Intro to topology homework 10. Due Thursday April 5, 11am.

1. Let $X$ be a set and $\mathfrak{b}'$ be a collection of sets such that $\cup\mathfrak{b}' = X$. Let $\mathfrak{b}$ be the collection of all intersections of finite families consisting of elements of $\mathfrak{b}'$, i.e.

$$\mathfrak{b} = \{\bigcap_{i=1}^{n} U_i \mid n \in \mathbb{N}, U_i \in \mathfrak{b}' \text{ for } i = 1, \ldots, n\}.$$

Show that $\mathfrak{b}$ is a base. In fact, show that $\mathfrak{b}$ is a base for a topology $\tau$ *on* $X$, and $\tau$ is the smallest topology $\tau'$ such that $\mathfrak{b}' \subseteq \tau'$. (That is, $\mathfrak{b}' \subseteq \tau$, and if $\tau'$ is any topology such that $\mathfrak{b}' \subseteq \tau'$, then $\tau \subseteq \tau'$.)

2. Suppose that $(X, \tau)$ is a topological space with a countable base. Show that every sequentially closed subset of $X$ is closed.

3.(a) Let $(X_1, \tau_1), (X_2, \tau_2)$ be top spaces. Let $\pi_1 : X_1 \times X_2 \to X_1$ be the projection map, i.e.

$$\pi_1(x_1, x_2) = x_1.$$

Let $\tau$ be the product topology on $X = X_1 \times X_2$.

(i) Show that $\pi_1$ is continuous as a map from $(X, \tau)$ to $(X_1, \tau_1)$. (ii) Use symmetry to infer that the same holds for the projection map $\pi_2 : X \to X_2$ ($\pi_2(x_1, x_2) = x_2$). (iii) Show that, in fact, $\tau$ is the smallest topology $\tau'$ on $X_1 \times X_2$ such that both $\pi_1$ is continuous from $(X, \tau')$ to $(X_1, \tau_1)$, and $\pi_2$ is continuous from $(X, \tau')$ to $(X_2, \tau_2)$.

(b) Let $(X, \tau)$ be a top space and let $A \subseteq X$. Show that $\pi : A \to X$, given by $\pi(a) = a$ for each $a \in A$, is continuous from $(A, \tau \restriction A)$ to $(X, \tau)$. (Recall that $\tau \restriction A$ is the subspace topoology on $A$ induced by $\tau$.) (In fact, $\tau \restriction A$ is the smallest topology on $A$ with this property; this can be argued like in part (a), but you needn't.)

(c) Let $(Z, \sigma)$, $(X_1, \tau_1)$, $(X_2, \tau_2)$ be top spaces and let $(X, \tau)$ be the product topology on $X = X_1 \times X_2$ from $\tau_1, \tau_2$. Let $f_1 : Z \to X_1$ and $f_2 : Z \to X_2$. Let $f : Z \to X$ then be given by $f(z) = (f_1(z), f_2(z))$.

Show that $f$ is continuous iff both $f_1$ and $f_2$ are continuous.

(d) Let $(X, \tau)$ be a top space and $A \subseteq X$. Let $(Z, \sigma)$ be another top space. Let $f : Z \to X$ be such that $\mathrm{rg}(f) \subseteq A$; (here $\mathrm{rg}(f)$ is the range of $f$) so in fact, $f : Z \to A$. Make and prove a statement talking about when $f$ is continuous as a map $(Z, \tau') \to (X, \tau)$ versus when it is continuous as a map $(Z, \tau') \to (A, \tau \restriction A)$. (I.e., are these conditions equivalent, or not, describe.)

(e) Use parts (c),(d) to infer that the example given in class is continuous, i.e. $f : \mathbb{R} \to \mathbb{R} \times S_1$, given by $f(a) = (a, (\cos(2\pi a), \sin(2\pi a)))$, is continuous. (You may take for granted that the trig functions $\mathbb{R} \to \mathbb{R}$ are continuous. Consider $\mathbb{R}^2$ as being the product $\mathbb{R} \times \mathbb{R}$. Use (c) to show that the map $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$, given by $a \mapsto (\cos(2\pi a), \sin(2\pi a))$ is continuous, and proceed from there.)