $\int \ln(x)dx$. So consider $\ln(x)^2$ as the product $1 \cdot \ln(x)^2$. We integrate the 1 and differentiate the $\ln(x)^2$, giving

$$= x\ln(x)^2 - \int x(2\ln(x)(1/x))dx$$

$$= x\ln(x)^2 - \int 2\ln(x)dx$$

Using the fact that $\int \ln(x)dx = x\ln(x) - x + d$, we have

$$= x\ln(x)^2 - 2(x\ln(x) - x) + c$$

$$= x\ln(x)^2 - 2x\ln(x) + 2x + c.$$

Alternatively, one can consider $\ln(x)^2$ as the product $\ln(x) \cdot \ln(x)$. Thus:

$$\int \ln(x) \cdot \ln(x)dx,$$

using integration by parts, using again what $\int \ln(x)dx$ is, written above:

$$= \ln(x)(x\ln(x) - x) - \int (1/x)(x\ln(x) - x)dx$$

$$= x\ln(x)^2 - x\ln(x) - \int \ln(x) - 1dx$$

$$= x\ln(x)^2 - x\ln(x) - (x\ln(x) - x) + x$$

$$= x\ln(x)^2 - 2x\ln(x) + 2x + c.$$

6.(a) Integrate

$$\int_0^{\pi/8} \sin^4(4x)\cos^2(4x)dx$$

(b) Antidifferentiate

$$\int \sec^{-1/3}(x)\tan^3(x)dx$$

Solution.

(a) We have an integral of a product of even powers of cosine and sine, so we use the identities $\cos^2 y = \frac{1}{2}(1+\cos(2y))$ and $\sin^2 y = \frac{1}{2}(1-\cos(2y))$:

$$\int = \int (\sin^2(4x))^2(\frac{1}{2}(1+\cos(2(4x))))dx$$

$$= \int (\frac{1}{2}(1-\cos(2(4x))))^2(\frac{1}{2}(1+\cos(8x)))dx$$

$$= \int \frac{1}{8}(1-\cos(8x))^2(1+\cos(8x))dx$$

$$= \frac{1}{8}\int (1-2\cos(8x)+\cos^2(8x))(1+\cos(8x))dx$$

We now expand the terms, as this will result in various powers of $\cos(8x)$, which we can integrate term by term:

$$= \frac{1}{8}\int 1-2\cos(8x)+\cos^2(8x)+\cos(8x)-2\cos^2(8x)+\cos^3(8x)dx$$

$$= \frac{1}{8}\int 1-\cos(8x)-\cos^2(8x)+\cos^3(8x)dx$$

$$= \frac{1}{8}\left[\int dx - \int \cos(8x)dx - \int \cos^2(8x)dx + \int \cos^3(8x)dx\right].$$

We can now directly integrate the first two terms. For the $\cos^2(8x)$ term, we have an even power of $\cos(8x)$, so we again use the identity $\cos^2(y) = \frac{1}{2}(1+\cos(2y))$.

$$= \frac{1}{8}\left[x - \frac{1}{8}\sin(8x) - \int \frac{1}{2}(1+\cos(2(8x)))dx + \int \cos^3(8x)dx\right].$$

$$= \frac{1}{8}\left[x - \frac{1}{8}\sin(8x) - \frac{1}{2}\int 1+\cos(16x)dx + \int \cos^3(8x)dx\right].$$

This results in (*):

$$= \frac{1}{8}\left[x - \frac{1}{8}\sin(8x) - \frac{1}{2}(x+\frac{1}{16}\sin(16x)) + \int \cos^3(8x)dx\right].$$

For the $\cos^3(8x)$ term, we have an odd power of $\cos(8x)$, so we sub $u = \sin(8x)$, separating one power of $\cos(8x)$ for the du term and writing the rest in terms of $u = \sin(8x)$:

$$\int \cos^3(8x)dx = \int \cos(8x)\cos^2(8x)dx = \int \cos(8x)(1-\sin^2(8x))dx$$

$$= \int (1-u^2)\cos(8x)dx$$

Since $du = 8\cos(8x)dx$, we have

$$= \int (1-u^2)(du/8) = \frac{1}{8}\int 1-u^2du$$

$$= \frac{1}{8}(u - \frac{1}{3}u^3) + c$$

$$= \frac{1}{8}(\sin(8x) - \frac{1}{3}\sin^3(8x)) + c.$$

Putting this back into line (*) we get the original integral is (and including the bounds again):

$$\int_0^{\pi/8} f(x)dx = \frac{1}{8}\left[x - \frac{1}{8}\sin(8x) - \frac{1}{2}(x + \frac{1}{16}\sin(16x)) + \frac{1}{8}(\sin(8x) - \frac{1}{3}\sin^3(8x))\right]\Big|_0^{\pi/8}$$

