Midterm 2

Instructions:
Section 1 (problems 1,2,3,4,5) [10 points each]
5600 students: Complete any four of problems of 1,2,3,4,5.
4500 students: Complete one of problems 1,2; and complete two problems from 3,4,5. (So three total.)
Extra problems may be completed.

Section 2 (problems 6,7,8) [20 points each]
All students: complete any two of problems 6,7,8.
(Note some parts are not required for 4500 students, but may be completed for extra credit.)

1. Let $X = \{0, 1, 2, 3, 4\}$.

(a) Give an example of a topology τ on X, such that $\{0, 1\}$ is closed in the topological space (X, τ).

(b) Let X be as above and $Y = \{5, 6, 7\}$. Let

$$\tau_X = \{\emptyset, X, \{0, 1, 2\}, \{3, 4\}\}$$

and

$$\tau_Y = \{\emptyset, Y, \{5\}, \{5, 6\}\}.$$

(Then (X, τ_X) and (Y, τ_Y) are topological spaces; you may assume this.)

Let $f : X \to Y$ be a continuous function such that $f(0) = 5$. Show that $f(2) = 5$ also. What values could $f(3)$ be?

(Here f is continuous with respect to the topologies τ_X, τ_Y.)

Solution. (a) For $\{0, 1\}$ to be closed, we need precisely that $X\backslash\{0, 1\}$ is open, i.e. $\{2, 3, 4\}$ is open. So we need $\{2, 3, 4\} \in \tau$. We also need $\emptyset, X \in \tau$ for τ to be a topology. Note that

$$\tau = \{\emptyset, X, \{2, 3, 4\}\}$$

is a topology on X, and since $\{2, 3, 4\} \in \tau$, $\{0, 1\}$ is closed w.r.t. τ, as required.

(b) Since $f(0) = 5$, we have $0 \in f^{-1}(\{5\})$. Since $\{5\} \in \tau_Y$ and f is continuous, we have $f^{-1}(\{5\}) \in \tau_X$. The only elements $U \in \tau_x$ such that $0 \in U$ are $U = \{0, 1, 2\}$ and $U = X$. So either $f^{-1}(\{5\}) = \{0, 1, 2\}$ or $f^{-1}(\{5\}) = X$. But $2 \in \{0, 1, 2\}$ and $2 \in X$, so either way, $2 \in f^{-1}(\{5\})$. So $f(2) = 5$.

$f(3)$ can be 5, 6, or 7. For consider the following functions:

$f_1 : X \to Y$, where $f_1(x) = 5$ for all x. Since f_1 is constant, f_1 is continuous, and $f_1(0) = 5$ and $f_1(3) = 5$.

$f_2 : X \to Y$, where $f_2(x) = 5$ for $x \in \{0, 1, 2\}$, and $f_2(3) = f_2(4) = 6$. Then $f_2^{-1}(\{5\}) = \{0, 1, 2\}$ and $f_2^{-1}(\{5, 6\}) = X$, both of which are in τ_X. (And $f_2^{-1}(\emptyset) = \emptyset$ and $f_2^{-1}(Y) = X$ is automatic since $f_2 : X \to Y$). So f_2 is continuous and $f_2(0) = 5$ and $f_2(3) = 6$.

$f_3 : X \to Y$, where $f_2(x) = 5$ for $x \in \{0, 1, 2\}$, and $f_3(3) = f_3(4) = 7$. Then $f_3^{-1}(\{5, \}) = \{0, 1, 2\}$ and $f_3^{-1}(\{5, 6\}) = \{0, 1, 2\}$, so like in the previous case, f_3 is continuous, and $f_3(0) = 5$ and $f_3(3) = 7$.

From what we know, f might be any of these three functions, so $f(3)$ could be 5, 6 or 7. (It can't be anything else as $Y = \{5, 6, 7\}$.) (In fact, the functions f_1, f_2, f_3 are the only continuous functions $f : X \to Y$ such that $f(0) = 5$.)

2. Let X be the set of all finite sets of non-negative integers, i.e.

$$X = \{A|\ A \subseteq \mathbb{N}\ \&\ A \text{ is finite}\}.$$

Show that X is countable.

Solution. Since a countable union of countable sets is countable, it suffices to show that

$$X = \bigcup_{n=0}^{\infty} X_n,$$

for some sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of countable sets X_n.

