Intro to Topology Homework 9 - due Friday Mar 30.
(Note: all solutions, including examples, should be explained, unless stated otherwise.)

1. Let B be the set of all open intervals $I \subseteq \mathbb{R}$ such that $I = (a, b)$, with $a < b$, $b - a < 1$, $a \in \mathbb{Q}$ and $b \in \mathbb{R} \backslash \mathbb{Q}$. Prove that B is a base for $(\mathbb{R}, \tau_{\text{std}})$.

2. Let (X_1, τ_1) and (X_2, τ_2) be top spaces. Let B_1 be a base for τ_1 and B_2 a base for τ_2. Let $f : X_1 \to X_2$. Show that f is continuous iff $f^{-1}(U) \in \tau_1$ for every $U \in B_2$. If f is continuous, does it follows that $f^{-1}(U) \in B_1$ for every $U \in B_2$?

3. Let τ_{std} be the standard topology on $\mathbb{R}$. Let τ be the collection of all $U \in \tau_{\text{std}}$ such that $\mathbb{Q} \subseteq U$ or $U = \emptyset$. This is a topology on $\mathbb{R}$; you may assume this.

(a) (With correction added.) Let (Y, τ') be a topological space. Suppose that for all points $y_0 \neq y_1 \in Y$, there is a set $Y_0 \in \tau'$ such that $y_0 \in Y_0$ but $y_1 \notin Y_0$. Show that $f : \mathbb{R} \to Y$ is continuous wrt τ, τ' iff f is constant.

(b) Describe all of the convergent sequences in $(\mathbb{R}, \tau)$.

(c) Given $U \subseteq \mathbb{R}$, describe the closure of U and the interior of U in $(\mathbb{R}, \tau)$.

4. Suppose (X, d) is a metric space, and $D \subseteq X$ is a dense set. Let B be the collection of all open balls centered at some point of D. (I.e., B is the set of all balls $\mathcal{B}(x, \varepsilon)$, where $x \in D$ and $\varepsilon > 0$.)

(a) Prove that B is a base for (X, τ_d).

(b) If D is also countable, prove that (X, τ_d) has a countable base.

5. Let X be the upper half-plane, including the x-axis. Let B_1 be the collection of all open balls $U = \mathcal{B}_{d_{\text{std}}}(x, \varepsilon)$ such that $U \subseteq X$. Let B_2 be the collection of all sets of the form $U \cup \{(x, 0)\}$, where $x \in \mathbb{R}$ and $U \subseteq X$ is a non-empty open ball, such that $(x, 0)$ is on the boundary of the ball (i.e., the ball is tangential to the x-axis, at the point $(x, 0)$). Let $B = B_1 \cup B_2$.

(a) Show that B is a base on X.

(b) Let τ be the topology generated by B. Describe the interior and closure operations for (X, τ).

6. Let $U \subseteq X$ for some top space X. Let U' be the interior of the closure of U. Show that U' is the interior of the closure of U'.