Now at $x = 0$ we have $\sin(8x) = \sin(16x) = 0$. At $x = \pi/8$ we have $\sin(8x) = \sin(8\pi/8) = \sin(\pi) = 0$ and $\sin(16x) = \sin(16(\pi/8)) = \sin(2\pi) = 0$. So all the sin terms evaluate to 0 when we plug in the bounds, leaving only the x terms remaining, and when $x = 0$ this term also evaluates to 0 (but not when $x = \pi/8$):

$$= \frac{1}{8}\left[\pi/8 - 0 - \frac{1}{2}(\pi/8 + 0) + \frac{1}{8}(0 - 0)\right] - \frac{1}{8}[0]$$

$$= \frac{1}{8}(\pi/8 - \frac{1}{2}(\pi/8)) = \frac{\pi}{128}.$$

(b) Antidifferentiate

$$\int \sec^{-1/3}(x)\tan^3(x)dx$$

We have a product of powers of sec and tan, with an odd positive power of tan. So we use the method of substituting $u = \sec(x)$, separating $\sec(x)\tan(x)$ from the integrand for the du term:

$$= \int \sec^{-1/3}(x)(\sec(x))^{-1}\sec(x)\tan(x)\tan^2(x)dx$$

Sub $u = \sec(x)$, so $du = \sec(x)\tan(x)dx$:

$$= \int \sec^{-4/3}(x)\tan^2(x)du$$

Use $\tan^2(x) + 1 = \sec^2(x)$, so $\tan^2(x) = \sec^2(x) - 1 = u^2 - 1$, and $\sec^{-4/3}(x) = u^{-4/3}$:

$$= \int u^{-4/3}(u^2 - 1)du$$

$$= \int u^{-4/3}u^2 - u^{-4/3}du$$

$$= \int u^{2/3} - u^{-4/3}du$$

$$= (3/5)u^{5/3} + 3u^{-1/3} + c$$

$$= (3/5)\sec(x)^{5/3} + 3\sec(x)^{-1/3} + c.$$

7.(a) Antidifferentiate, simplifying fully.

$$\int (1 - 3x^2)^{3/2}dx$$

(b) Integrate

$$\int_0^1 \frac{x^3}{1+x^2}dx.$$

(c) Antidifferentiate:

$$\int \frac{x}{(x^2-2)^{25/2}}dx.$$

Solution.

$$\int (1-3x^2)^{3/2}dx$$

There is no immediate substitution that will be useful for this integral. So we a trig sub. The form $a - bx^2$ appears in the integrand, with $1-3x^2 = 1-(\sqrt{3}x)^2$, so we use the sub $x = \frac{1}{\sqrt{3}}\sin(\theta)$. Thus, we get $dx = \frac{1}{\sqrt{3}}\cos(\theta)d\theta$, and $1-3x^2 = 1-3(\frac{1}{\sqrt{3}})^2\sin(\theta)^2 = 1-\sin(\theta)^2 = \cos(\theta)^2$, so

$$= \int (\cos(\theta)^2)^{3/2}\cos(\theta)d\theta$$

$$= \int |\cos(\theta)|^3\cos(\theta)d\theta.$$

For subbing $x = \sin(\theta)$, we use the interval $-\pi/2 \leq \theta \leq \pi/2$ (the right side of the circle, where $\sin(\theta)$ goes through all values from -1 to 1), and so $\cos(\theta) \geq 0$, so $|\cos(\theta)| = \cos(\theta)$:

$$= \int \cos(\theta)^3\cos(\theta)d\theta.$$

This is an even power of $\cos(\theta)$, so we use the identity $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$:

$$= \int (\cos(\theta)^2)^2 d\theta$$

$$= \int (\frac{1}{2}(1+\cos(2\theta)))^2 d\theta$$

$$= \int \frac{1}{4}(1+2\cos(2\theta)+\cos^2(2\theta))d\theta$$

For the $\cos^2(2\theta)$ term, we again use the same identity (but with 2θ in place of θ), giving:

$$= \int \frac{1}{4}(1+2\cos(2\theta)+\frac{1}{2}(1+\cos(2(2\theta))))d\theta$$

$$= \int \frac{1}{4}(1+2\cos(2\theta)+\frac{1}{2}+\frac{1}{2}\cos(4\theta))d\theta$$

$$= \frac{1}{4}\int \frac{3}{2}+2\cos(2\theta)+\frac{1}{2}\cos(4\theta)d\theta$$

$$= \frac{1}{4}\left[\frac{3}{2}\theta+2(\frac{1}{2}\sin(2\theta))+\frac{1}{2}(\frac{1}{4}\sin(4\theta))\right]+c$$

$$= \frac{3}{8}\theta+\frac{1}{4}\sin(2\theta)+\frac{1}{32}\sin(4\theta)+c$$

Now we need to rewrite everything in terms of x. We subbed $x = \sin(\theta)$, with the interval $-\pi/2 \leq \theta \leq \pi/2$, so this is equivalent to $\arcsin(x) = \theta$. So

$$= \frac{3}{8}\arcsin(x) + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + c$$

