Exact Hausdorff dimension in random recursive constructions

(iteration/fractals/probability)

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ABSTRACT **The exact Hausdorff dimension function is determined for sets in $\mathbb{R}^m$ constructed by using a recursion that is governed by some given law of randomness.**

We present a method of determining the exact Hausdorff dimension function for a wide class of random recursive constructions. Let us recall the setting. Fix the compact subset J of $\mathbb{R}^m$ with $J = \mathrm{cl}(\mathrm{int}(J))$ and a positive integer n. An n-ary random recursion modeled on J is a probability space (Ω, Σ, P) and a family of random subsets of $\mathbb{R}^m$;

$$\mathbf{J} = \left\{ J_\sigma | \sigma \in \{1, \ldots, n\}^* = \bigcup_{\nu=0}^{\infty} \{1, \ldots, n\}^\nu \right\}$$

satisfying three properties.

(*i*) $J_\phi(\omega) = J$ for almost all $\omega \in \Omega$. For every $\sigma \in \{1, \ldots, n\}^*$ and, for almost all ω, if $J_\sigma(\omega)$ is nonempty, then $J_\sigma(\omega)$ is geometrically similar to J and the map $\omega \to J_\sigma(\omega)$ is measurable with respect to the Hausdorff metric on the space of compact subsets of $\mathbb{R}^m$.

(*ii*) For almost every $\omega \in \Omega$ and for every $\sigma \in \{1, \ldots, n\}^*$, $J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*n}$ are nonoverlapping subsets of $J_\sigma(\omega)$.

(*iii*) The random vectors $\tau_\sigma = (T_{\sigma*1}, \ldots, T_{\sigma*n})$, $\sigma \in \{1, \ldots, n\}^*$ are independent and identically distributed, where $T_{\sigma*i}(\omega)$ equals the ratio of the diameter of $J_{\sigma*i}(\omega)$ to the diameter of $J_\sigma(\omega)$ [for convenience $T_\phi(\omega) = \mathrm{diam}(J)$].

The system $\mathbf{J}$ is called an n-ary random recursive construction. We now define the random set K by

$$K(\omega) = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in \{1, \ldots, n\}^k} J_\sigma(\omega).$$

Our interest centers on the Hausdorff dimension function of K. We will consider those constructions such that $E(T_1^0 + \ldots + T_n^0) > 1$. These are really the only interesting constructions as is well known from the theory of branching processes (for discussion, see ref. 1). The Hausdorff dimension was determined by Mauldin and Williams.

THEOREM 1. *Assume* $\mathrm{E}(\mathrm{T}_1^0 + \ldots + \mathrm{T}_n^0) > 1$. *Let α be the unique number such that*

$$\mathrm{E}(\mathrm{T}_1^\alpha + \ldots + \mathrm{T}_n^\alpha) = 1.$$

Then $\mathrm{P}(\mathrm{K} \neq \phi) > 0$ *and for* P-*a.e.* ω, $\mathcal{H}^\alpha(\mathrm{K}(\omega)) < \infty$. *Moreover, if* $\mathrm{K}(\omega) \neq \phi$, *then the Hausdorff dimension of* $\mathrm{K}(\omega)$ *is* α.

The fact that $\mathcal{H}^\alpha(K(\omega)) < \infty$, for almost all ω, was proved by considering the natural estimates

$$S_{k,\alpha} = \sum_{\sigma \in \{1, \ldots, n\}^k} (\mathrm{diam}(J_\sigma))^\alpha$$

and noticing that $\{S_{n,\alpha}\}_{n=1}^\infty$ forms a martingale. Therefore, there is a finite random variable X such that for almost all ω

$$X(\omega) = \lim_{k \to \infty} S_{k,\alpha}(\omega).$$

Since, for almost all ω,

$$\lim_{k \to \infty} \sup_{\sigma \in \{1, \ldots, n\}^k} \mathrm{diam}(J_\sigma) = 0,$$

it follows that $\mathcal{H}^\alpha(K) \leq X(\omega) < +\infty$. The proof that α is the dimension of $K(\omega)$ was completed by constructing an appropriate family of random measures.