So note that for each $A \in X$, since A is finite, A is bounded in $\mathbb{N}$. So there is some $n \in \mathbb{N}$ such that $A \subseteq \{0, 1, \ldots, n-1\}$, i.e. $A \in \mathcal{P}(\{0, 1, \ldots, n-1\})$. So let

$$X_n = \mathcal{P}(\{0, 1, \ldots, n-1\}).$$

Certainly $X_n \subseteq X$. This combined with the above remarks show that

$$X = \bigcup_{n=0}^{\infty} X_n.$$

And X_n is countable: In fact since $\{0, 1, \ldots, n-1\}$ is finite, so is X_n, because in fact, X_n has exactly 2^n elements. (I.e., there's a bijection between X_n and the finite set $\{0, 1, 2, \ldots, 2^n - 1\}$.) Since X_n is finite, it's countable, as required.

(Addendum: Proof that X_n has exactly 2^n elements: Let F_n be the set of functions $f : \{0, 1, \ldots, n-1\} \to \{0, 1\}$. There's a natural bijection π from F_n to $\mathcal{P}(\{0, 1, \ldots, n-1\})$: given $f \in F_n$, let $\pi(f)$ be the set A such that f is its characteristic function. I.e., for $i < n$, $i \in A$ iff $f(i) = 1$. This is a bijection: it's onto since given $A \subseteq \{0, 1, \ldots, n-1\}$, just let $f_A : \{0, 1, \ldots, n-1\} \to \{0, 1\}$ be the characteristic function of A, i.e. $f(i) = 1$ iff $i \in A$ (so $f(i) = 0$ iff $i \notin A$). Then $\pi(f_A) = A$. And it's 1 to 1: given $f \neq g$, we have $f(i) \neq g(i)$ for some $i < n$, so for such i, we have $i \in \pi(f)$ iff $i \notin \pi(g)$, so $\pi(f) \neq \pi(g)$.

Therefore the number of elements in X_n is the same as the number of elements in F_n, so it suffices to show that the latter is 2^n.

By induction: F_0 has just one element (the empty function, i.e. the function with empty domain), so it has 2^0 elements. Suppose F_n has 2^n elements. Now there is a bijection σ from F_{n+1} to $\{0, 1\} \times F_n$: just let $\sigma(f) = (f(n), f \restriction \{0, 1, \ldots, n-1\})$. For σ is 1-1 as given $f \neq g$, we have $f(i) \neq g(i)$ for some $i < n$, so either $f(n) \ngeq g(n)$, or $f \restriction \{0, 1 \ldots, n-1\} \neq g \restriction \{0, 1, \ldots, n-1\}$, and therefore $\sigma(f) \neq \sigma(g)$. And σ is onto as given $(j, f') \in \{0, 1\} \times F_n$, the function $f \in F_{n+1}$, defined by $f(n) = j$ and $f(m) = f'(m)$ for $m < n$, is such that $\sigma(f) = (j, f')$.

But now by induction there are 2^n elements in F_n. Therefore there are 2^{n+1} elements in $\{0, 1\} \times F_n$, since this set is essentially 2 disjoint copies of F_n. (I.e., if $\rho : F_n \to \{0, 1, 2, \ldots, 2^n - 1\}$ is a bijection, then $\rho' : \{0, 1\} \times F_n \to \{0, 1, \ldots, 2^{n+1} - 1\}$ is a bijection, where $\rho'(j, f) = j * 2^n + \rho(f)$.) Since F_{n+1} is bijectable with $\{0, 1\} \times F_n$, F_{n+1} also has 2^{n+1}, elements. This completes the induction.)

In the true or false questions, if you're giving a counterexample, you don't need to prove that it is a counterexample, but state it precisely. If you know the name of a counterexample, you can just give its name; otherwise define it.

3. True or false: "For every uncountable closed set $D \subseteq \mathbb{R}$, there's $a < b \in \mathbb{R}$ such that $[a, b] \subseteq D$". (State "true" or "false" and either explain briefly or state a counterexample.)

Solution. False. The Cantor set is a counterexample.

4. True or false: "Let (X, τ), (Y, σ) be topological spaces and $f : X \to Y$. Suppose for every $V \subseteq Y$, we have $f^{-1}(\mathrm{Int}(V)) = \mathrm{Int}(f^{-1}(V))$. Then f is continuous."