For $\sin(2\theta)$, use the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, and, similarly,

$$\sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = 2\sin(\theta)\cos(\theta)\cos(2\theta)$$

giving (*):

$$= 2\sin(\theta)\cos(\theta)(\cos^2(\theta) - \sin^2(\theta))$$

(using also the identity $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$). Thus, everything has been written in terms of $\sin(\theta)$ and $\cos(\theta)$. We need to write these in terms of x. We have $\sin(\theta) = x$, so that's fine. For $\cos(\theta)$, use a reference triangle, or the Pythagorean identity

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$x^2 + \cos^2(\theta) = 1$$
$$\cos^2(\theta) = 1 - x^2$$
$$|\cos(\theta)| = \sqrt{1 - x^2}$$

and again, as mentioned above, $\cos(\theta) \geq 0$, so

$$\cos(\theta) = \sqrt{1 - x^2}.$$

So:

$$\sin(4\theta) = 2x\sqrt{1-x^2}(\sqrt{1-x^2}^2 - x^2) = 2x\sqrt{1-x^2}(1 - 2x^2).$$

So putting everything together, the antiderivative is

$$\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + c$$

$$= \frac{3}{8}\arcsin(x) + \frac{1}{4}2x\sqrt{1-x^2} + \frac{1}{32}2x\sqrt{1-x^2}(1 - 2x^2).$$

(b)

$$\int_0^1 \frac{x^3}{1 + x^2}dx.$$

Here there is no direct substitution for the integral. The term $1 + x^2$ appears in the denominator, so we try a trig sub with $x = \tan(\theta)$. This gives $dx = \sec^2(\theta)d\theta$ and $1 + x^2 = 1 + \tan^2(\theta) = \sec^2(\theta)$. So:

$$= \int_{x=0}^1 \frac{\tan^3(\theta)}{\sec^2(\theta)}\sec^2(\theta)d\theta.$$

$$= \int_{x=0}^1 \tan^3(\theta)d\theta$$

This is an odd power of tan, which we deal with as for products of powers of tan and sec. So, since it's an odd power, we sub $u = \sec(\theta)$, separating $\sec(\theta)\tan(\theta)$ to facilitate this:

$$= \int_{x=0}^1 \sec(\theta)^{-1}\tan^2(\theta)\tan(\theta)\sec(\theta)d\theta$$

We have $du = \tan(\theta)\sec(\theta)d\theta$. Use $\tan^2(\theta) + 1 = \sec^2(\theta)$, to write $\tan^2(\theta) = \sec^2(\theta) - 1 = u^2 - 1$:

$$= \int_{x=0}^{1} u^{-1}(u^2 - 1)du$$

$$= \int_{x=0}^{1} u - u^{-1}du$$

$$= \frac{1}{2}u^2 - \ln(|u|)|_{x=0}^{1}$$

$$= \frac{1}{2}\sec^2(\theta) - \ln(|\sec(\theta)|)|_{x=0}^{1}$$

Now we subbed $x = \tan(\theta)$, which means we are restricted to $-\pi/2 < \theta < \pi/2$. Our actual interval of integration is $x = 0$ to $x = 1$. For $x = 0$, this gives $0 = \tan(\theta)$, and $-\pi/2 < \theta < \pi/2$, i.e. $\theta = \arctan(0)$, so $\theta = 0$. And for $x = 1$, we have $\theta = \arctan(1)$, i.e. $-\pi/2 < \theta < \pi/2$ and $\tan(\theta) = 1$, i.e. $\theta = \pi/4$. So

$$= \frac{1}{2}\sec^2(\theta) - \ln(|\sec(\theta)|)|_{\theta=0}^{\pi/4}$$

Now when $\theta = \pi/4$ we have $\sec(\theta) = \sec(\pi/4) = 1/\cos(\pi/4) = 2/\sqrt{2}$. When $\theta = 0$ we have $\sec(\theta) = 1/\cos(0) = 1/1 = 1$. So

$$= \frac{1}{2}(2/\sqrt{2})^2 - \ln(|2/\sqrt{2}|) - \frac{1}{2}(1^2) + \ln(|1|)$$

$$= \frac{1}{2}(4/2) - \ln(\sqrt{2}) - \frac{1}{2} + 0$$

$$= 1 - \frac{1}{2}\ln(2) - \frac{1}{2}$$

$$= \frac{1}{2}(1 - \ln(2)).$$

(Note $1 > \ln(2)$ so this is positive, as it should be, since the integrand was positive over $0 < x < 1$.)

(c)

$$\int \frac{x}{(x^2 - 2)^{25/2}}dx.$$

Here we can just use a regular sub of $u = x^2 - 2$. For this gives $du = 2xdx$, so $du/2 = xdx$, so

$$= \int \frac{du/2}{u^{25/2}}$$

$$= \frac{1}{2}\int u^{-25/2}du$$

$$= \frac{1}{2}\frac{1}{-23/2}u^{-23/2} + c.$$

$$= \frac{-1}{23}(x^2 - 2)^{-23/2} + c.$$