The formula determining α generalizes the formula given by Moran (2) in the deterministic case and, in that case, reduces to his formula. Moran also showed, in the deterministic case, that $\mathcal{H}^\alpha(K) > 0$, where $\mathcal{H}^\alpha$ is Hausdorff's measure in dimension α. Graf (3) showed that this remains true for the random case, provided $P(\Sigma_{i=1}^n T_i^\alpha = 1) = 1$ and there is some δ such that $0 < \delta < \min_{1 \leq i \leq n} T_i$ P-a.s. Actually, for most constructions $P(\Sigma_{i=1}^n T_i^\alpha \neq 1) > 0$. Graf showed $H^\alpha(K) = 0$ a.s. under this assumption. S. J. Taylor conjectured the general form of the exact dimension in a letter.

Our main theorems give upper and lower estimates on the exact Hausdorff dimension function associated with constructions such that $P(\Sigma_{i=1}^n T_i^\alpha \neq 1) > 0$. These estimates depend on (*i*) r_β, the radius of convergence of the moment generating function of X^β, (*ii*) the probabilistic behavior of the construction, and (*iii*) the geometric properties of the modeling set, J.

UPPER-BOUND THEOREM. *Assume that* $\mathrm{E}(\mathrm{T}_1^0 + \ldots + \mathrm{T}_n^0) > 1$. *Suppose $\beta > 0$ is such that* $\mathrm{r}_\beta < \infty$. *Let* $\mathrm{h}_\beta(\mathrm{t}) = \mathrm{t}^\alpha(\log(\log(1/\mathrm{t})))^{1/\beta}$. *Then there is a constant* c *such that*

$$\mathcal{H}^{h_\beta}(\mathrm{K}(\omega)) \leq \mathrm{c}X(\omega) < +\infty$$

for P-*a.e.* ω.

Thus, an analysis of r_β is essential for obtaining upper bounds on the exact dimension function.

RADIUS THEOREM. *Assume that* $\mathrm{E}(\mathrm{T}_1^0 + \ldots + \mathrm{T}_n^0) > 1$ *and* $\mathrm{P}(\Sigma_{i=1}^n \mathrm{T}_i^\alpha \neq 1) > 0$. *Set*

$$\beta_0 = sup\left\{\beta > 1 \left| \sum_{i=1}^n \mathrm{T}_i^{\alpha/(1-(1/\beta))} \leq 1, \text{ P-}a.s.\right.\right\}.$$

Then, $1 < \beta_0 < +\infty$ *and*

$$\mathrm{r}_\beta = \begin{cases} +\infty, & 0 \leq \beta < \beta_0 \\ > 0, & \beta = \beta_0 \\ 0, & \beta_0 < \beta \end{cases}.$$

Since $\Sigma_{i=1}^n \mathrm{T}_i^m \leq 1$, P-*a.s.* (1), *it follows that*

$$\beta_0 \geq 1/(1 - (\alpha/m)).$$

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Although it is not always true that $r_{\beta_0} < \infty$, there are several general conditions that ensure that $\beta_0 = 1/(1 - (\alpha/m))$ and $r_{\beta_0} < \infty$.

The harder problem of obtaining good lower bounds on the dimension function leads us to a combination of restrictions on the construction. However, it is still true that almost all commonly occurring constructions satisfy these restrictions. A geometric condition is given in the following definition.

Definition. A compact subset J of $\mathbb{R}^m$ has the neighborhood boundedness property means that there is a positive integer n_0 such that, if $\mathrm{diam}(J) < \varepsilon$ and $J_1, \ldots, J_k$ are nonoverlapping sets, each similar to J with $\mathrm{dist}(J, J_i) < \varepsilon \le \mathrm{diam}(J_i)$, $i = 1, \ldots, k$, then $k \le n_0$.

For example it can be shown that, if J is the union of finitely many nonempty convex sets and $J = \mathrm{cl}(\mathrm{int}(J))$, then J has this property.

