Intro to Topology Homework 4 - due Friday Feb 17 (you can hand it in at my office, or slide it under my office door, 416 GAB)
Any problem is good for homework discussion.

1. Let (X, d) be a metric space and let $\langle x_i \rangle_{i \in \mathbb{N}}$ be a sequence in X. Prove that $\langle x_i \rangle_{i \in \mathbb{N}}$ converges to at most one element of X.

2. Let (X, d) be a metric space and let $\langle x_i \rangle_{i \in \mathbb{N}}$ be a sequence in X converging to $x \in X$. Suppose $z \in X$, $a > 0$ and $d(z, x_i) > a$ for all $i \in \mathbb{N}$. Show that $d(z, x) \geq a$.

3. Let X be the collection of all functions f such that $f : \mathbb{N} \to \mathbb{R}$, f is bounded (i.e. there is $M > 0$ such that $|f(n)| < M$ for all n) and

$$\sum_{n=0}^{\infty} |f(n)| < \infty.$$

For $f, g \in X$, define

$$d_{\sum}(f, g) = \sum_{n=0}^{\infty} |f(n) - g(n)|.$$

and

$$d_{\sup}(f, g) = \sup\{|f(n) - g(n)| \mid n \in \mathbb{N}\}.$$

(Given $A \subseteq \mathbb{R}$ with $A \neq \emptyset$, $\sup(A)$ is the least $r \in \mathbb{R} \cup \{\infty\}$ such that $a \leq r$ for all $a \in A$; for any such A there is a least such r.)

(a) Show that $d_{\sum}(f, g) < \infty$ and $d_{\sup}(f, g) < \infty$, and prove the triangle inequality for $d_{\sup}$.

In fact, both $(X, d_{\sum})$ and $(X, d_{\sup})$ are metric spaces, which you may now assume.

(b) Let f_n be the function such that $f_n(n) = 1$ and $f_n(m) = 0$ for $m \neq n$. Show that the sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ converges to 0 (i.e. the constantly 0 function) pointwise, but the sequence does not converge in either $(X, d_{\sum})$ or $(X, d_{\sup})$.

(c) Let $\langle f_i \rangle_{i \in \mathbb{N}}$ be a sequence in X. Show that if the sequence converges (to some function) in $(X, d_{\sum})$, then the sequence converges in $(X, d_{\sup})$, and that if it converges in $(X, d_{\sup})$, then it converges pointwise.

(d) Give an example of a sequence $\langle f_i \rangle_{i \in \mathbb{N}}$ such that the sequence converges to 0 in $(X, d_{\sup})$, but the same sequence does not converge to 0 in $(X, d_{\sum})$.

Let (X, d) be a metric space and let $\langle x_i \rangle_{i \in \mathbb{N}}$ be a sequence in X. We say that the sequence is *Cauchy* iff for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $d(x_m, x_n) < \varepsilon$.

4. (a) Let (X, d) be a metric space. Prove that every convergent sequence in (X, d) is Cauchy.

(b) Give an example of a sequence in $\mathbb{Q}$ which is Cauchy but not convergent in $(\mathbb{Q}, d_{\text{standard}})$.

(c) Give an example of a metric d on $\mathbb{Q}$, such that every Cauchy sequence of $(\mathbb{Q}, d)$ is convergent in $(\mathbb{Q}, d)$.