(Note: Here $\mathrm{Int}(A)$ is the interior of the set A. The interior $\mathrm{Int}(V)$ is with respect to (Y, σ), and the interior $\mathrm{Int}(f^{-1}(V))$ is with respect to (X, τ).)

(State "true" or "false" and either explain briefly or state a counterexample.)

Solution. True. Let $V \subseteq Y$ such that $V \in \sigma$. Then $\mathrm{Int}(V) = V$ (since $\mathrm{Int}(V)$ is the largest open subset of V by definition). So by hypothesis, we have $f^{-1}(V) = f^{-1}(\mathrm{Int}(V)) = \mathrm{Int}(f^{-1}(V))$, and the latter is open since $\mathrm{Int}(U)$ is open for any U, by definition of Int. So we have shown that $f^{-1}(V) \in \tau$ for all $V \in \sigma$, i.e. f is continuous.

5. True or false: "Let (X, τ) be a topological space and $\langle x_i \rangle_{i \in \mathbb{N}}$ a sequence of points such that $x_i \in X$ for each $i \in \mathbb{N}$. Then there is at most one $x \in X$ such that $x_i \to x$ as $i \to \infty$."

(State "true" or "false" and either explain briefly or state a counterexample.)

Solution. False. E.g. if $X = \{0, 1\}$ and τ is the trivial topology on X, then every sequence in X converges to both 0 and 1, and so the constant sequence $x_n = 0$ for all $n \in \mathbb{N}$ is such that $x_n \to 0$ and $x_n \to 1$.

Also the naturals $X = \mathbb{N}$ with the cofinite topology τ_{cof}, and the sequence $x_n = n$, is such that $x_n \to k$ for every $k \in \mathbb{N}$.

Section 2. Complete any two of problems 6, 7, 8.

6. Let $\mathfrak{b}$ be the set of all left-closed, right-open intervals $\subseteq \mathbb{R}$, i.e.,

$$\mathfrak{b} = \{[a, b) \mid a, b \in \mathbb{R} \ \& \ a < b\}.$$

(a) Show that $\mathfrak{b}$ is a base, for some topology on $\mathbb{R}$. (Let τ be this topology for parts (b) and (c) below.)

(b) Let τ_{std} be the standard topology on $\mathbb{R}$. Show that $\tau \not\subseteq \tau_{\text{std}}$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the identity function $f(x) = x$. Show that f is *not* continuous as a function $(\mathbb{R}, \tau_{\text{std}}) \to (\mathbb{R}, \tau)$.

Part (d) is not required for 4500 students (but may be completed for extra credit).

(d) Compute the closure of the interval $I = (0, 1)$, i.e. $\text{Cl}(I)$, with respect to τ.

Solution.

(a) Note that $\bigcup \mathfrak{b} = \mathbb{R}$ since every $U \in \mathfrak{b}$ is of the form $[a, b) \subseteq \mathbb{R}$, and for each $x \in \mathbb{R}$, $x \in [x, x + 1)$ and $[x, x + 1) \in \mathfrak{b}$. So if $\mathfrak{b}$ is a base at all, then $\mathfrak{b}$ is a base for a topology on $X = \mathbb{R}$.

To see $\mathfrak{b}$ is a base, we use the characterization from lectures that: $\mathfrak{b}$ is a base iff for all $U, V \in \mathfrak{b}$, we have $U \cap V$ is a union of some set of elements of $\mathfrak{b}$, i.e. $U \cap V = \bigcup C$ for some $C \subseteq \mathfrak{b}$.

So let $U = [a, b) \in \mathfrak{b}$ and $V = [c, d) \in \mathfrak{b}$. Note that

$$[a, b) \cap [c, d) = [\max(a, c), \min(b, d)).$$

(For the left side is just the set of all $x \in \mathbb{R}$ such that $x \geq a$ and $x < b$ and $x \geq c$ and $x < d$, which is equivalent to saying $x \geq \max(a, c)$ and $x < \min(b, d)$.)

