Summary of topics covered in course:

Open/closed subsets of $\mathbb{R}$
Metric spaces
Countability and uncountability
Cantor set
Topological spaces
Bases
Interior, Closure
Continuity
Separation axioms
Product, subspace topologies
Connectedness
Compactness
Density (this came up in a few exercises; you should know the definition - it's in homework 7, problem 6 - that particular problem was not required for 4500, but you should still be comfortable with the definition).

Review Problems - I may add some more over the next few days

On Thursday, Steve will discuss (1) any questions that you have on the material and/or related exercises from homeworks or whatever; (2) a selection of these problems. I doubt there will be time to discuss them all so you should give the questions a look over and think about which ones you would prefer to discuss on Thursday, and in general, any questions that you want to ask. (The problems focus more on more recent material; all material covered will be examinable but you should look more at the older reviews/midterms and/or in Munkres for more problems on that material. There are a few problems here on the older material though, but some of them you've seen before.)

1. Let τ be the topology on $\mathbb{R}$ with base $\tau_{\text{std}} \cup \{\{q\} | q \in \mathbb{Q}\}$. Let $(\mathbb{R}^2, \rho)$ be the product of the spaces $(\mathbb{R}, \tau) \times (\mathbb{R}, \tau)$. How does the closure of a subset of $\mathbb{R}^2$, with respect to ρ, relate to its closure with respect to the standard topology on $\mathbb{R}^2$?

Solution.

Let $A \subseteq \mathbb{R}^2$. Then for any $x \in \mathbb{R}^2$, $x \in \text{Cl}(A)$ iff for every open set W such that $x \in A$, we have $A \cap W \neq \emptyset$. (Here "open" means $W \in \rho$.)

Let $\tau_{\text{std},1}$ be the standard top on $\mathbb{R}$ and $\tau_{\text{std},2}$ the standard top on $\mathbb{R}^2$. Then $\tau_{\text{std},2}$ is the product of $\tau_{\text{std},1}$ with itself.

Claim 1: $\tau_{\text{std},2} \subseteq \rho$. Proof: Let $W \in \tau_{\text{std},2}$. Given $x \in W$, there are $W_1, W_2 \in \tau_{\text{std},1}$ such that $x \in W_1 \times W_2 \subseteq W$, but $\tau_{\text{std},1} \subseteq \tau$ by the definition of τ, so $W_1, W_2 \in \tau$. But then $W_1 \times W_2 \in \rho$. Since $x \in W$ was arbitrary, we have that W is a union of elements of ρ, and since ρ is a topology, $W \in \rho$, as required.

Claim 2: $\text{Cl}(A) \subseteq \text{Cl}_{\text{std},2}(A)$, where $\text{Cl}_{\text{std},2}(A)$ is the closure of A with respect to the standard topology on $\mathbb{R}^2$. Proof: Suppose $x \in \text{Cl}(A)$. Let $W \in \tau_{\text{std},2}$ such that $x \in W$. By Claim 1, $W \in \rho$. Since $x \in \text{Cl}(A)$ and $x \in W \in \rho$, we have $W \cap A \neq \emptyset$.

Now let $x \in \text{Cl}_{\text{std},2}(A)$; we will determine under what conditions $x \in \text{Cl}(A)$.

Case 1. $x \in A$. Then $x \in \text{Cl}(A)$ as we always have $A \subseteq \text{Cl}(A)$ (since $\text{Cl}(A)$ is the smallest closed set which is a superset of A).

Case 2. $x \notin A$ and $x \in \mathbb{Q}^2$. Then $x \notin \text{Cl}(A)$.

Proof: Note that $\{x\} \in \rho$, since if $x = (q, r)$ we have $q, r \in \mathbb{Q}$, and $\{q\} \in \tau$ and $\{r\} \in \tau$, so $\{x\} = \{q\} \times \{r\} \in \rho$. But since $x \notin A$, we have $\{x\} \cap A = \emptyset$, and since $x \in \{x\} \in \rho$, this implies so $x \notin \text{Cl}(A)$.

Case 3. $x \notin A$ and $x \in (\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})$. Then $x \in \text{Cl}(A)$.

Proof: Let $W \in \rho$ be such that $x \in W$. Since $x \in W \in \rho$, we have $W_1, W_2 \in \tau$ such that $x \in W_1 \times W_2 \subseteq W$ and $W_1, W_2 \in \tau$. We have $x = (y, z)$ with $y, z \in \mathbb{R} - \mathbb{Q}$. Since $y \in W_1 \in \tau$, we have some W_1' in the given base for τ, such that $y \in W_1' \subseteq W_1$. But since $y \notin \mathbb{Q}$, this implies $W_1' \in \tau_{\text{std},1}$. Similarly, we have $W_2' \in \tau_{\text{std},1}$ such that $z \in W_2' \subseteq W_2$. But then $x = (y, z) \in W_1' \times W_2' \subseteq W$ ($\subseteq W$ since $W_1' \subseteq W_1$ and $W_2' \subseteq W_2$ and $W_1 \times W_2 \subseteq W$). And $W_1' \times W_2' \in \tau_{\text{std},2}$. Since $x \in W_1' \times W_2'$ and $x \in \text{Cl}_{\text{std},2}(A)$, this implies $W_1' \times W_2' \cap A \neq \emptyset$, and since $W_1' \times W_2' \subseteq W$, this implies $W \cap A \neq \emptyset$, as required.

Case 4. $x \notin A$ and $x \in \mathbb{Q} \times (\mathbb{R} - \mathbb{Q})$.
In this case, we don't have enough information to determine whether $x \in \text{Cl}(A)$. But let $x = (q, z)$, so we have $q \in \mathbb{Q}$ and $z \in \mathbb{R} - \mathbb{Q}$. Let $A_q = A(\cap\{q\} \times \mathbb{R})$. Then Claim: $x \in \text{Cl}(A)$ iff $x \in \text{Cl}_{\text{std},2}(A_q)$.

Proof: Suppose $x \in \text{Cl}(A)$. Let $W \in \tau_{\text{std},2}$ such that $x \in W$. We must show that $W \cap A_q \neq \emptyset$. We have $W_1, W_2 \in \tau_{\text{std},1}$ such that $x \in W_1 \times W_2 \subseteq W$. Note that $\{q\} \times \mathbb{R} \in \rho$ and $W_1 \times W_2 \in \rho$, and since $x \in W_1 \times W_2$, we have $q \in W_1$. And also $x \in \{q\} \times \mathbb{R}$. So

$$x \in (\{q\} \times \mathbb{R}) \cap (W_1 \times W_2) = \{q\} \times W_2 \in \rho.$$

Since $x \in \text{Cl}(A)$, this implies $\{q\} \times W_2 \cap A \neq \emptyset$, but $\{q\} \times W_2 \subseteq \{q\} \times \mathbb{R}$, so $\{q\} \times W_2 \cap A_q \neq \emptyset$, and $q \in W_1$ and $W_1 \times W_2 \subseteq W$, so $W \cap A_1 \neq \emptyset$, as required.

Now suppose $x \in \text{Cl}_{\text{std},2}(A_q)$. Let $W \in \rho$ with $x \in W$. We must show $W \cap A \neq \emptyset$. There are $W_1, W_2 \in \tau$ such that $x = (q, z) \in W_1 \times W_2 \subseteq W$. So there are W_1', W_2' in the given base for τ such that $q \in W_1' \subseteq W_1$ and $z \in W_2' \subseteq W_2$. Since $z \notin \mathbb{Q}$, we must have $W_2' \in \tau_{\text{std}}$. Therefore $\mathbb{R} \times W_2' \in \tau_{\text{std},2}$. Since $x \in \text{Cl}_{\text{std},2}(A_q)$, this implies $A_q \cap (\mathbb{R} \times W_2') \neq \emptyset$. Since $A_q \subseteq \{q\} \times \mathbb{R}$, this implies $A_q \cap (\{q\} \times W_2') \neq \emptyset$. But $(\{q\} \times W_2') \subseteq (W_1' \times W_2')$ since $q \in W_1'$ since $x = (q, z) \in W_1' \times W_2'$. Therefore $A_q \cap (W_1' \times W_2') \neq \emptyset$. But $A_q \subseteq A$ and $W_1' \times W_2' \subseteq W_1 \times W_2 \subseteq W$, so therefore $A \cap W \neq \emptyset$, as required.

Case 5. $x \notin A$ and $x \in ((\mathbb{R} - \mathbb{Q}) \times \mathbb{Q})$.
By symmetry, this is the reflected version of Case 4.

This completes all cases.

(So for example, in the case that A is the open unit disk, then $\text{Cl}(A)$ consists of all points x such that either: (1) $x \in A$, or (2) x is on the unit circle, and $x \notin \mathbb{Q} \times \mathbb{Q}$.

