Intro to topology homework 10. Due Thursday April 5, 11am.

If you're looking for the separation axioms in Munkres, note that he uses "Hausdorff" for T_2, "regular" for T_3, and "normal" for T_4.

1. Prove that any finite T_1 space is in fact the discrete topology. (Remark: So any T_1, not T_2 space has to be infinite.) (Remark: also the discrete topology on any set X is the topology of a metric, so by Theorem L1.18, is T_4.)

2. Say a top space X is $T_{1.5}$ iff every sequence of points in X converges to at most one point in X. (Remark: I don't believe that this is standard terminology.)

(a) Prove that $T_2 \implies T_{1.5} \implies T_1$.

(b) Give an example of a space which is T_1 but not $T_{1.5}$. (Remark: There are also example of $T_{1.5}$, not T_2 spaces, but by part (b), they can't have a countable base.)

(c) Prove that if X is $T_{1.5}$ and X has a countable base then X is T_2.

3.(a) Let (X, d) be a metric space. Let C_1, C_2 be closed disjoint subsets of X. Explicitly define a continuous function $f : X \to \mathbb{R}$ such that

- $f(x) = 0$ for all $x \in C_1$,
- $f(x) = 1$ for all $x \in C_2$,
- $0 < f(x) < 1$ for all $x \in X \backslash (C_1 \cup C_2)$.

Prove that your function is continuous and has these properties.

(Hint: make liberal use of the measurement $d(x, C)$ of distance between a point x and set C.) (Hint: if $f, g : X \to \mathbb{R}$ are continuous functions, then so are $f + g$, $f * g$, and if $g(x) \neq 0$ for all $x \in X$, then so is f/g.)

(b) Suppose (X, τ) is a T_1 topological space, and that for any two closed disjoint sets $C_1, C_2 \subseteq X$, there is a continuous function $f : X \to \mathbb{R}$ with the properties described in (a). Prove that X is T_4.

4. Suppose X is a T_3 space and $x \neq y \in X$. Prove that there are $U_1, U_2 \in \tau$ such that $x \in U_1$, $y \in U_2$, and $\mathrm{Cl}(U_1) \cap \mathrm{Cl}(U_2) = \emptyset$.

5.[required for 4500 only; extra credit for 5600] Let X be a T_3 space and $A \subseteq X$. Show that the subspace topology induced on A is also T_3.

5.[required for 5600 only; extra credit for 4500] Let X_1, X_2 be two spaces whose product is T_3. Prove that both X_1 and X_2 are T_3.