Intro to Topology Homework 1A - due Tuesday Jan 24.
(Part 1B will be posted Thursday.)
Note: Problems with a * are those suitable for the homework discussion requirement.

1. Show that neither $(0, 4]$ nor $\mathbb{Q}$ are open sets.

2. Show directly from the definition of "open" that if $A, B \subseteq \mathbb{R}$ are two open sets then $A \cup B$ is also open. (Don't use any theorems.)

3*. For $i \in \mathbb{N}$ let A_i be the interval $[\frac{1}{2^{i+1}}, \frac{1}{2^i})$.
(a) Show that

$$\emptyset = \bigcap_{i \in \mathbb{N}} A_i$$

and

$$(0, 1) = \bigcup_{i \in \mathbb{N}} A_i.$$

(b) Let $K_i = A_i \backslash \{\frac{1}{2^{i+1}}\}$, let L_i be the interval $(-i, -i + 2\sin(i))$ and $B_i = K_i \cup L_i$. Show directly from the definition of "open" (without using any theorems) that

$$\bigcup_{i \in \mathbb{N}} B_i$$

is open.

4. Recall that $A \backslash B = \{x \in A | x \notin B\}$. (a1) Prove

$$X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B),$$

and (a2) for any indexed family of sets $\langle A_j \rangle_{j \in J}$, that

$$X \backslash (\bigcup_{j \in J} A_j) = \bigcap_{j \in J} (X \backslash A_j).$$

(b) State and prove a law analogous to (a2), but in which the expression on the left side of the equality is

$$X \backslash (\bigcap_{j \in J} A_j).$$

5. Given $a, b \in \mathbb{R}$ with $a < b$, there is a $q \in \mathbb{Q}$ such that $q \in (a, b)$. (We say the rationals are *dense* in the reals.) Discuss this seem plausible? (You are not required to prove it; I just want you to think about it and determine whether it fits with your intuitions about the reals. It is in fact true, and from the next problem onward you can assume it as fact.)

We write $\mathbb{Z}^+ = \mathbb{N}^+ = \mathbb{N}\backslash\{0\}$.

6*. For $q \in \mathbb{Q}$ and $n \in \mathbb{Z}^+$ let $B_{q,n} = (q - \frac{1}{n}, q + \frac{1}{n})$.
(a) Let A be the set

$$A = \bigcup_{q\in\mathbb{Q}} \left(\bigcap_{n\in\mathbb{Z}^+} B_{q,n} \right). \tag{1}$$

Prove that for all $x \in \mathbb{R}$:

$$[x \in A]$$

$$\text{if \& only if}$$

$$[\text{there exists } q \in \mathbb{Q} \text{ such that for all } n \in \mathbb{Z}^+, \quad x \in B_{q,n}]. \tag{2}$$

Clarification: Equation (1) means: for $q \in \mathbb{Q}$, let $C_q = \bigcap_{n\in\mathbb{Z}^+} B_{q,n}$. So we have a family $\langle C_q \rangle_{q\in\mathbb{Q}}$. Then A is the union of this family:

$$A = \bigcup_{q\in\mathbb{Q}} C_q.$$

(b) Compute the set A, and write it in more familiar terms. (Hint: A is one of the sets $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\emptyset$, $\mathbb{R}\backslash\mathbb{Q}$, $\mathbb{N}$.)
(c) Now let

$$B = \bigcap_{q\in\mathbb{Q}} \left(\bigcup_{n\in\mathbb{Z}^+} B_{q,n} \right).$$

Give a condition for membership in B, analogous to the condition in (2), and prove that it describes membership in B exactly. That is, give a condition *************** such that for all reals x,

$$x \in B \iff ***************. \tag{3}$$

The condition *************** should be analogous to the condition in (2). Prove that the resulting statement (3) is true for all reals x.

Then compute the set B (like done in part (b) for A; again B is one of the sets $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\emptyset$, $\mathbb{R}\backslash\mathbb{Q}$, $\mathbb{N}$.)

(d) Let C be the set of all $x \in \mathbb{R}$ such that for all $n \in \mathbb{Z}^+$ there exists $q \in \mathbb{Q}$ such that $x \in B_{q,n}$. Write C in terms of unions and intersections, and repeat part (b) for the set C.

For the next problem:

A *rational interval* is an interval (a, b) or $[a, b]$ or $(a, b]$ or $[a, b)$ such that $a, b \in \mathbb{Q}$.

Two sets B, C are *disjoint* iff $B \cap C = \emptyset$. A family of sets $\langle A_j \rangle_{j\in J}$ is a *disjoint family* iff for every $j, k \in J$, if $j \neq k$ then A_j, A_k are disjoint.

7*. Let $A \subseteq \mathbb{R}$. Prove that the following statements are equivalent.

(a) A is open

(b) (This condition was "A is a union of open itervals", but it has been removed as the equivalence with openness was proved in class.)

(c) A is a union of open rational intervals. (I.e., there is a family $\langle I_j \rangle_{j \in J}$ such that each I_j is a rational interval, and $A = \cup_{j \in J} A_j$.)

(d) A is a union of a disjoint family of open intervals

Then give an example of an open set $A \subseteq [0, 1]$ which is *not* the union of any disjoint family of open rational intervals.