But for example, if A is the unit disk subtract the vertical line $x = 0.1$ (note that the points $(0.1, \pm\sqrt{0.99})$ lie on the unit circle and $\sqrt{0.99}$ is irrational), then $\text{Cl}(A)$ consists of all points p such that either: (1) $p \in A$, or (2) p is on the unit circle, and $p \neq (0.1, \pm\sqrt{0.99})$, or (3) p is on the vertical line $x = 0.1$, with $p = (0.1, r)$ for some irrational r, such that $-\sqrt{0.99} < r < \sqrt{0.99}$ (i.e. so that p is in the interior of the unit disk).)

2. Let $C \subseteq \mathbb{R}$ and $A = \{(xy, y/\sin(x)) \mid (x,y) \in [\pi/4, 3\pi/4] \times C\}$. Show that if $C = [0,5]$ then A is both compact and connected. Show that if C is the Cantor set then A is compact but not connected.

Solution.

The product of closed sets is closed in the product topology. (This was in a problem in midterm 2; but here there's less to verify: if $C_1 \subseteq X_1$ and $C_2 \subseteq X_2$ are both closed sets w.r.t. (X_1, τ_1) and (X_2, τ_2) respectively, and $(x_1, x_2) \in (X_1 \times X_2) - (C_1 \times C_2)$ then since $X_1 - C_1 \in \tau_1$ and $X_2 - C_2 \in \tau_2$, then $W = (X_1 - C_1) \times (X_2 - C_2) \in \rho$, where ρ is the product topology given by τ_1, τ_2, and $(C_1 \times C_2) \cap W = \emptyset$, and $(x_1, x_2) \in W$. Therefore $C_1 \times C_2$ is closed w.r.t. ρ.)

So for either C the Cantor set or $C = [0,5]$, it follows that $C \times [\pi/4, 3\pi/4]$ is closed in $\mathbb{R}^2$, since both C and $[\pi/4, 3\pi/4]$ are closed (recall the standard topology on $\mathbb{R}^2$ is identical to the product of $\tau_{\text{std},1}$ with $\tau_{\text{std},1}$).

Moreover, $C \times [\pi/4, 3\pi/4] \subseteq \mathcal{B}((0,0), 10)$, so this set is bounded.

Since a subset of $\mathbb{R}^2$ is compact iff it is closed and bounded, we have that this set is compact.

Now suppose $C = [0,5]$. Then both C and $[\pi/4, 3\pi/4]$ are connected since intervals $\subseteq \mathbb{R}$ are connected. Since the product of connected sets is connected, $C \times [\pi/4, 3\pi/4]$ is a connected subset of $\mathbb{R}^2$.

Let $A = C \times [\pi/4, 3\pi/4]$. Since A is a connected set w.r.t. $(\mathbb{R}^2, \tau_{\text{std},2})$, it follows that the subspace $(A, \tau_{\text{std},2} \restriction A)$ is a connected space. It is also a compact space, i.e. A is a compact subset of this subspace. This is a general fact:

Lemma. Let (X, τ) be a top space and $B \subseteq X$. Then B is a compact set w.r.t. (X, τ) iff B is compact w.r.t. $(B, \tau \restriction B)$.

Proof. Suppose B is compact w.r.t. (X, τ). Let $\langle U_i \rangle_{i \in I}$ be an open cover coming from the subspace topology. Then for each $i \in I$ there is $V_i \in \tau$ such that $U_i = V_i \cap B$. Fix such V_i's. Then since $B \subseteq \cup_{i \in I} U_i$ and $U_i \subseteq V_i$, we have $B \subseteq \cup_{i \in I} V_i$. So $\langle V_i \rangle_{i \in I}$ is an open cover of B. Since B is compact w.r.t. (X, τ), there are finitely many i's in I, $i_1, \ldots, i_n$, such that $B \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. But then

$$B \cap B \subseteq (V_{i_1} \cup \ldots \cup V_{i_1}) \cap B$$

and $B \cap B = B$ so

$$B \subseteq (V_{i_1} \cap B) \cup \ldots \cup (V_{i_n} \cap B)$$

and $U_i = V_i \cap B$ so

$$B \subseteq U_{i_1} \cup \ldots \cup U_{i_n}.$$

Therefore $\{U_{i_1}, \ldots, U_{i_n}\}$ is a finite subcover of B from the family $\langle U_i \rangle_{i \in I}$. Therefore B is compact w.r.t. $(B, \tau \restriction B)$.

Conversely, suppose B is compact w.r.t. $(B, \tau \restriction B)$. Let $\langle V_i \rangle_{i \in I}$ be an open cover of B w.r.t. τ, i.e. $V_i \in \tau$ for each $i \in I$ and $B \subseteq \cup_{i \in I} V_i$. Let $U_i = V_i \cap B$. Then $U_i \in \tau \restriction B$ for each $i \in I$, and since $B \subseteq \cup_{i \in I} V_i$, we have

$$B \cap B \subseteq (\bigcup_{i \in I} V_i) \cap B$$

$$B \subseteq \bigcup_{i \in I} (V_i \cap B) = \bigcup_{i \in I} U_i.$$

Therefore by compactness w.r.t. $(B, \tau \restriction B)$, there are finitely many $i_1, \ldots, i_n \in I$ such that

$$B \subseteq U_{i_1} \cup \ldots \cup U_{i_n},$$

but $U_i \subseteq V_i$ for each i, so

$$B \subseteq V_{i_1} \cup \ldots \cup V_{i_n},$$

so $\{V_{i_1}, \ldots, V_{i_n}\}$ form a finite subcover of B from $\langle V_i \rangle_{i \in I}$, as required.

This proves the Lemma. So $(A, \tau_{\mathrm{std},2} \restriction A)$ is a compact space (i.e. A is a compact subset of this space), and is also a connected space. Therefore it suffices to show that the function $f : A \to \mathbb{R}^2$, given by $f(x,y) = (xy, y/\sin(x))$, is continuous, since the continuous image of a connected set is connected, and the continuous image of a compact set is compact, and the set in question is $f``A$.

The continuity of f can be shown directly with the ε-δ definition, or as follows. By a homework exercise (from the homework on product topology), it suffices to show that each component is continuous into $\mathbb{R}$; i.e. that $f_1 : A \to \mathbb{R}$ by $f_1(x,y) = xy$ is continuous, and that $f_2 : A \to \mathbb{R}$ by $f_2(x,y) = y/\sin(x)$, are both continuous. And by a fact given in class, we can combine continuous functions by the arithmetic operations, and produce more continuous functions, as long as we don't divide by 0. So, if we know that $f_{1,1} : A \to \mathbb{R}$, $f_{1,1}(x,y) = x$ and $f_{1,2} : A \to \mathbb{R}$, $f_{1,2}(x,y) = y$, are both continuous, then so is their product $f_1(x,y) = xy = f_{1,1}(x,y)f_{1,2}(x,y)$. But if we define $f'_{1,1} : \mathbb{R}^2 \to \mathbb{R}$ by $f'_{1,1}(x,y) = x$, then by the same homework on the product topology, $f'_{1,1}$ is continuous. And I think by another homework problem, this implies $f_{1,1}$ is also continuous. (That is, we have the following: Lemma: if $h : X \to Y$ is continuous w.r.t. the topological spaces (X, τ) and (Y, σ) and $B \subseteq X$, then $(f \restriction B) : B \to Y$ is continuous w.r.t. $(B, \tau \restriction B)$ and (Y, σ); this is straightforward to prove, much like the lemma above on compactness.) Since $f'_{1,1}$ is continuous, the lemma stated in parentheses implies that $f_{1,1}$ is also continuous. Similarly, $f_{1,2}$ is continuous. Therefore f_1 is continuous. We similarly have that f_2 is continuous, using also that $f_2(x,y) = \frac{f_{1,2}(x,y)}{\sin(f_{1,1}(x,y))}$, and that $\sin : \mathbb{R} \to \mathbb{R}$ is continuous, and that compositions of continuous functions are continuous, and that $\sin(x) \neq 0$ for $x \in [\pi/3, 3\pi/4]$. So finally, f is continuous, as required.

