Midterm 1 Review Problems
(Note: all solutions, including examples, should be explained, unless indicated otherwise.)

1. (a) Give an example of a set $A \subseteq (0,1)$, such that: A is closed in $(\mathbb{R}, d_{\text{std}})$ and A is infinite, but for all $a < b \in \mathbb{R}$, $(a,b) \not\subseteq A$.

Hint: Compare with one of the first examples of closed sets we looked at in class (an example which was not a closed interval).

(b) Give an example of a metric d on $\mathbb{R}$, such that $[0,1)$ is both closed and open in $(\mathbb{R}, d)$.

Hint: First just try to make sure it's open. Use a metric which makes lots of open sets.

2. Prove that if $A \subseteq \mathbb{R}$ is open then A is the union of a family of open intervals $\langle I_j \rangle_{j \in J}$, such that for all $j \in J$, I_j has rational endpoints.

Hint: Given an open interval I and $x \in I$, the density of the rationals in $\mathbb{R}$ implies there's $p, q \in \mathbb{Q}$ such that $(p,q) \subseteq I$ and $p < x < q$.

3. Prove that in any metric space, a set is closed iff it is sequentially closed.

Hint: We went through this in class.

4. Let $X = \mathbb{R}$ and d be the function defined as follows. Given $x, y \in \mathbb{R}$, we define $d(x,y)$.

Case 1: $x, y \in \mathbb{R} \backslash \mathbb{Z}$. Then $d(x,y) = d_{\text{std}}(x,y)$.

Case 2: $x \in \mathbb{Z}$ and $y \in \mathbb{R} \backslash \mathbb{Z}$. Then $d(x,y) = 1 + d_{\text{std}}(x,y)$.

Case 3: $x \in \mathbb{R} \backslash \mathbb{Z}$ and $y \in \mathbb{Z}$. Then $d(x,y) = 1 + d_{\text{std}}(x,y)$.

Case 4: $x, y \in \mathbb{Z}$. Then $d(x,y) = 2 + d_{\text{std}}(x,y)$.

This completes all cases.

(a). Show that $(\mathbb{R}, d)$ is a metric space. (You may assume that $(\mathbb{R}, d_{\text{std}})$ is a metric space.)

Hint: No hint.

(b). Show that for every $x \in \mathbb{R}$, $\{x\}$ is open in $(\mathbb{R}, d)$ iff $x \in \mathbb{Z}$.

Hint: If $x \in \mathbb{Z}$, find $\varepsilon > 0$ such that $\{x\} = \mathcal{B}(x, \varepsilon)$.

(c). Show that if $A \subseteq \mathbb{R}$ is open in $(\mathbb{R}, d_{\text{std}})$ then A is open in $(\mathbb{R}, d)$.

Hint: Start by assuming $A = \mathcal{B}_{\text{std}}(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$ for some x, ε. Let $z \in (x - \varepsilon, x + \varepsilon)$. Find $\varepsilon' > 0$ such that $\mathcal{B}_d(z, \varepsilon') \subseteq (x - \varepsilon, x + \varepsilon)$. Conclude that $(x - \varepsilon, x + \varepsilon)$ is open in $(\mathbb{R}, d)$. Now generalize to any A open in $(\mathbb{R}, d_{\text{std}})$.

(d). Let $\langle x_i \rangle_{i \in \mathbb{N}}$ be a sequence of points $x_i \in \mathbb{R}$ such that $i \neq j$ implies $x_i \neq x_j$, and let $x \in \mathbb{R}$. Show that $x_i \to x$ in $(\mathbb{R}, d)$ (as $i \to \infty$) iff both (i) $x_i \to x$ in $(\mathbb{R}, d_{\text{std}})$, and (ii) $x \notin \mathbb{Z}$.

Hint: Think about the cases $x \in \mathbb{Z}$, $x \notin \mathbb{Z}$, separately. Use the result of (b) and the extent to which d and d_{std} agree.

(e). Let $g : \mathbb{R} \to \mathbb{R}$, $f_0 : (0,1) \to \mathbb{R}$, $f_1 : (1,2) \to \mathbb{R}$ be continuous (in the standard metric on these intervals and $\mathbb{R}$). Let $a \in \mathbb{R}$. Define

$$f(x) = \begin{cases} g(x) & \text{if } x \leq 0 \\ f_0(x) & \text{if } 0 < x < 1 \\ a & \text{if } x = 1 \\ f_1(x) & \text{if } 1 < x < 2 \\ g(x) & \text{if } 2 \leq x. \end{cases}$$

Show that f is continuous in $(\mathbb{R}, d)$.

Hint: In showing f is continuous at various points x, break into cases depending on which interval x is in: $x \leq 0$, $0 < x < 1$, $x = 1$, etc. Again use the result of (b) and the extent to which d and d_{std} agree.

5.

(a) Give an example of a sequence $\langle f_i \rangle_{i \in \mathbb{N}}$ in $X = C([0,1])$, which converges in $(X, d_{\int})$, but not in $(X, d_{\max})$.

Hint: Think about some of the examples we've looked at of sequences of functions with integrals shrinking to 0.

(b) Show that for any sequence in X, if the sequence converges in $(X, d_{\max})$ then the sequence converges in $(X, d_{\int})$.

Hint: (The X here is still $C([0,1])$.) Show that if $d_{\max}(f,g) < \delta$ then $\int_0^1 |f(x) - g(x)| dx < \delta$. Apply this fact to convergence.

6. Let (X, d) be a metric space and let $A \subseteq X$ be closed. Suppose that for every $U \subseteq X$, if $U \neq \emptyset$ and U is open then $U \cap A \neq \emptyset$. Prove that $A = X$.

Hint: This is similar to a homework problem we had: Suppose $\mathbb{Q} \subseteq A \subseteq \mathbb{R}$ and A is closed; show $A = \mathbb{R}$.

7. Let U be an open subset of a metric space (X, d). Prove that U is the union of a family of closed balls of (X, d).

Hint: Given $\varepsilon > 0$ and $x \in X$, find a closed ball $\bar{B} \subseteq \mathcal{B}(x, \varepsilon)$ with $x \in \bar{B}$.