Moreover, the latter set is itself of the form $I = [e, f)$ (with $e = \max(a, c)$ and $f = \min(b, d)$), so $I \in \mathfrak{b}$. Therefore $U \cap V = I$ is a union of elements of $\mathfrak{b}$ (in fact $I = \bigcup C$ where $C = \{I\}$), as required.

(b) Note that $[0, 1) \in \tau$, since $[0, 1) \in \mathfrak{b} \subseteq \tau$. But $[0, 1) \notin \tau_{\text{std}}$ (since at the point $x = 0$, there's no $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq [0, 1)$). Therefore $\tau \not\subseteq \tau_{\text{std}}$.

(c) To show that f is not continuous from (X, τ_{std}) to (X, τ), we need to find a set $U \in \tau$ such that $f^{-1}(U) \notin \tau_{\text{std}}$.

Let $U \subseteq \mathbb{R}$. Then note: $f^{-1})(U) = U$. For given $x \in \mathbb{R}$, $x \in f^{-1}(U)$ iff $f(x) \in U$ iff $x \in U$, the latter since $f(x) = x$. Therefore in particular, $f^{-1}([0, 1)) = [0, 1)$. But as in (b), we have $[0, 1) \in \tau$, but $[0, 1) \notin \tau_{\text{std}}$, so $f^{-1}([0, 1)) = [0, 1) \notin \tau_{\text{std}}$. Therefore $U = [0, 1)$ is as required. So f is not continuous.

(d) $\text{Cl}((0, 1))$ is the smallest closed set with $(0, 1)$ a subset. So $(0, 1) \subseteq \text{Cl}((0, 1))$. If $(0, 1)$ itself is closed then $\text{Cl}((0, 1)) = (0, 1)$. The complement of $(0, 1)$ is $V = (-\infty, 0] \cup [1, \infty)$. But V is not open, since $0 \in V$, but there is no $W \in \mathfrak{b}$ such that $0 \in W \subseteq V$. For if $[a, b) \in \mathfrak{b}$ and $0 \in [a, b)$, then $a \leq 0 < b$, so $[a, b) \cap (0, 1) \neq \emptyset$, so $[a, b) \not\subseteq V$. So $(0, 1)$ is not closed, so $(0, 1)$ is a proper subset of $\text{Cl}((0, 1))$.

The complements of closed sets extending $(0,1)$ are open sets contained in $\mathbb{R}\backslash(0,1)$, i.e. contained in $(-\infty,0] \cup [1,\infty)$. Note that $[1,\infty) \in \tau$ since $[1,\infty) = \cup_{n\in\mathbb{N}}[1,n)$, and each $[1,n) \in \mathfrak{b}$. Also note that $(-\infty,0) \in \tau$ since $(-\infty,0) = \cup_{n\in\mathbb{N}}[-n,0)$ and each $[-n,0) \in \mathfrak{b}$. So $(-\infty,0)\cup[1,\infty) \in \tau$. Therefore its complement, $[0,1)$, is closed.

Since $[0,1)$ is closed but $(0,1)$ is not closed, and $[0,1) = (0,1) \cup \{0\}$, i.e. there's just one more point added, we have that $[0,1)$ must be the smallest closed set extending $(0,1)$. So $\mathrm{Cl}((0,1)) = [0,1)$.

Alternative: use the characterization of closure given in one of the review problems. This tells us that $\mathrm{Cl}((0,1))$ is the set of all $x \in \mathbb{R}$ such that for all $U \in \tau$ with $x \in U$, we have $U \cap (0,1) \neq \emptyset$. Certainly for all $x \in (0,1)$, if $U \in \tau$ and $x \in U$ then $x \in U \cap (0,1)$, so $U \cap (0,1) \neq \emptyset$. So $(0,1) \subseteq \mathrm{Cl}((0,1))$. If $x \geq 1$ then note that $[1,x+1)$ is an element of $\mathfrak{b}$, so an element of τ, and $[1,x+1) \cap U = \emptyset$. Therefore $x \notin \mathrm{Cl}((0,1))$. Now let $x \leq 0$. If $x < 0$, then $U = [x-1, x+|x/2|)$ is in $\mathfrak{b}$ and $U \cap (0,1) = \emptyset$, so again $x \notin \mathrm{Cl}((0,1))$. If $x = 0$, then if $U \in \tau$ and $x \in U$, then there is some $W \in \mathfrak{b}$ such that $x = 0 \in W \subseteq U$. So we have some $a, b \in \mathbb{R}$ and $x = 0 \in [a,b) \subseteq U$. But then $a \leq 0 < b$, so $[a,b) \cap (0,1) \neq \emptyset$ (since $\frac{1}{2}\min(1,b)$ is in this intersection). So $U \cap (0,1) \neq \emptyset$. This was true for an arbitrary $U \in \tau$ with $x \in U$. So $x = 0 \in \mathrm{Cl}((0,1))$. So overall, we have $(0,1) \subseteq \mathrm{Cl}((0,1))$, $0 \in \mathrm{Cl}((0,1))$, and $[1,\infty)$ avoids $\mathrm{Cl}((0,1))$, and $(-\infty,0)$ avoids $\mathrm{Cl}((0,1))$. So $\mathrm{Cl}((0,1)) = [0,1)$.