Now suppose C is the Cantor set. Then as discussed earlier, $A = C \times [\pi/4, 3\pi/4]$ is compact. So again using the continuity of f, we have that the set in question, $f``A$, is compact. However, $C$ is not connected: the sets $C \cap (1/2, \infty)$ and $C \cap (-\infty, 1/2)$ form a separation of the Cantor space $(C, \tau_{\mathrm{std},1} \restriction C)$. And in fact $f``A$ is also not connected. (But the continuous image of a non-connected set *can* be connected, so we need to establish this.) For $y \in C$, let $A_y = A \cap \mathbb{R} \times \{y\}$, and $B_y = f``A_y = \{(xy, y/\sin(x)) | x \in [\pi/4, 3\pi/4]\}$. Then if $y = 0$ then $B_y = \{(0,0)\}$, and if $y > 0$ then the vertically lowest point of B_y is $((\pi/2)y, y/\sin(\pi/2)) = (y\pi/2, y)$, since $\sin(x)$ reaches its maximum value 1 over the interval $[\pi/4, 3\pi/4]$ at $x = \pi/2$. Likewise, B_y has its vertically highest points at the endpoints $((\pi/4)y, y/\sin(\pi/4)) = (y\pi/4, y\sqrt{2})$, and $((3\pi/4)y, y/\sin(3\pi/4)) = (3y\pi/4, y\sqrt{2})$. Therefore if $y \leq 1/3$, then every point in B_y has vertical height $\leq$ the highest points of $B_{1/3}$, i.e. height at most $(1/3)\sqrt{2}$. So if we let $U_1 = \mathbb{R} \times (-\infty, \frac{\sqrt{2}}{3} + 0.000001)$ then for $y \leq 1/3$ we have $B_y \subseteq U_1$ and U_1 is open in $\mathbb{R}^2$. And if $y \geq 2/3$, then every point in B_y has vertical height $\geq$ the lowest point of $B_{2/3}$, i.e. height at least $(2/3)$. So letting $U_2 = \mathbb{R} \times ((2/3) - 0.000001, \infty)$, then for $y \geq 2/3$, $B_y \subseteq U_2$. But $B = f``A$ is the union of the various sets B_y, for $y \in C$. And each $y \in C$ is such that either $y \leq 1/3$ or $y \geq 2/3$. Therefore $B = \bigcup_{y \in C, y \leq 1/3} B_y \cup \bigcup_{y \in C, y \geq 2/3} B_y$, which is $\subseteq U_1 \cup U_2$. And $\frac{\sqrt{2}}{3} + 0.000001 < \frac{2}{3} - 0.000001$, so $U_1 \cap U_2 = \emptyset$, and $U_1, U_2 \in \tau_{\text{std},2}$. Therefore U_1, U_2 separate B. So B is disconnected.

3. (a) Let $X = C([0,1])$ with "max difference" metric topology. Let $A \subseteq X$ be the set of quadratic functions (with domain $[0,1]$) and linear functions (with domain $[0,1]$). Show that A is connected.
(b) Give an example of a non-connected subset of $C([0,1])$.
(c) Same topology, show that it's $T4$.
Solution.
(a) (Originally I had written just quadratic functions, but I should have said quadratic functions together with linear functions.)
We will in fact show that A is path-connected. This suffices since every path-connected set is connected.

Let f, g be two quadratic or linear functions, mapping $[0,1] \to \mathbb{R}$. Then there are constants a_0, a_1, a_2 such that

$$f(z) = a_0 + a_1 z + a_2 z^2$$

and constants b_0, b_1, b_2 such that

$$g(z) = b_0 + b_1 z + b_2 z^2.$$

Since $\mathbb{R}$ is path connected, we can let $h_0 : [0,1] \to \mathbb{R}$ be continuous and such that $h_0(0) = a_0$ and $h(1) = b_0$, and $h_1 : [0,1] \to \mathbb{R}$ continuous such that $h_1(0) = a_1$ and $h_1(1) = b_1$, and $h_2 : [0,1] \to \mathbb{R}$ continuous such that $h_2(0) = a_2$ and $h_2(1) = b_2$. Let $h : [0,1] \to A$ be the function

$$h(x) = p_x$$

where p_x is the function

$$p_x : [0,1] \to \mathbb{R},$$

$$p_x(z) = h_0(x) + h_1(x) z + h_2(x) z^2.$$

Then Claim: h is well-defined (mapping into A), h is continuous, $h(0) = f$ and $h(1) = g$.
Proof. Let $x \in [0,1]$. Then for any $z \in [0,1]$, the formula given for $p_x(z)$ defines a specific value, so the function p_x is well-defined. And p_x is in A since the formula for $p_x(z)$ in terms of z is the standard form for a quadratic (if $h_2(x) \neq 0$) or linear function (if $h_2(x) = 0$).

$h(0) = f$ since

$$h(0)(z) = p_0(z) = h_0(0) + h_1(0) z + h_2(0) z^2$$

(by definition of p_0)

$$= a_0 + a_1 z + a_2 z^2$$

(by choice of h_0, h_1, h_2)

$$= f(z).$$

Likewise $h(1) = g$.

Continuity: let $x \in [0, 1]$ and $\varepsilon > 0$. Since h_0, h_1, h_2 are each continuous there are $\delta_0, \delta_1, \delta_2 > 0$ such that for all $x' \in [0, 1]$,

$$|x - x'| < \delta_i \implies |h_i(x) - h_i(x')| < \varepsilon/3$$

for each $i = 0, 1, 2$. Let $\delta = \min(\delta_0, \delta_1, \delta_2)$. We claim that for all $x' \in [0, 1]$,

$$|x - x'| < \delta \implies d_{\max}(h(x), h(x')) < \varepsilon.$$

For let $x' \in [0, 1]$ with $|x - x'| < \delta$. We have

$$d_{\max}(h(x), h(x')) = \max\{|h(x)(z) - h(x')(z)| \mid z \in [0, 1]\}.$$

So let $z \in [0, 1]$. Then

$$|h(x)(z) - h(x')(z)|$$
$$= |h_0(x) + h_1(x)z + h_2(x)z^2 - h_0(x') - h_1(x')z - h_2(x')z^2|$$
$$= |(h_0(x) - h_0(x')) + (h_1(x) - h_1(x'))z + (h_2(x) - h_2(x'))z^2|$$

By the (version of the) triangle inequality $|a + b + c| \leq |a| + |b| + |c|$:

$$\leq |h_0(x) - h_0(x')| + |z||h_1(x) - h_1(x')| + |z^2||h_2(x) - h_2(x')|$$

And since $|x - x'| < \delta$, we have $|h_i(x) = h_i(x')| < \varepsilon/3$ for each $i = 0, 1, 2$, so

$$< (\varepsilon/3)(1 + |z| + |z^2|)$$

And since $z \in [0, 1]$, $|z| \leq 1$ and $|z^2| \leq 1$:

$$\leq (\varepsilon/3)(3) = \varepsilon,$$

so

$$|h(x)(z) - h(x')(z)| < \varepsilon,$$

and since $h(x)$ and quadratic/linear functions they're continuous, so the absolute difference function $|h(x) - h(x')|$ is also continuous, so attains its maximum value over the closed interval $[0, 1]$, so

$$d_{\max}(h(x), h(x')) < \varepsilon,$$

as required. Therefore h is continuous.

(b) Any set of the form $\{f, g\}$ with $f \neq g \in C([0, 1])$ is disconnected, since if $\varepsilon = d_{\max}(f, g)$, then $\varepsilon > 0$, so $U_1 = \mathcal{B}(f, \varepsilon/2)$ and $U_2 = \mathcal{B}(g, \varepsilon/2)$ separate $\{f, g\}$ (since $f \in U_1$ and $g \in U_2$, so both $U_1 \cap \{f, g\} \neq \emptyset$ and $U_2 \cap \{f, g\} \neq \emptyset$, and $\{f, g\} \subseteq U_1 \cup U_2$, and $U_1 \cap U_2 \cap \{f, g\} = \emptyset$ since in fact $U_1 \cap U_2 = \emptyset$ since if $h \in U_1 \cap U_2$ then $d(f, h) < \varepsilon/2$ and $d(g, h) < \varepsilon/2$ which by the triangle inequality implies $d(f, g) \leq d(f, h) + d(h, g) < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, contradicting that $d(f, g) = \varepsilon$). So e.g. let f be the constantly 0 function and g the constantly 1 function, with domain $[0, 1]$. Then $f, g \in C([0, 1])$ and $f \neq g$ and so $\{f, g\}$ is disconnected.

(c) The max-metric topology is the topology of the max metric. The topology of a metric space is always T4. So this topology is T4.

4. Let (X, τ) be a T_2 topological space such that X is compact. Show that the space is T_3.

Solution.

Let $z \in X$ and $C \subseteq X$ such that C is closed and $z \notin C$. Since C is closed, $C \subseteq X$ and X is compact, we have (by a theorem from class) that C is compact. For each $c \in C$, let $U_c, V_c \in \tau$ be such that $z \in U_c$, $c \in V_c$, and $U_c \cap V_c = \emptyset$ (they exist b/c (X, τ) is T2). Then $C \subseteq \bigcup_{c \in C} V_c$ and each $V_c \in \tau$, so $\langle V_c \rangle_{c \in C}$ forms an open cover of V_c. Since C is compact, there is a finite subcover from the open cover: we have $c_1, \ldots, c_n \in C$ such that $\{V_{c_1}, \ldots, V_{c_n}\}$ is a (finite) subcover of C. Now let $U = \cap_{i=1}^n U_{c_i}$, and let $V = \bigcup_{i=1}^n V_{c_i}$. Then: $C \subseteq V$, by choice of the finite subcover. And $z \in U$, since $z \in U_c$ for each $c \in C$; in particular $z \in U_{c_i}$ for each $i = 1, \ldots, n$. And $U \cap V = \emptyset$, since if for some $x \in X$ we have $x \in U \cap V$, then $x \in V_{c_k}$ for some $k \in \{1, \ldots, n\}$, and $x \in U_{c_l}$ for *every* $l = 1, \ldots, n$ But then in particular, $x \in U_{c_k}$. But then $x \in U_{c_k} \cap V_{c_k}$, contradicting that $U_c \cap V_c = \emptyset$ for every $c \in C$.