LOWER-BOUND THEOREM. *Assume that* $\mathrm{E}(\mathrm{T}_1^0 + \ldots + \mathrm{T}_n^0) > 1$. *Suppose* $\mathrm{J} = cl(int(\mathrm{J}))$ *has the neighborhood boundedness property and there is some* $\xi > 0$ *such that* $\mathrm{E}[1/min\{\mathrm{T}_i^\xi | \mathrm{T}_i > 0\}] < \infty$. *For* P-*a.e.* ω, *if* $\mathrm{K}(\omega) \ne \phi$, *then*

$$\mathcal{H}^h(\mathrm{K}(\omega)) > 0,$$

where

$$\mathrm{h(t)} = \mathrm{t}^\alpha(log(log(1/\mathrm{t})))^{1-(\alpha/m)}.$$

Our main theorem shows how to calculate the exact Hausdorff dimension function in the general case. More specifically, we have the following.

EXACT-DIMENSION THEOREM. *Assume that* $\mathrm{E}(\mathrm{T}_1^0 + \ldots + \mathrm{T}_n^0) > 1$. *Let* $\mathrm{J} = cl(int(\mathrm{J})) \ne \phi$ *have the neighborhood boundedness property. Let* $\alpha > 0$ *be defined by* $\mathrm{E}(\Sigma_{i=1}^n \mathrm{T}_i^\alpha) = 1$. *Suppose that* $\mathrm{P}(\Sigma_{i=1}^n \mathrm{T}_i^\alpha \ne 1) > 0$ *and that* $\mathrm{E}[1/min\{\mathrm{T}_i^\xi | \mathrm{T}_i > 0\}] < \infty$, *for some* $\xi > 0$. *Suppose, in addition, that one of the following three conditions is satisfied:*

(i) *The distribution of* $(\mathrm{T}_1^m, \ldots, \mathrm{T}_n^m)$ *has a derivative* $\mathrm{f} \ge 0$ *with respect to Lebesgue measure on* $\{\vec{y} \in [0, 1]^n | \Sigma_{i=1}^n \mathrm{y}_i < 1\}$ *and there is a point* $\vec{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_n) \in [0, 1]^n$ *with* $\mathrm{x}_i > 0$ *for* $\mathrm{i} = 1, \ldots, \mathrm{n}$ *and* $\Sigma_{i=1}^n \mathrm{x}_i = 1$, *a neighborhood* U *of* $\vec{x}$, *and a constant* $\mathrm{c} > 0$ *such that* $\mathrm{f}(\vec{y}) \ge \mathrm{c}$ *for all* $\vec{y} = (\mathrm{y}_1, \ldots, \mathrm{y}_n) \in \mathrm{U} \cap [0, 1]^n$ *with* $\Sigma_{i=1}^n \mathrm{y}_i < 1$ *and* $\mathrm{y}_i \le \mathrm{x}_i$ *for* $\mathrm{i} = 1, \ldots, \mathrm{n}$.

(ii) *There exists* $\delta \in (0, 1)$ *and* $\mathrm{q} \in \mathbb{N}$, *such that, for large* t,

$$\mathrm{E}\left[\left(\sum_{i=1}^n \mathrm{T}_i^m\right)^t \prod_{i=1}^n 1_{\{\mathrm{T}_i^m \le 1-\delta\}}\right] \ge 1/\mathrm{t}^q.$$

(iii) *There exists some* $\mathrm{a} \in (1/\mathrm{n}, 1) \setminus \{1/\nu | \nu = 1, \ldots, \mathrm{n} - 1\}$ *such that*

$$\prod_{\nu=0}^\infty \left(\mathrm{E}\left[\left(\sum_{\rho=1}^n \mathrm{T}_\rho^m\right)^{\mathrm{a}^{-\nu}} \prod_{i=1}^n 1_{\{\mathrm{T}_i^m/(\Sigma_{\rho=1}^n \mathrm{T}_\rho^m) \le \mathrm{a}\}}\right]\right)^{\mathrm{a}^\nu} > 0.$$

Then, for P-*a.e.* ω, *if* $\mathrm{K}(\omega)$ *is nonempty, then*

$$0 < \mathcal{H}^h(\mathrm{K}(\omega)) < \infty,$$

where

$$\mathrm{h(t)} = \mathrm{t}^\alpha(log|log\mathrm{t}|)^{1-(\alpha/m)}.$$

Remarks. The assumption that $E[1/\min\{T_i^\xi | T_i > 0\}] < \infty$ is a probabilistic condition needed to obtain the lower bound. The assumption that $P(\Sigma_{i=1}^n T_i^\alpha \ne 1) > 0$ ensures that $\beta < \infty$. Any one of conditions *i*, *ii*, and *iii* of the *Exact-Dimension Theorem* ensure that the radius of convergence of the moment generating function of X^β is positive and finite where $1/\beta = 1 - (\alpha/m)$. We apply our results to two examples.