7. Let C be the Cantor set.

(a) Explain briefly why there's an irrational number in C.

(b) Let $A \subseteq \mathbb{R}$ be a countable set. Give a construction/definition of a specific number in $C\backslash A$. (Just give the construction. It should identify one specific number. You don't have to prove that it's in $C\backslash A$.)

(Note: Here by "one specific number" I mean that given the set A, and given certain choices made in the proof/construction, exactly one number is produced. *Which* number that it is will depend in general on A and the choices made.)

Solution.

(a) C is uncountable but the rationals $\mathbb{Q}$ are countable. Any subset of a countable set is countable, so C cannot be a subset of $\mathbb{Q}$. So C has an irrational element.

(b) Since A is countable, we may fix $f : \mathbb{N} \to A$ a surjection. Let C_n be the usual sets defined to produce the Cantor set, i.e. $C = \cap_{n\in\mathbb{N}} C_n$, and $C_0 = [0, 1]$, $C_1 = [0, 1/3] \cup [2/3, 1]$, and so on, with C_n a union of 2^n disjoint closed intervals each of length $1/3^n$. (So, as defined in detail in class.)

Index the subintervals of the C_n's as I did in class using finite sequences σ of 0's and 2's of length n, i.e. with $\langle\rangle$ the empty sequence, $I_{\langle\rangle} = [0, 1]$, then $I_{\langle 0\rangle} = [0, 1/3]$, $I_{\langle 1\rangle} = [2/3, 1]$, $I_{\langle 00\rangle} = [0, 1/9]$, $I_{\langle 01\rangle} = [2/9, 3/9 = 1/3]$, and so on.

Now define an infinite sequence σ of 0's and 2's as follows. We also define a nested sequence of intervals J_n with J_n of length $1/3^n$; it will be one of the 2^n subintervals of C_n. At the 0^{th} stage, the very beginning, we start with $J_0 = [0, 1]$, and none of the sequence σ defined.

We first define $\sigma(0)$ and J_1. If $f(0) \in I_{\langle 0\rangle}$ then we set $\sigma(0) = 2$; otherwise $f(0) \notin I_{\langle 0\rangle}$ and we set $\sigma(0) = 0$. Now that we have defined $\sigma(0)$ we also set $J_1 = I_{\langle\sigma(0)\rangle}$. So note that $f(0) \notin J_1$.

We are now up to the beginning of the 1^{th} stage.

From now on we maintain the following inductive hypotheses: at the beginning of stage n we have already defined integers $\sigma(0), \ldots, \sigma(n-1)$, and intervals $J_0, \ldots, J_n$, such that for each $i \neq n$, $J_i = I_{\langle\sigma(0),\ldots,\sigma(i-1)\rangle}$, (so therefore $J_{i+1} \subseteq J_i$ by definition of the I_σ's) and for each $i < n$, $f(i) \notin J_{i+1}$.

(Note that these hypotheses are indeed true at the beginning of the 1^{th} stage.) So now suppose we're at the beginning of the n^{th} stage and the inductive hypotheses hold.

We define $\sigma(n)$ as follows: if $f(n) \in I_{\langle\sigma(0),\ldots,\sigma(n-1),0\rangle}$ then set $\sigma(n) = 2$; otherwise set $\sigma(n) = 0$. Then set $J_{n+1} = I_{\langle\sigma(0),\ldots,\sigma(n)\rangle}$; note that $f(n) \notin J_{n+1}$. Note that we have attained the inductive hypotheses for the beginning of the $(n + 1)^{\text{th}}$ stage.

