Intro to Topology Homework 6 - due Friday Mar 2.
(Note: all solutions, including examples, should be explained, unless stated otherwise.)

For (1) and (2), let (X, d) and (Y, e) be metric spaces and $f : X \to Y$. Continuous and sequentially continuous mean w.r.t. d and e.
Given $x \in X$ and $U \subseteq X$, we say U is a *neighbourhood of* x iff there's an open $U' \subseteq U$ st $x \in U'$ (all w.r.t. d).

(1) Prove that f is continuous iff sequentially continuous.

(2) Let $x \in X$. Prove that

f is continuous at x

iff

for all open neighbourhoods U of $f(x)$, we have that

$f^{-1}(U)$ contains a neighbourhood of x.

In the following, given a function $f : X \to Y$ and a family $\langle A_i \rangle_{i \in J}$ with each $A_i \subseteq Y$, we write e.g.

$$\bigcap_{i \in J} f^{-1}(A_i)$$

for the intersection

$$\bigcap_{i \in J} B_i,$$

where $\langle B_i \rangle_{i \in J}$ is the family with each $B_i = f^{-1}(A_i)$. We'll use notation like this from now on without specific mention.

(3) Prove the following two laws (i.e. prove that for all sets X, Y (no metrics involved), functions $f : X \to Y$, all sets J and families $\langle A_i \rangle_{i \in J}$ with $A_i \subseteq Y$ for each $i \in J$, and for all sets $A \subseteq Y$, the following equations hold):

$$f^{-1}(\bigcap_{i \in J} A_i) = \bigcap_{i \in J} f^{-1}(A_i).$$

$$f^{-1}(Y \backslash A) = X \backslash f^{-1}(A).$$

For each of the following 6 statements, either prove it in general or give a counter-example. If giving a counter-example, give a counter-example in which $f : X \to Y$, (X, d), (Y, e) are metric spaces, and f is continuous (w.r.t. d and e):

$$f``\bigcup_{i \in J} A_i \subseteq \bigcup_{i \in J} f``A_i.$$

$$f``\bigcap_{i \in J} A_i \subseteq \bigcap_{i \in J} f``A_i.$$

$$f``\bigcup_{i\in J} A_i \supseteq \bigcup_{i\in J} f``A_i.$$

$$f``\bigcap_{i\in J} A_i \supseteq \bigcap_{i\in J} f``A_i.$$

$$f``\bigcup_{i\in J} A_i = \bigcup_{i\in J} f``A_i.$$

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