So U and V are as required for T3-ness with respect to z, C. This shows (X, τ) is T3.

5. Let C be the Cantor set.

Is there

(a) an onto function $f : \mathbb{N} \to C$?

(b) an onto function $f : \mathbb{R} \to C$?

(c) an onto function $f : C \to \mathbb{R}$?

(d) a continuous onto function $f : \mathbb{N} \to C$? (where it's the discrete topology on $\mathbb{N}$ and restriction of standard on C)

(e) a continuous onto function $f : \mathbb{R} \to C$? (standard topology on $\mathbb{R}$)

(f) a continuous onto function $f : C \to \mathbb{R}$?

Solution.

(a) No, C is uncountable.

(b) Yes. Since $C \subseteq \mathbb{R}$ and $C \neq \emptyset$, we can just let $f : \mathbb{R} \to C$ be defined by $f(x) = x$ for $x \in C$ and $f(x) = 0$ for $x \notin C$ (note $0 \in C$). Then f is onto.

(c) Yes. There was a homework problem which constructed an onto function $f : C \to [0, 1]$. (E.g. for $x \in C$, let $x = 0.t_1t_2t_3\ldots$ be the base 3 representation of x which uses only 0's and 2's, i.e. each $t_i \in \{0, 2\}$; this representation exists by one of the definitions of C. Then let $f(x) = 0.(\frac{t_1}{2})(\frac{t_2}{2})(\frac{t_3}{2})\ldots$, in base 2. One can show that $f : C \to [0, 1]$ is onto.)

Now there's an onto function $g : [0, 1] \to \mathbb{R}$: let $g(0) = 0$, $g(1) = 0$, and let $g(x) = tan(-\pi/2 + \pi x)$ for $x \in (0, 1)$. This is onto.

Then $g \circ f : C \to \mathbb{R}$ is onto.

(d) No, by (a).

(e) No. $\mathbb{R}$ is connected but C is not connected (e.g. $(-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$ form a separation of C). But the continuous image of a connected set is connected. So there cannot be a continuous function $\mathbb{R} \to C$ whose range is C.

(f) No, since $(C, \tau_{\text{std},1} \restriction C)$ is a compact space (since C is closed and bounded $\subseteq \mathbb{R}$, C is a compact subset of $\mathbb{R}$; this implies that $(C, \tau_{\text{std},1} \restriction C)$ is a compact space, i.e. C is a compact subset of this space, as proved in the Lemma in Problem 1), but $\mathbb{R}$ is non-compact, and the continuous image of a compact set is compact, there is no continuous function from C to $\mathbb{R}$.

6. Let (X, d) be a metric space. Recall that a subset $D \subseteq X$ is *dense* iff for

every non-empty open set U, $U \cap D \neq \emptyset$. Show that X has a countable dense subset iff X has a countable base.

Solution. Suppose there is a countable dense subset $D \subseteq X$. Let $\mathfrak{b} = \{\mathcal{B}(d,q) | d \in D, q \in \mathbb{Q}\}$. Then since $D, \mathbb{Q}$ are both countable, so is $D \times \mathbb{Q}$, and the function $f : D \times \mathbb{Q} \to \mathfrak{b}$ given by $f(d,q) = \mathcal{B}(d,q)$ is onto, so $\mathfrak{b}$ is countable. And $\mathfrak{b} \subseteq \tau_d$ since it consits of open balls.

Claim. $\mathfrak{b}$ forms a base for τ_d.

Proof: Let $W \subseteq X$ be such that $W \in \tau_d$. Let $z \in W$. Then there is $\varepsilon > 0$ such that $\mathcal{B}(z,\varepsilon) \subseteq W$. Then $\mathcal{B}(z,\varepsilon/2)$ is a non-empty set (z is in it) and is in τ_d. Therefore $D \cap \mathcal{B}(z,\varepsilon/2) \neq \emptyset$, since D is dense. Let $d \in D \cap \mathcal{B}(z,\varepsilon/2)$.

Then Subclaim: $z \in \mathcal{B}(d,\varepsilon/2) \subseteq \mathcal{B}(z,\varepsilon)$. For $d \in \mathcal{B}(z,\varepsilon/2)$ so $d(d,z) < \varepsilon/2$ so $z \in \mathcal{B}(d,\varepsilon/2)$, giving the first part of the Subclaim. And if $w \in \mathcal{B}(d,\varepsilon/2)$ then $d(d,w) < \varepsilon/2$, and $d(d,z) < \varepsilon/2$ since $d \in \mathcal{B}(z,\varepsilon/2)$. Therefore $d(w,z) \leq d(w,d) + d(d,z) < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, using the triangle inequality. Therefore $w \in \mathcal{B}(z,\varepsilon)$, finishing the proof of the Subclaim.

Now $d(d,z) < \varepsilon/2$. Let $q \in \mathbb{Q} \cap (d(d,z), \varepsilon/2]$. Then $\mathcal{B}(d,q) \in \mathfrak{b}$, and by the Subclaim, since $q \leq \varepsilon/2$, $\mathcal{B}(d,q) \subseteq \mathcal{B}(d,\varepsilon/2) \subseteq \mathcal{B}(z,\varepsilon)$, and by choice of ε, $\mathcal{B}(z,\varepsilon) \subseteq W$, so we have $\mathcal{B}(d,q) \subseteq W$. And since $q > d(d,z)$, we have $z \in \mathcal{B}(d,q)$. Since $z \in W$ was arbitrary, it follows that W is a union of elements of $\mathfrak{b}$. So $\mathfrak{b}$ is a base for τ_d as required.

Since $\mathfrak{b}$ is countable, the Claim completes the proof that there is a countable base.

Now for the converse. Suppose there is a countable base $\mathfrak{b}$. For each $U \in \mathfrak{b}$ such that $U \neq \emptyset$, let $d_U \in U$. Let $D = \{d_U \mid U \in \mathfrak{b},\ U \neq \emptyset\}$.

Claim: D is countable and dense.

Proof. Since $\mathfrak{b}$ is countable, and the function $f : \mathfrak{b} \to D$, given by $f(U) = d_U$, is onto, we have that D is countable. D is dense, since if $U \in \tau_d$ is non-empty, then let $x \in U$. Then there is $W \in \mathfrak{b}$ such that $x \in W \subseteq U$, since $\mathfrak{b}$ is a base. Then $d_W \in W \subseteq U$, so $d_W \in U$, and $d_W \in D$, so $D \cap U \neq \emptyset$, as required.

7. Let X_1, X_2 be top spaces and $A_1 \subseteq X_1$, $A_2 \subseteq X_2$. Show that $\text{Cl}(A_1 \times A_2) = \text{Cl}(A_1) \times \text{Cl}(A_2)$.

Solution. This was in midterm 2; see solution on website.

8. Let $A \subseteq \mathbb{R}$ be an open set. Show that there is some $J \subseteq \mathbb{N}$ and a family $\langle I_i \rangle_{i \in J}$ of pairwise disjoint open intervals (i.e. $i \neq j \in J \implies I_i \cap I_j = \emptyset$) such that $A = \cup_{i \in \mathbb{N}} I_i$ (note the family is required to be countable).

Solution. An early homework problem showed that any open subset of $\mathbb{R}$ is the union of a family of pairwise disjoint non-empty open intervals $\langle U_i \rangle_{i \in K}$ It follows that the family is countable. For given $i \in K$, choose $q_i \in \mathbb{Q} \cap U_i$. Then the function $K \to \mathbb{Q}$ by $i \mapsto q_i$ is 1-1, since if $i \neq j$ then $U_i \cap U_j = \emptyset$ (they're pairwise disjoint), but $q_i \in U_i$ and $q_j \in U_j$, so $q_i \neq q_j$ (since $q_i = q_j$ implies $q_i \in U_i \cap U_j$, but this intersection is empty, contradiction). It follows that K is countable (a 1-1 function into a countable set has countable domain; by a homework problem). So either K is finite or countably infinite. If K is finite, let $J \subseteq \mathbb{N}$ have the same number of elements as does K, and if K is infinite, let $J = \mathbb{N}$. Let $f : J \to K$ be a bijection. Then for $i \in J$ let I_i be $U_{f(i)}$. Then if $i \neq j$ with $i, j \in J$ then $U_{f(i)} \cap U_{f(j)} = \emptyset$, since the family $\langle U_i \rangle_{k \in K}$

consists of pairwise disjoint intervals. So the family $\langle I_i \rangle_{i\in J}$ is a collection of pairwise disjoint open intervals (and the index set is $J \subseteq \mathbb{N}$, as required), and $A = \cup_{i\in J} I_i$, as required.