This completes the definition.

We now claim that the intersection $\cap_{n\in\mathbb{N}} J_n$ contains exactly one point x and that $x \in C\backslash A$. So our construction produces this particular point. (Note that which point we construct depends on not just the original set A, but also the particular surjection $f : \mathbb{N} \to A$ that we used. Given f (and therefore A), there is a uniquely determined point produced.)

(You weren't asked to prove that the construction actually works. But it does because: the closed intervals J_n are nested, i.e. $J_{n+1} \subseteq J_n$ for all n, and are bounded in $[0,1]$, so by the lemma proved in class, the intersection $\cap_{n\in\mathbb{N}}J_n \neq \emptyset$. The intersection is a subset of the Cantor set C since each $J_n \subseteq C_n$ - so if $x \in \cap_{n\in\mathbb{N}}J_n$ then $x \in J_n$ for each n, so $x \in C_n$ for each n, so $x \in C$. The intersection consists of at most a singleton since for any n, it is a subset of J_n, which has length $1/3^n$. So if x, y are both in the intersection, then for all $n \in \mathbb{N}$, $x, y \in J_n$, so for all $n \in \mathbb{N}$, $|x - y| \leq 1/3^n$, so in fact $|x - y| \leq 0$, so $x = y$. So the intersection produces a singleton $\{x\} \subseteq C$. And $x \notin A$ since if it was, then $x = f(n)$ for some $n \in \mathbb{N}$ but we ensured that $f(n) \notin J_{n+1}$, but $x \in J_{n+1}$, contradiction.)

8. Let (X, τ), (X_1, τ_1) and (X_2, τ_2) be topological spaces. Let ρ be the product topology on $X_1 \times X_2$ (given by τ_1, τ_2).

(a) Let $f_1 : X \to X_1$ be continuous (with respect to τ and τ_1) and $f_2 : X \to X_2$ continuous (with respect to τ and τ_2). Let $g : X \to X_1 \times X_2$ be the map

$$g(x) = (f_1(x), f_2(x)).$$

Show that g is continuous (with respect to τ and ρ).

Part (b) is not required for 4500 students (but may be completed for extra credit).

(b) Let $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$. Show that

$$\mathrm{Cl}(A_1 \times A_2) = \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2).$$

(Here $\mathrm{Cl}(A_1 \times A_2)$ is the closure of $A_1 \times A_2$ with respect to ρ, $\mathrm{Cl}(A_1)$ with respect to τ_1, and $\mathrm{Cl}(A_2)$ with respect to τ_2). (Partial credit for showing just one direction i.e. $\subseteq$ or $\supseteq$).

Solution. (a) Let $U \in \rho$. We must show that $g^{-1}(U) \in \tau$. First, we have that

$$U = \bigcup C = \bigcup_{V \in C} V$$

for some $C \subseteq \mathfrak{b}$, where $\mathfrak{b}$ is the conventional basis for the product topology. That is, $\mathfrak{b}$ is the set of cross products $U_1 \times U_2$ where $U_1 \in \tau_1$ and $U_2 \in \tau_2$. So C is a collection of some such cross products. Now

$$g^{-1}(U) = g^{-1}\Big(\bigcup_{U_1 \times U_2 \in C} U_1 \times U_2)\Big) = \bigcup_{U_1 \times U_2 \in C} g^{-1}(U_1 \times U_2).$$

We need to see $g^{-1}(U) \in \tau$, so it suffices to show that $g^{-1}(U_1 \times U_2) \in \tau$ for each $U_1 \times U_2 \in C$, since then the union on the right is also in τ, since unions of elements of τ produce more elements of τ.

