Midterm 1 Review Problems
(Note: all solutions, including examples, should be explained, unless indicated otherwise.)

1. (a) Give an example of a set $A \subseteq (0,1)$, such that: $A$ is closed in $(\mathbb{R}, d_{\text{std}})$ and $A$ is infinite, but for all $a < b \in \mathbb{R}$, $(a,b) \not\subseteq A$.
(b) Give an example of a metric $d$ on $\mathbb{R}$, such that $[0,1)$ is both closed and open in $(\mathbb{R}, d)$.

2. Prove that if $A \subseteq \mathbb{R}$ is open then $A$ is the union of a family of open intervals $\langle I_j \rangle_{j \in J}$, such that for all $j \in J$, $I_j$ has rational endpoints.

3. Prove that in any metric space, a set is closed iff it is sequentially closed.

4. Let $X = \mathbb{R}$ and $d$ be the function defined as follows. Given $x, y \in \mathbb{R}$, we define $d(x,y)$.
Case 1: $x, y \in \mathbb{R}\backslash\mathbb{Z}$. Then $d(x,y) = d_{\text{std}}(x,y)$.
Case 2: $x \in \mathbb{Z}$ and $y \in \mathbb{R}\backslash\mathbb{Z}$. Then $d(x,y) = 1 + d_{\text{std}}(x,y)$.
Case 3: $x \in \mathbb{R}\backslash\mathbb{Z}$ and $y \in \mathbb{Z}$. Then $d(x,y) = 1 + d_{\text{std}}(x,y)$.
Case 4: $x, y \in \mathbb{Z}$, $x \neq y$. Then $d(x,y) = 2 + d_{\text{std}}(x,y)$.
Case 5: $x = y \in \mathbb{Z}$. Then $d(x,y) = 0$.
This completes all cases.

(a). Show that $(\mathbb{R}, d)$ is a metric space. (You may assume that $(\mathbb{R}, d_{\text{std}})$ is a metric space.)

(b). Show that for every $x \in \mathbb{R}$, $\{x\}$ is open in $(\mathbb{R}, d)$ iff $x \in \mathbb{Z}$.

(c). Show that if $A \subseteq \mathbb{R}$ is open in $(\mathbb{R}, d_{\text{std}})$ then $A$ is open in $(\mathbb{R}, d)$.

(d). Let $\langle x_i \rangle_{i \in \mathbb{N}}$ be a sequence of points $x_i \in \mathbb{R}$ such that $i \neq j$ implies $x_i \neq x_j$, and let $x \in \mathbb{R}$. Show that $x_i \to x$ in $(\mathbb{R}, d)$ (as $i \to \infty$) iff both (i) $x_i \to x$ in $(\mathbb{R}, d_{\text{std}})$, and (ii) $x \notin \mathbb{Z}$.

(e). Let $g : \mathbb{R} \to \mathbb{R}$, $f_0 : (0,1) \to \mathbb{R}$, $f_1 : (1,2) \to \mathbb{R}$ be continuous (in the standard metric on these intervals and $\mathbb{R}$). Let $a \in \mathbb{R}$. Define

$$f(x) = \begin{cases} g(x) & \text{if } x \leq 0 \\ f_0(x) & \text{if } 0 < x < 1 \\ a & \text{if } x = 1 \\ f_1(x) & \text{if } 1 < x < 2 \\ g(x) & \text{if } 2 \leq x. \end{cases}$$

Show that $f$ is continuous in $(\mathbb{R}, d)$.

5.
(a) Give an example of a sequence $\langle f_i \rangle_{i \in \mathbb{N}}$ in $X = C([0,1])$, which converges in $(X, d_{\int})$, but not in $(X, d_{\max})$.
(b) Show that for any sequence consisting of functions in $X = C([0,1])$, if the sequence converges in $(X, d_{\max})$ then the sequence converges in $(X, d_{\int})$.

(c) Let $A$ be the set of all $f \in X = C([0,1])$ such that $f$ is linear. Show that $A$ is not open in $(X, d_{\max})$, nor in $(X, d_{\int})$. Show that $A$ *is* closed in $(X, d_{\max})$.

6. Let $(X, d)$ be a metric space and let $A \subseteq X$ be closed. Suppose that for every $U \subseteq X$, if $U \neq \emptyset$ and $U$ is open then $U \cap A \neq \emptyset$. Prove that $A = X$.

7. Let $U$ be an open subset of a metric space $(X, d)$. Prove that $U$ is the union of a family of closed balls of $(X, d)$.

8. Suppose $(X, d)$, $(Y, e)$ are metric spaces and $f : X \to Y$, $g : X \to Y$ are continuous w.r.t. the metrics $d, e$. Suppose $d'$ and $e'$ are also metrics on $X, Y$ respectively.

(a) Suppose $d'(x, y) \geq d(x, y)$ for all $x, y \in X$ and $e'(x, y) \leq e(x, y)$ for all $x, y \in Y$. Show that $f$ is also continuous w.r.t. $d', e'$. (I.e., continuous as a function from the metric space $(X, d')$ to the metric space $(Y, e')$.)

(b) Suppose $A \subseteq X$ and for every $U \subseteq X$ which is open in $(X, d)$, $A \int U \neq \emptyset$ ($A$ is *dense*). Suppose $f(x) = g(x)$ for all $x \in A$. Prove that $f = g$.

(c) Suppose that $\langle x_i \rangle_{i<\omega}$ is a sequence in $X$ and $x \in X$ and $x_i \to x$ as $i \to \infty$ (with respect to the metric $d$). Prove that $f(x_i) \to f(x)$ as $i \to \infty$ (with respect to the metric $e$).