9. (a) Using the original definition of "closure" (i.e. the closure of a set A is the intersection of all closed sets B such that $A \subseteq B$), prove the characterization of closure given in class, i.e. prove that a point x is in the closure of A iff for every open set W such that $x \in W$, we have that $W \cap A \neq \emptyset$.
Solution.
If $x \notin \mathrm{Cl}(A)$ then since $\mathrm{Cl}(A)$ is closed, $W = X - \mathrm{Cl}(A)$ is open, and we have $x \in W$. Therefore there is an open set W such that $x \in W$, but $W \cap A = \emptyset$. If there is an open set W such that $x \in W$, but $W \cap A = \emptyset$, then $X - W$ is closed, and $A \subseteq X - W$, and therefore $\mathrm{Cl}(A) \subseteq X - W$ (since $\mathrm{Cl}(A)$ is the smallest closed set C such that $A \subseteq C$). The characterization follows.

(b) The boundary of a set A is $B = \mathrm{Cl}(A) - \mathrm{Int}(A)$. Is it possible for the boundary to have non-empty interior, i.e. for $\mathrm{Int}(B) \neq \emptyset$?
Solution.
Yes, e.g. $A = \mathbb{Q} \subseteq \mathbb{R}$. Then $\mathrm{Cl}(A) = \mathbb{R}$ (since by the characterization, $x \in \mathrm{Cl}(A)$ iff for every open $W \subseteq \mathbb{R}$ such that $x \in W$, we have $W \cap \mathbb{Q} \neq \emptyset$, and this is true for any open $W \neq \emptyset$, since $\mathbb{Q}$ is dense in $\mathbb{R}$). But $\mathrm{Int}(A) = \emptyset$, since if $U \subseteq A = \mathbb{Q}$ is open, and if $U \neq \emptyset$, then U contains some irrational, by the density of $\mathbb{R} - \mathbb{Q}$, contradicting that $U \subseteq \mathbb{Q}$.

10. Show that if A is connected subset of a topological space, and $A \subseteq B \subseteq \mathrm{Cl}(A)$, then B is also connected.
Solution.
By contradiction, let U, V be a supposed separation of B; so we have $B \subseteq U \cup V$, $U \cap V \cap B = \emptyset$, $U \cap B \neq \emptyset$, and $V \cap B \neq \emptyset$, and $U, V \in \tau$, where τ is the topology we're dealing with. Since $A \subseteq B$, it follows that $A \subseteq U \cup V$ and $U \cap V \cap A = \emptyset$, and we have $U, V \in \tau$. Since A is connected, we must have either $U \cap A = \emptyset$ or $V \cap A = \emptyset$. Suppose $V \cap A = \emptyset$. Now let $z \in V \cap B$. Then since $B \subseteq \mathrm{Cl}(A)$ we have $z \in \mathrm{Cl}(A)$, and therefore since $z \in V \in \tau$, we have $V \cap A \neq \emptyset$. This contradicts that we deduced $V \cap A = \emptyset$. The case that $U \cap A = \emptyset$ is just the same, by symmetry. This shows there's no separation of B, so B is connected.

11.(a) Prove that in $\mathbb{R}^n$, every open ball is connected.
Solution.
We can in fact show that every open ball is path-connected (which implies it's connected).

For given $x \in \mathcal{B}(p, \varepsilon) \subseteq \mathbb{R}^n$, let $f_1^x : [0, \frac{1}{2}] \to \mathbb{R}^n$ be the linear function satisfying $f_1^x(0) = x$ and $f_1^x(1) = p$. (I.e., $f_1^x(a) = 2(\frac{1}{2} - a)x + 2ap$.) Then f_1^x is continuous (like proof of continuity of function in problem 1); (in fact every polynomial function $\mathbb{R}^m \to \mathbb{R}^n$ is continuous).

Claim: f_1^x"$[0, \frac{1}{2}] \subseteq \mathcal{B}(p, \varepsilon)$.

Proof. For convenient notation, for $y, z \in \mathbb{R}^n$, note that

$$d(y, z) = \sqrt{(y - z) \cdot (y - z)}$$

where the $\cdot$ is the usual dot product, i.e. $y\dot{z} = \sum_{i=1}^{n} y_i z_i$, since

$$d(y,z) = \sqrt{\sum_{i-1}^{n} (y_i - z_i)^2}.$$

Also recall that for scalars $a, b \in \mathbb{R}$ and $y, z \in \mathbb{R}^n$, $(ay) \cdot (bz) = (ab)(y \cdot z)$; this follows directly from the definition of $\cdot$.

So let $a \in [0, \frac{1}{2}] = \mathrm{dom}(f_1^x)$. Then

$$d(p, f_1^x(a)) = \sqrt{(p - f_1^x(a)) \cdot (p - f_1^x(a))}$$

$$= \sqrt{(p - 2(\frac{1}{2} - a)x - 2ap) \cdot (p - 2(\frac{1}{2} - a)x - 2ap)}$$

$$= \sqrt{((1-2a)(p-x)) \cdot ((1-2a)(p-x))}$$

$$= \sqrt{(1-2a)^2 (p-x) \cdot (p-x)}$$

$$= |1-2a| \sqrt{(p-x) \cdot (p-x)}$$

$$= |1-2a| d(p,x)$$

Since $a \in [0, \frac{1}{2}]$:

$$\leq d(p,x) < \varepsilon.$$

So $f_1^x(a) \in \mathcal{B}(p, \varepsilon)$, as required.

This proves the Claim.

For $x \in \mathcal{B}(p, \varepsilon)$, define $f_2^x : [\frac{1}{2}, 1] \to \mathbb{R}^n$ to be the linear function such that $f_2^x(\frac{1}{2}) = p$ and $f_2^x(1) = x$. (I.e. $f_2^x(a) = 2(1-a)p + 2(a - \frac{1}{2})x$.) Then similarly to f_1^x, we also have f_2^x is continuous, and $f_2^x``[\frac{1}{2}, 1] \subseteq \mathcal{B}(p, \varepsilon)$.

Now given $x, y \in \mathcal{B}(p, \varepsilon)$, let $f : [0,1] \to \mathbb{R}^n$ be defined by $f \restriction [0, \frac{1}{2}] = f_1^x$ and $f \restriction [\frac{1}{2}, 1] = f_2^y$. (Note that this is well defined as $f_1^x(\frac{1}{2}) = p = f_2^y(\frac{1}{2})$.) Then f is continuous: for $a \in [0,1]$, if $a \in (\frac{1}{2}, 1]$ and $\varepsilon > 0$ then there's $\delta > 0$ as required for continuity of f, since just let $\delta = \min((a - \frac{1}{2}), \delta')$, where δ' is such that for all b, $|b - a| < \delta'$ implies $d(f_2^y(b), f_2^y(a)) < \varepsilon$ (exists by continuity of f_2^y). Then if $|b - a| < \delta$, then $b \in (\frac{1}{2}, 1]$, so $f(b) = f_2^y(b)$, and $|b - a| < \delta'$, so $d(f_2^y(b), f_2^y(a)) < \varepsilon$, and $f_2^y(b) = f(b)$ and $f_2^y(a) = f(a)$, so $d(f(b), f(a)) < \varepsilon$, as required. Similarly if $a \in [0, \frac{1}{2})$. If $a = \frac{1}{2}$ and $\varepsilon > 0$ then choose $\delta = \min(\delta', \delta'')$, where δ' are as required for continuity of f_1^x w.r.t. $\frac{1}{2}, \varepsilon$, and δ'' as required for f_2^y w.r.t. $\frac{1}{2}, \varepsilon$. Then δ is as required for f w.r.t. $\frac{1}{2}, \varepsilon$. (Note again here that $f(\frac{1}{2}) = p = f_1^x(\frac{1}{2}) = f_2^y(\frac{1}{2})$.)

And we have $f_1^x(0) = x$ and $f_2^y(1) = y$, so $f(0) = x$ and $f(1) = y$. And $f``[0,1] = f_1^x``[0, \frac{1}{2}] \cup f_2^y``[\frac{1}{2}, 1] \subseteq \mathcal{B}(p, \varepsilon)$. So f is as required for x, y.

Therefore $\mathcal{B}(p, \varepsilon)$ is path-connected, as required.

(b) Now if you did (a) by showing that every open ball is in fact path-connected, then do it again, without using path-connectedness. (Hint: show first that $[a,b] \times [c,d]$ is connected for any reals $a < b$ and $c < d$. Combine this with Munkres' §23 exercise 2.)

Solution.

There are actually a couple of ways to do this; one follows the hint, and one directly uses the theorem on the union of a family of connected sets, where the family's intersection is non-empty.

Method 1 (not following hint):

Claim 1. Let $p \in \mathbb{R}^n$ and $\varepsilon > 0$. Let $z \in \mathcal{B}(p, \varepsilon)$. Then there is a set of the form $W = [a_1, b_1] \times \ldots \times [a_n, b_n]$ such that $p, z \in W$ and $W \subseteq \mathcal{B}(p, \varepsilon)$.