Let's in fact show that $g^{-1}(U_1 \times U_2) \in \mathfrak{b}$ for each $U_1 \times U_2 \in \mathfrak{b}$ (this does it as $C \subseteq \mathfrak{b}$). So, we have $U_1 \in \tau_1$ and $U_2 \in \tau_2$. Now

$$g^{-1}(U_1 \times U_2) = \{x \in X \mid g(x) \in U_1 \times U_2\}$$

$$= \{x \in X \mid (f_1(x), f_2(x)) \in U_1 \times U_2\}$$

$$= \{x \in X \mid f_1(x) \in U_1 \ \&\ f_2(x) \in U_2\}$$

$$= \{x \in X \mid f_1(x) \in U_1\} \cap \{x \in X \mid f_2(x) \in U_2\}$$

$$= f_1^{-1}(U_1) \cap f_2^{-1}(U_2).$$

But f_1 is continuous from (X, τ) to (X_1, τ_1), and $U_1 \in \tau_1$, so $f_1^{-1}(U_1) \in \tau$. Similarly, f_2 is continuous and $U_2 \in \tau_2$, so $f_2^{-1}(U_2) \in \tau$. And τ is closed under finite intersections, so the above intersection is in τ. Therefore $g^{-1}(U_1 \times U_2) \in \tau$, as required.

(b) I'll do this a couple of ways. The first uses the characterization of the closure given in one of the problems on the review.

By that characterization, we have that "(i) $(x_1, x_2) \in \mathrm{Cl}(A_1 \times A_2)$" iff "(ii) for every open set U including (x_1, x_2), we have $U \cap A_1 \times A_2 \neq \emptyset$."

So let $(x_1, x_2) \in \mathrm{Cl}(A_1 \times A_2)$, so (i), and therefore (ii), holds. We want to show that $(x_1, x_2) \in \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$, i.e. that $x_1 \in \mathrm{Cl}(A_1)$ and $x_2 \in \mathrm{Cl}(A_2)$. We use the characterization again, so we need to show that for all $U_1 \in \tau_1$, if $x_1 \in U_1$ then $U_1 \cap A_1 \neq \emptyset$, and likewise with "1" replaced with "2". Let's just show the "1" case; the other will then hold by symmetry. So let $U_1 \in \tau_1$ such that $x_1 \in U_1$. Then $U_1 \times X_2 \in \rho$, since ρ is the product topology. And $(x_1, x_2) \in U_1 \times X_2$. Therefore by (ii), $U_1 \times X_2 \cap A_1 \times A_2 \neq \emptyset$. So let (a_1, a_2) be in this non-empty intersection. Since $(a_1, a_2) \in U_1 \times X_2$, we have $a_1 \in U_1$, and likewise, $a_1 \in A_1$. Therefore $a_1 \in U_1 \cap A_1$, so $U_1 \cap A_1 \neq \emptyset$, as required. This proves that $x_1 \in \mathrm{Cl}(A_1)$. As mentioned, the fact that $x_2 \in \mathrm{Cl}(A_2)$ follows by symmetry.

Now let $(x_1, x_2) \in \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$, i.e. $x_1 \in \mathrm{Cl}(A_1)$ and $x_2 \in \mathrm{Cl}(A_2)$. We want to see that $(x_1, x_2) \in \mathrm{Cl}(A_1 \times A_2)$. For this we verify (ii). So let U be an open set including (x_1, x_2); we must show $U \cap A_1 \times A_2 \neq \emptyset$. Since we're using the product topology there is $U_1 \in \tau_1$ and $U_2 \in \tau_2$ such that $(x_1, x_2) \in U_1 \times U_2 \subseteq U$. So it suffices to show that $U_1 \times U_2 \cap A_1 \times A_2 \neq \emptyset$. But since $(x_1, x_2) \in U_1 \times U_2$, we have $x_1 \in U_1$ and $x_2 \in U_2$. Since $x_1 \in \mathrm{Cl}(A_1)$ and $x_1 \in U_1 \in \tau_1$, we have that $U_1 \cap A_1 \neq \emptyset$ (by the characterization of closure again). So let $a_1 \in U_1 \cap A_1$. Similarly, $U_2 \cap A_2 \neq \emptyset$; let a_2 be in this non-empty set. Then $(a_1, a_2) \in U_1 \times U_2 \cap (A_1 \times A_2)$, so the latter set is non-empty, as required.

This completes the proof.

Second method:

First: we show $\mathrm{Cl}(A_1 \times A_2) \subseteq \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$.
This follows from the facts that (i) $A_1 \times A_2 \subseteq \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$ and (ii) $\mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$ is closed in the product topology and (iii) $\mathrm{Cl}(A_1 \times A_2)$ is the smallest closed set (closed w.r.t. ρ) which has $A_1 \times A_2$ a subset.

