Intro to Topology Homework 2 - due Thursday Feb 2

(Note: All open/closed sets in this homework are open/closed subsets of $\mathbb{R}$.)

1. Show that $\mathbb{Z}$ is closed.

2. Let $\langle A_j \rangle_{j \in J}$ be a family of closed sets. Prove that the intersection of the family is closed.

3. (a) Prove the following: Suppose A is a closed set such that $\mathbb{Q} \subseteq A$. Then $A = \mathbb{R}$.
(b) By considering complements of the sets mentioned in (a), translate the fact proved in (a) into a statement about the irrationals and open sets (you don't need to prove it).

4.
(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.
(i) Let $a \in \mathbb{R}$. Show that $\{x \in \mathbb{R} | f(x) = a\}$ is closed. (You may use results from earlier homeworks. If you use a definition of continuity directly, then use the $\varepsilon - \delta$ definition.)
(ii) Let $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Use (i) and a theorem from lectures to conclude that

$$\{x \in \mathbb{R} | f(x) \in \{a_1, a_2, a_3, a_4\}\}$$

is closed. (The set $\{a_1, a_2, a_3, a_4\}$ is the set whose elements are precisely a_1, a_2, a_3, and a_4.)
(b) Let $f(x) = \sin(\tan(x))$. Let $A = \{x \in \mathbb{R} | f(x) = 0\}$. What is the smallest closed set B such that $A \subseteq B$? (Careful.)

5. We say a sequence $\langle A_i \rangle_{i \in \mathbb{N}}$ of sets is *decreasing* iff for each $i \in \mathbb{N}$, $A_{i+1} \subseteq A_i$.
(a) Give an example of a decreasing sequence $\langle A_i \rangle_{i \in \mathbb{N}}$ of non-empty closed sets A_i such that $\cap_{i \in \mathbb{N}} A_i = \emptyset$.
(b) Give an example of a decreasing sequence $\langle A_i \rangle_{i \in \mathbb{N}}$ of non-empty open sets A_i, such that each $A_i \subseteq [0, 1]$, and $\cap_{i \in \mathbb{N}} A_i = \emptyset$.
(c) Try for a few minutes to find a sequence as in (a), such that each $A_i \subseteq [0, 1]$. (You don't need to write anything for this part, but feel free to.)

6. Let's say an interval is *non-singleton* iff it has at least 2 elements in it.
(a) Show that every open set is the union of a family of non-singleton closed intervals.
(b) Translate (a) into a statement about closed sets (i.e., every closed set is the ....).