Proof. For each $i \leq n$, let

$$a_i = \min(p_i, z_i),$$

$$b_i = \max(p_i, z_i).$$

Note then that for each i,

$$a_i \leq p_i, z_i \leq b_i,$$

and therefore

$$p, z \in W = [a_1, b_1] \times \ldots \times [a_n, b_n].$$

So we just need to see that $W \subseteq \mathcal{B}(p, \varepsilon)$.

Note that z, p are at opposite corners of the prism W. (That is, if $a_i \neq b_i$ then either [$p_i = a_i$ and $z_i = b_i$] or [$p_i = b_i$ and $z_i = a_i$], and if $a_i = b_i$ then $p_i = z_i = a_i = b_i$.) We have $d(p, z) < \varepsilon$. Let $y \in W$; we'll show $d(p, y) \leq d(p, z)$, so $d(p, y) < \varepsilon$ too, so $y \in \mathcal{B}(p, \varepsilon)$, as required.

Well,

$$d(p, y) = \sqrt{\sum_{i=1}^{n} (p_i - y_i)^2}$$

and for each i, $y_i \in [a_i, b_i]$. So $|y_i - a_i| \leq |b_i - a_i|$ and $|y_i - b_i| \leq |b_i - a_i|$. We have that either [$p_i = a_i$ and $z_i = b_i$] or [$p_i = b_i$ and $z_i = a_i$] (even if $p_i = z_i = a_i = b_i$) so in any case, we get $|y_i - p_i| \leq |b_i - a_i|$, so $|y_i - p_i| \leq |z_i - p_i|$. Therefore $(y_i - p_i)^2 \leq (z_i - p_i)^2$. Therefore

$$\sum_{i=1}^{n} (p_i - y_i)^2 \leq \sum_{i=1}^{n} (z_i - p_i)^2.$$

Therefore

$$d(p, y) = \sqrt{\sum_{i=1}^{n} (p_i - y_i)^2} \leq \sqrt{\sum_{i=1}^{n} (z_i - p_i)^2} = d(p, z).$$

So

$$d(p, y) \leq d(p, z) < \varepsilon$$

so $d(p, y) < \varepsilon$, as required.

This proves Claim 1.

Claim 2. W is connected $\subseteq \mathbb{R}^n$.

Proof. First $[a_1, b_1] \times [a_2, b_2]$ is connected in $(\mathbb{R}^2, \tau_{\text{std},2})$, since $[a_i, b_i]$ is connected $\subseteq \mathbb{R}$, and the product of connected sets is a connected subset of the product topology, and $(\mathbb{R}^2, \tau_{\text{std},2}) = (\mathbb{R} \times \mathbb{R}, \rho)$, where ρ is the product topology of $(\mathbb{R}, \tau_{\text{std},1})$ with itself.

Now $(\mathbb{R}^3, \tau_{\text{std},3})$ is the 3-fold product of $(\mathbb{R}, \tau_{\text{std},1})$ with itself. (I.e. the product of $(\mathbb{R} \times \mathbb{R}, \rho_2)$ with $(\mathbb{R}, \tau_{\text{std},1})$, where ρ_2 is the product of $(\mathbb{R}, \tau_{\text{std},1}) \times (\mathbb{R}, \tau_{\text{std},1})$. Since $[a_1, b_1] \times [a_2, b_2]$ is connected $\subseteq \mathbb{R}^2$ and $[a_3, b_3]$ connected $\subseteq \mathbb{R}$, we therefore get that $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ is connected $\subseteq \mathbb{R}^3$.

And so on, by induction: $\mathbb{R}^n$ is the n-fold product of $(\mathbb{R}, \tau_{\text{std},1})$ with itself, n times:

$$(\mathbb{R}^n, \tau_{\text{std},1}) = (\mathbb{R} \times \ldots \times \mathbb{R}, \rho_n).$$

By induction, one proves that $[a_1, b_1] \times \ldots \times [a_n, b_n]$ is a connected subset of $(\mathbb{R}^n, \tau_{\text{std},1})$.

This proves Claim 2.

Now for $z \in \mathcal{B}(p, \varepsilon)$ let W_z be the set $[a_1, b_1] \times \ldots \times [a_n, b_n]$ described above. Then the family $\langle W_z \rangle_{z \in \mathcal{B}(p,\varepsilon)}$ is such that:

$$\bigcup_{z \in \mathcal{B}(p,\varepsilon)} W_z = \mathcal{B}(p, \varepsilon),$$

since by Claim 1, for each $z \in \mathcal{B}(p, \varepsilon)$, we have $z \in W_z$, and $W_z \subseteq \mathcal{B}(p, \varepsilon)$); each W_z is connected, by Claim 2;

$$\cap_{z \in \mathcal{B}(p,\varepsilon)} W_z \neq \emptyset$$

since $p \in W_z$ for each z, by Claim 1.

Therefore by the theorem from lectures on unions of families of connected sets, where the intersection of the family is non-empty, we have that $\bigcup_{z \in \mathcal{B}(p,\varepsilon)} W_z$ is connected, so $\mathcal{B}(p, \varepsilon)$ is connected.

Method 1 (following hint, sketch):

Now we want to use Munkres' §23 exercise 2. So we would like to come up with a countable family $\langle U_n \rangle_{n \in \mathbb{N}}$ of connected sets U_n, such that for each n, $U_n \cap \bigcup_{i=1}^n U_i$ is non-empty, and $\mathcal{B}(p, \varepsilon) = \bigcup_{i \in \mathbb{N}} U_i$. Then the Munkres exercise implies that $\mathcal{B}(p, \varepsilon)$ is connected.

For each $m \in \mathbb{N}^+$, let $X_m = \{r_{m,1}, \ldots, r_{m,k_m}\}$ enumerate all points of the form $p + x$, such that $p + x \in \mathcal{B}(p, \varepsilon)$, and x has the form $(\frac{i_1}{m}, \ldots, \frac{i_n}{m})$, for some integers $i_1, \ldots, i_n$. Note that since $\mathcal{B}(p, \varepsilon)$ is bounded, m is a fixed denominator, and $i_1, \ldots, i_n$ are required to be integers, there are only finitely many such points $p + x$. Note $X_m \neq \emptyset$ since $p \in X_m$. Note that for each such x, there is a finite sequence $p + x_0, p + x_1, p + x_2 \ldots, p + x_j$, such that $x_0 = (0, 0, \ldots, 0)$, $x = x_j$, each $p + x_i \in X_m$, and for each $i < j$, $d(p + x_i, p + x_{i+1}) = 1/m$. (Each "step" from $p + x_i$ to $p + x_{i+1}$ is given by changing just one coordinate by $1/m$.

E.g. if $n = 3$ and $p = (0, 1, 2)$ and $\varepsilon = 0.5$ and $m = 5$ and $p + x = (0.4, 0.8, 2) = p + (\frac{2}{5}, \frac{-1}{5}, \frac{0}{5})$ (which is in X_5 then; note $d(p, p+x) = \sqrt{(0.4)^2 + (0.2)^2} < 0.5$), then we'd get such a path by first stepping 2 times up along the first coordinate, then down once along the second coordinate:

$$p = p + (0, 0, 0),\ p + (\frac{1}{5}, 0, 0),\ p + (\frac{2}{5}, 0, 0),\ p + (\frac{2}{5}, \frac{-1}{5}, 0) = p + x.$$

Now let

$$X'_m = \{r \in X_m \mid d(p, r) < \varepsilon - (2n/m)\}.$$

Given $r \in X'_m$, there is always a finite sequence of steps $p = p + x_0, p + x_1, \ldots, p + x_j = r$ from p to r, like discussed above, but such that each $p + x_i$ is in X'_m. Now enumerate X'_m in an order $q_0, \ldots, q_{j_m}$, without repetitions, such that $q_0 = p$, and for each $i < j_m$, there is $j \leq i$ such that $d(q_j, q_{i+1}) = 1/m$ (so q_{i+1} is just a single "step" from some already enumerated q_j). (This is always possible - after enumerating $q_0, \ldots, q_i$, if there are still some elements of X'_m left to enumerate, choose q_{i+1} to be a point $p + x$ a minimal number of "steps" away from p - i.e. minimal distance in the taxicab metric - which is yet to be enumerated. Then if the sequence $p = p + x_0, p + x_1, \ldots, p + x_j = p + x$ has this minimal number of steps, then the second last element, $p + x_{j-1}$, is in X'_m, and there is a shorter sequence of steps to reach $p + x_{j-1}$ than the shortest sequence to reach $p + x$ (since throwing away $p + x$ from the given sequence already reaches $p + x_{j-1}$). Therefore $p + x_{j-1}$ must already have been enumerated, by choice of $p + x$. I.e $p + x_{j-1} = q_{j'}$ for some $j' \leq i$. So $q_{j'}$ is a single step away from q_{i+1}, as required.)