For (i) this is because $A_1 \subseteq \mathrm{Cl}(A_1)$ and $A_2 \subseteq \mathrm{Cl}(A_2)$, so if $(x, y) \in A_1 \times A_2$ then $x \in A_1$ and $y \in A_2$, so $x \in \mathrm{Cl}(A_1)$ and $y \in \mathrm{Cl}(A_2)$, so $(x, y) \in \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$.

For (ii): Let's verify that the product of two closed sets is closed. For note that for any $D_1 \subseteq X_1$ and $D_2 \subseteq X_2$ (not necessarily closed), if $(x_1, x_2) \in X_1 \times X_2$ then:

$$\begin{aligned} &(x_1, x_2) \in X_1 \times X_2 \backslash (D_1 \times D_2) \\ \iff\ & (x_1, x_2) \notin D_1 \times D_2 \\ \iff\ & x_1 \notin D_1 \text{ or } x_2 \notin D_2 \\ \iff\ & x_1 \in X_1 \backslash D_1 \text{ or } x_2 \in X_2 \backslash D_2 \\ \iff\ & (x_1, x_2) \in (X_1 \backslash D_1) \times X_2 \text{ or } (x_1, x_2) \in X_1 \times (X_2 \backslash D_2). \end{aligned}$$

Therefore

$$X_1 \times X_2 \backslash (D_1 \times D_2) = ((X_1 \backslash D_1) \times X_2) \cup (X_1 \times (X_2 \backslash D_2)).$$

Now assume that D_1, D_2 are closed (w.r.t. τ_1, τ_2 respectively). Then $X_1 \backslash D_1$ is in τ_1, so $(X_1 \backslash D_1) \times X_2$ is in ρ (the product topology), and likewise $X_1 \times (X_2 \backslash D_2)$

is in ρ, so the union of these two sets is also in ρ. So the above calculation shows that $X_1 \times X_2 \backslash (D_1 \times D_2)$ is therefore open, so $D_1 \times D_2$ is closed (still assuming D_1, D_2 are each closed).

Applying this to $D_1 = \mathrm{Cl}(A_1)$ and $D_2 = \mathrm{Cl}(A_2)$, we see that $\mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$ is closed w.r.t. ρ.

And (iii) is a standard fact.

Second: $\mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2) \subseteq \mathrm{Cl}(A_1 \times A_2)$.
I'll instead prove the contrapositive, i.e. that the complement of $\mathrm{Cl}(A_1 \times A_2)$ is a subset of the complement of $\mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$.
Proof of this: Let $(x_1, x_2) \in X_1 \times X_2$ such that $(x_1, x_2) \notin \mathrm{Cl}(A_1 \times A_2)$. Since the latter closure is closed, there's an open set $U \in \rho$ such that $(x_1, x_2) \in U$ and $U \cap \mathrm{Cl}(A_1 \times A_2) = \emptyset$. Therefore $U \cap A_1 \times A_2 = \emptyset$. Let $U_1 \in \tau_1$ and $U_2 \in \tau_2$ be such that $(x_1, x_2) \in U_1 \times U_2 \subseteq U$.

Claim: Either $U_1 \cap A_1 = \emptyset$ or $U_2 \cap A_2 = \emptyset$.
For otherwise, let $a_1 \in U_1 \cap A_1$ and $a_2 \in U_2 \cap A_2$. Then $(a_1, a_2) \in U_1 \times U_2$ and $(a_2, a_2) \in A_1 \times A_2$, so the intersection of these two sets is non-empty, contradicting that $U \cap A_1 \times A_2 = \emptyset$ and $U_1 \times U_2 \subseteq U$.

Now if $U_1 \cap A_1 = \emptyset$ then $x_1 \notin \mathrm{Cl}(A_1)$, since $x_1 \in U_1$ and $\mathrm{Cl}(A_1) \subseteq X_1 \backslash U_1$, since $X_1 \backslash U_1$ is closed and contains A_1 as a subset. Therefore $(x_1, x_2) \notin \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$.

Similarly if $U_2 \cap A_2 = \emptyset$ then $x_2 \notin \mathrm{Cl}(A_2)$, so $(x_1, x_2) \notin \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$.

So in either case we have $(x_1, x_2) \notin \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_2)$, as required.