Now let $V_{m,i}$ be the closed box, centered at q_i, with sides of length $2/m$. Note that if $z \in V_{m,i}$ then $d(z, q_i) \leq n/m$; this this can be shown from the triangle inequality, since we're working in $\mathbb{R}^n$. Then again using the triangle inequality, since $q_i \in X'_m$, $d(q_i, p) \leq \varepsilon - (n/m)$, one can show that $V_{m,i} \subseteq \mathcal{B}(p, \varepsilon)$.

Now enumerate $\{V_{m,i} | m \in \mathbb{N}^+, 0 \leq i \leq j_m\}$ as $U_1, U_2, U_3, \ldots$, in the order

$$V_{1,1}, V_{1,2}, V_{1,3}, \ldots, V_{1,j_1}, V_{2,1}, V_{2,2}, \ldots, V_{2,j_2}, V_{3,1}, \ldots$$

Then Claim: the family $\langle U_n \rangle_{n \in \mathbb{N}}$ has the right properties.

Proof of this is omitted.

12. Let $(X_1, \tau_1), (X_2, \tau_2)$ be two topological spaces such that $X_1 \cap X_2 = \emptyset$. Fix $x_1 \in X_1$ and $x_2 \in X_2$. We'll describe a way to "join" these two spaces, to form a single space, in which x_1 and x_2 represent the same point. We are joining the spaces "at" the points x_1 and x_2. Let a be some object such that $a \notin X_1 \cup X_2$.

Define a topological space (X'_1, τ'_1) to be essentially the same as (X_1, τ_1), except that we replace the point x_1 with a. That is, first define X'_1 as:

$$X'_1 = (X_1 - \{x_1\}) \cup \{a\}.$$

Let $f : X'_1 \to X_1$ be the map given by $f(x) = x$ for $x \in X'_1 - \{a\}$, and $f(a) = x_1$. Now define τ'_1 by:

$$\tau'_1 = \{f^{-1}(W) \mid W \in \tau_1\}.$$

This defines (X'_1, τ'_1). Similarly, let (X'_2, τ'_2) be the space given by replacing $x_2 \in X_2$ with a.

It's straightforward to check that (X'_1, τ'_1) and (X'_2, τ'_2) are topological spaces, and $X'_1 \cap X'_2 = \{a\}$.

Now define a new topological space (X, τ) by "joining" the two spaces at their common point a. That is, let

$$X = X'_1 \cup X'_2,$$

and given $W \subseteq X$, let $W \in \tau$ iff both $(W \cap X'_1) \in \tau'_1$ and $(W \cap X'_2) \in \tau'_2$.

So (X, τ) is the result of "joining" (X_1, τ_1) to (X_2, τ_2), "at" the points x_1 and x_2. (Remark: in terms of the topologies, it doesn't matter what the underlying points actually *are*, it just matters what the topological structure is. Although we've changed the identities of x_1 and x_2 to a, we've preserved the topological structures of the original X_1 and X_2, on each of their "sides" of the space X. Now for the problems:

(a) Show that (X, τ) is a topological space.

(b) Describe the interior and closure operations of X in terms of those for X_1 and X_2.

(c) Show that X is compact (or connected, or path-connected) (w.r.t. τ) iff both X_1 and X_2 are compact (or connected, or path-connected) (w.r.t. τ_1 and τ_2)

(d) Suppose $X_1 = \mathbb{R}$ and $X_2 = \mathbb{R}^2$, $x_1 = 0$ and $x_2 = (0, 0)$. Give a subset of $A \subseteq \mathbb{R}^3$, and a bijection $f : X \to A$, and f is continuous, and f^{-1} is continuous. (This is a *homeomorphism*, a bijection which exactly preserves topological structure; i.e. U is open iff $f``U$ is open.)

Solution.
(a) We have $\emptyset \in \tau$ since $\emptyset \cap X_1' = \emptyset \in \tau_1'$ and likewise for "2" in place of "1". We have $X \in \tau$ since $X \cap X_1' = X_1' \in \tau_1'$ and likewise for "2".

Let $V, W \in \tau$. Then $(V \cap W) \cap X_1' = (V \cap X_1') \cap (W \cap X_1') \in \tau_1'$ since $V \cap X_1' \in \tau_1'$ and $W \cap X_1' \in \tau_1'$ and τ_1' is a topology. Likewise for "2". So $V \cap W \in \tau$.

Let $\langle V_\alpha \rangle_{\alpha \in I}$ be a family of sets $V_\alpha \in \tau$. Let U be the union of the family. Then $U \cap X_1' = \bigcup_{\alpha \in I} (V_\alpha \cap X_1')$, and each $V_\alpha \cap X_1' \in \tau_1'$, so the latter union is in τ_1'. Likewise for "2". Therefore $U \in \tau$, as required.

This shows (X, τ) is a topological space.

(b) First note that if $A \subseteq X$, then $\mathrm{Int}(A) \subseteq \mathrm{Int}_1(A \cap X_1') \cup \mathrm{Int}_2(A \cap X_2')$, where Int_1 is the interior operation on X_1', and likewise for "2". For $W = \mathrm{Int}(A)$ is open w.r.t. τ, so $W \cap X_1' \in \tau_1'$, and $W \subseteq A$, so $W \cap X_1' \subseteq A \cap X_1'$. Therefore $W \cap X_1' \subseteq \mathrm{Int}_1(A \cap X_1')$. Likewise $W \cap X_2' \subseteq \mathrm{Int}_2(A \cap X_2')$. But $W = W \cap X_1' \cup W \cap X_2'$.

Now note that $a \in \mathrm{Int}(A)$ iff $a \in \mathrm{Int}_1(A \cap X_1')$ and $a \in \mathrm{Int}_2(A \cap X_2')$. Proof: if $a \in \mathrm{Int}(A)$ then there's $W \in \tau$, such that $a \in W \subseteq A$. But then $W \cap X_1' \in \tau_1'$, and $a \in W \cap X_1'$, so $a \in \mathrm{Int}_1(A)$. Likewise for "2". Now suppose $a \in \mathrm{Int}_1(A \cap X_1')$ and $a \in \mathrm{Int}_2(A \cap X_2')$. Let $W_1 \in \tau_1$ such that $a \in W_1 \subseteq X_1'$ and likewise for W_2. Then $W = W_1 \cup W_2$ is in τ (since $W \cap X_1' = W_1$ and likewise for "2") and $a \in W$. So $a \in \mathrm{Int}(A)$.

Now suppose $x \in X_1'$, and $x \in \mathrm{Int}_1(A \cap X_1')$. We want to determine whether $x \in \mathrm{Int}(A)$.

Case 1. $a \in \mathrm{Int}(A)$ (note we've already described this condition in terms of Int_1 and Int_2).
Then $x \in \mathrm{Int}(A)$. For let $U \in \tau$, such that $a \in U \subseteq A$, and let $U' \in \tau_1'$, such that $x \in U' \subseteq A \cap X_1'$. Then $U' \cup U \in \tau$, since

$$(U' \cup U) \cap X_2' = (U' \cap X_2') \cup (U \cap X_2')$$

but $U' \cap X_2' \subseteq \{a\}$ and $\{a\} \subseteq U \cap X_2'$, so in fact

$$(U' \cup U) \cap X_2' = U \cap X_2'$$

and $U \cap X_2' \in \tau_2'$ since $U \in \tau$. And

$$(U' \cup U) \cap X_1' = (U' \cap X_1') \cup (U \cap X_1')$$
$$= U' \cup (U \cap X_1')$$

and $U' \in \tau_1'$, and $U \cap X_1' \in \tau_1'$ since $U \in \tau$, so $U' \cup (U \cap X_1') \in \tau_1'$.

Case 2. $a \notin \mathrm{Int}(A)$.

Subcase 1. There's $V \in \tau_1'$ such that $x \in V$ and $a \notin V$.
Then $x \in \mathrm{Int}(A)$. For $V \in \tau$ and $V \subseteq A$ and $x \in V$.

Subcase 2. For every $V \in \tau_1'$, if $x \in V$ then $a \in V$ (including the case that $x = a$).
Then $x \notin \mathrm{Int}(A)$. For if $x \in \mathrm{Int}(A)$ then there's $W \in \tau$ such that $x \in W \subseteq A$. But then $W \cap X_1' \in \tau_1'$ and $x \in W \cap X_1'$, but then $a \in W \cap X_1'$ by subcase hypothesis, so $a \in W$, but $W \in \tau$ and $W \subseteq A$, and therefore $a \in \mathrm{Int}(A)$, contradicting case hypothesis.

Things work symmetrically when "1" is replaced by "2".

This gives a description of Int in terms of the interior operations for X_1' and X_2', but these translate directly to the interior operations for X_1 and X_2, just replacing "a" with "x_1" for X_1, and likewise for "2".

This completes the description.

Remark: Note this implies that if X_1, X_2 are both T_1 (and therefore X_1', X_2' are both T_1), then for $x \in X_1'$, $x \in \mathrm{Int}(A)$ iff $x \neq a$ and $x \in \mathrm{Int}_1(A \cap X_1')$, or $x = a$ and $a \in \mathrm{Int}_1(A \cap X_1')$ and $a \in \mathrm{Int}_2(A \cap X_2')$.

The closure operation works similarly. (It can be inferred from the interior operation, in fact, since closed sets are just complements of open sets.) Further detail is omitted.

(c) First X_1 is compact w.r.t. τ_1 iff X_1' is compact w.r.t. τ_1', since these spaces are identical except for the replacement of x_1 with a, essentially just a renaming of that point; the open sets are preserved through this renaming. Likewise for "2".

Suppose X_1 and X_2 are both compact. So X_1' and X_2' both are.

Let $\langle U_i \rangle_{i \in I}$ be an open cover of X. Then $V_i = U_i \cap X_1' \in \tau_1'$ for each i, so $\langle V_i \rangle_{i \in I}$ forms an open cover of X_1', since also $X_1' \subseteq X$, so $X_1' = X \cap X_1'$, and $X = \bigcup_{i \in I} U_i$. By compactness of X_1 w.r.t. τ_1', there is a finite subcover, so we have $V_{i_1}, \ldots, V_{i_n}$ such that

$$X_1' \subseteq V_{i_1} \cup \ldots \cup V_{i_n}.$$

Similarly, if we set $W_i = U_i \cap X_2'$, then there are finitely many $W_{j_1}, \ldots, W_{j_m}$ such that

$$X_2' \subseteq W_{j_1} \cup \ldots \cup W_{j_m}$$

(using compactness of X_2'). But then

$$X = X_1' \cup X_2' \subseteq V_{i_1} \cup \ldots \cup V_{i_n} \cup W_{j_1} \cup \ldots \cup W_{j_m}.$$

But $V_i \subseteq U_i$ and $W_i \subseteq U_i$ for each i, so

$$X \subseteq U_{i_1} \cup \ldots \cup U_{i_n} \cup U_{j_1} \cup \ldots \cup U_{j_m}.$$

Therefore the list of sets $U_{i_1}, \ldots, U_{i_n}, U_{j_1}, \ldots, U_{j_m}$ form a cover of X, there are only finitely many of them (there are $n + m$ of them), and each is one of the sets from the original family $\langle U_i \rangle_{i \in I}$. So this is is a finite subcover of X from this family, as required. So X is compact.

Now suppose X_1 is not compact; so X_1' is not compact either. We will show X is not compact. (By symmetry then, if X_2 is not compact, then X is not compact either.)

Let $\langle U_i \rangle_{i \in I}$ be an open cover of X_1' which admits no finite subcover of X_1'; i.e. for any finite list $U_{i_1}, \ldots, U_{i_n}$ of sets from this family, we have $X \not\subseteq U_{i_1} \cup \ldots \cup U_{i_n}$.

For $i \in I$, define $V_i \in \tau$ as follows: if $a \notin U_i$ then let $V_i = U_i$. If $a \in U_i$ then let $V_i = U_i \cup X_2'$. Note that in either case, $V_i \in \tau$ (we have $V_i \cap X_2'$ is either $\emptyset$ or X_2', which is in τ_2', and $V_i \cap X_1' = U_i$, which is in τ_1'). So $\langle V_i \rangle_{i \in I}$ is a family of sets each in τ. This family is also a cover of X, since there is $i_0 \in I$ such that $a \in U_{i_0}$ (exists since $\langle U_i \rangle_{i \in I}$ is a cover of X_1' and $a \in X_1'$), and therefore we have $X_2' \subseteq V_{i_0}$, so $U_{i_0} \subseteq \bigcup_{i \in I} V_i$, and $U_i \subseteq V_i$ for each i, so $X_1' = \bigcup_{i \in I} U_i \subseteq \bigcup_{i \in I} V_i$. So the union of the V_i's covers both X_1' and X_2', so covers X.

We claim there is a no finite subcover of X from the family of $\langle V_i \rangle_{i \in I}$. For let $V_{i_1}, \ldots, V_{i_n}$ be a finite list of sets from that family. We must show these sets do not form a cover of X.

We know $U_{i_1}, \ldots, U_{i_n}$ does not cover X_1', by choice of the family $\langle U_i \rangle_{i \in I}$. Let $x \in X_1'$ such that $x \notin U_{i_1} \cup \ldots \cup U_{i_n}$. By definition of V_i, $V_{i_k} \cap X_1' = U_{i_k}$ for $k = 1, \ldots, n$. But $x \in X_1'$, and therefore $x \notin V_{i_1} \cup \ldots \cup V_{i_n}$, but $x \in X$, so these sets $V_{i_1}, \ldots, V_{i_k}$ do not cover X, as required.

So the family $\langle V_i \rangle_{i \in I}$ is a cover of X with no finite subcover of X, so X is not compact.

The connectedness is a similar type of argument (except that it deals with connectedness instead of compactness). Further detail is omitted.

For path-connectedness, here is an outline. If X_1' and X_2' are both path-connected, and x, y are points in X, if $x \neq a$ and $y \neq a$, then one can path-connect x to a in X_1', and then path-connect a to y in X_2'. Conactenate these two paths to path-connect x to y. If X is path-connected, prove that X_1' is path-connected as follows: given $x, y \in X_1'$, let $f : [0,1] \to X$ be a function as required for path-connecting x to y. One must show that you can replace f with a function whose image is contained with X_1'. We have $f(0) = x$ and $f(1) = y$. Suppose $f``[0,1] \not\subseteq X_1'$. Let $z_0 = \inf\{z \in [0,1] | f(z) \notin X_1'\}$. Similarly, let $z_1 = \sup\{z \in [0,1] | f(z) \notin X_1'\}$. Now define $g : [0,1] \to X_1'$ by: given $z \in [0,1]$, if $z_0 < z < z_1$ then let $g(z) = a$; if $z \leq z_0$ or $z \geq z_1$ then, if $f(z) \in X_1'$ let $g(z) = f(z)$, whereas if $f(z) \notin X_1'$ let $g(z) = a$. (Note that if $z < z_0$ or $z > z_1$ then certainly $f(z) \in X_1'$ so $g(z) = f(z)$; the last clause is only relevant when $z = z_0$ or $z = z_1$.)

Prove that g is continuous from $[0,1]$ to X_1', in particular $g``[0,1] \subseteq X_1'$, $g(0) = x$ and $g(1) = y$. The main issue is to see that g is continuous at z_0 and at z_1. Consider z_0. Suppose first that $f(z_0) \in X_1'$, so then $g(z_0) = f(z_0)$. Note then that $z_0 < z_1$, since if $z_0 = z_1$, since $f(z_0) \in X_1'$ and then $f(z_0) = f(z_1)$, it follows that $f``[0,1] \subseteq X_1'$ from choice of z_0, z_1, contradiction. Now let $U \in \tau_1'$ such that $f(z_0) \in U$. We claim that $a \in U$. For otherwise $U \in \tau$, and therefore $f^{-1}(U)$ is open in $[0,1]$, and $z_0 \in U$, but therefore $(z_0 - \varepsilon, z_0 + \varepsilon) \subseteq f^{-1}(U)$ for some $\varepsilon > 0$, which implies $f``(z_0 - \varepsilon, z_0 + \varepsilon) \subseteq U \subseteq X_1'$, contradicting the choice of z_0. So $a \in U$. So let δ be such that $0 < \delta < z_1 - z_0$ and for all $z \in [0,1]$, $|z - z_0| < \delta$ implies $f(z) \in U$. Then if $z \in [0,1]$ and $|z - z_0| < \delta$ then $g(z) \in U$, as required for continuity at z_0.

Now suppose that $f(z_0) \notin X_1'$. So $g(z_0) = a$. Let $U \in \tau_1'$ such that $a \in U$. Then $W = U \cup X_2' \in \tau$ and $f(z_0) \in W$. Therefore there is $\delta > 0$ such that for $z \in [0,1]$, $|z - z_0| < \delta$ implies $f(z) \in W$. But then if $|z - z_0| < \delta$ then: if $z < z_0$ then $g(z) = f(z) \in W$ and $f(z) \in X_1'$, so in fact $g(z) \in U$; likewise if $z > z_1$; if $z_0 \leq z < z_1$ then $g(z) = a \in U$; and if $z = z_1$ then either $g(z) = a \in U$, or $g(z) = f(z) \in W$, and $f(z) \in X_1'$, so again $g(z) \in U$.

(d) Let $A \subseteq \mathbb{R}^3$ be the union of the xy-plane with some line through the origin, which does not lie in the xy-plane. Let p be some non-origin point on the line. We have $X_1 = \mathbb{R}_2$ and $X_2 = \mathbb{R}$; we have $x_1 = (0,0)$ and $x_2 = 0$; we have $a \notin \mathbb{R}^2 \cup \mathbb{R}$ replacing x_1, x_2. Let $f : X \to \mathbb{R}^3$ be: given $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, let $f(x,y) = (x,y,0)$; given $x \in \mathbb{R} - \{0\}$, let $f(x) = xp$; and let $f(a) = (0,0,0)$.

We omit the proof that A, f have the required properties.