Example 1 (The zero-set of the Brownian bridge). Let $(B_t)_{t \ge 0}$ be the one-dimensional Brownian motion on $[0, 1]$. Let $B_t^0 = B_t - tB_1$. Then $(B_t^0)_{0 \le t \le 1}$ is called the Brownian bridge. Define $\tau_1 = \sup\{t \le 1/2 | B_t^0 = 0\}$ and $\tau_2 = \inf\{t \ge 1/2 | B_t^0 = 0\}$. Set $J_\phi = [0, 1]$, $J_1 = [0, \tau_1]$, and $J_2 = [\tau_2, 1]$. Continue this process by rescaling to each of the intervals already obtained. Because of the scaling and invariance properties of the Brownian bridge, the random set K obtained by this recursive construction is the zero set of the Brownian bridge. Our results can be used to determine the exact Hausdorff dimension of this zero set. Thereby we reprove a result due to Taylor and Wendel (4). Note that we have $T_1 = \tau_1$ and $T_2 = 1 - \tau_2$. We calculate the density function of the distribution of (T_1, T_2) with respect to two-dimensional Lebesgue measure. Since the distribution of $tB_{1/t}$ is the same as that of B_t, we deduce that τ_2 has the same distribution as $\inf\{t \ge 1/2 | tB_{1/t} - tB_1 = 0\} = \inf\{t \ge 1/2 | B_{1/t} - B_1 = 0\} = 1/\sup\{s \in [1, 2] | B_s - B_1 = 0\}$. Since $B_s - B_1$ has the same distribution as B_{s-1}, we obtain that τ_2 has the same distribution as $1/\sup\{s \in [1, 2] | B_{s-1} = 0\}$, which, in turn, is distributed as $1/(1 + \sup\{s \in [0, 1] | B_s = 0\})$. Similarly, we obtain that τ_1 is distributed as $1/(1 + \inf\{s \in [1, \infty) | B_s = 0\})$. Following Chung (5), we define $\gamma(t) = \sup\{s | s \le t, B_s = 0\}$ and $\beta(t) = \inf\{s | s \ge t, B_s = 0\}$. Then T_1 is distributed as $1/(1 + \beta(1))$ while T_2 is distributed as $\gamma(1)/(1 + \gamma(1))$. For f: $[0, 1]^2 \to \mathbb{R}_+$ measurable we, therefore, obtain

$$E(f(T_1, T_2)) = E[f(1/(1 + \beta(1)), 1/(1 + \gamma(1))].$$

Following Chung (5), we obtain

$$E(f(T_1, T_2)) = (1/2\pi) \int_0^1 \int_1^\infty f(1/(1 + u), s/(1 + s))(1/(s(u - s))^{1/2})du\, ds.$$

By a change of variables, we find that

$$E(f(T_1, T_2)) = (1/2\pi) \int_0^{1/2} \int_0^{1/2} f(v, t)(1/(vt(1 - v - t)^3)^{1/2})dv\, dt.$$

Thus the distribution of (T_1, T_2) has the density function

$$\rho(v, t) = (1/2\pi)1_{[0,1/2]\times[0,1/2]}(v, t)(1/(vt(1 - v - t)^3)^{1/2}).$$

To determine the main dimension α, we solve

$$1 = E(T_1^\alpha + T_2^\alpha) = (1/2\pi) \int_0^{1/2} \int_0^{1/2} (v^\alpha + t^\alpha)(1/(vt(1 - v - t)^3)^{1/2})dv\, dt.$$

It can be checked that $\alpha = 1/2$ is the solution. We obviously have $P(T_1^\alpha + T_2^\alpha \ne 1) > 0$. For $\xi \in (0, 1/2)$,

$$E(1/\min(T_1^\xi, T_2^\xi)) = (1/2\pi) \int_0^{1/2} \int_0^{1/2} (1/v^\xi \wedge t^\xi)(1/(vt(1 - v - t)^3)^{1/2})dv\, dt < \infty.$$

Since $m = 1$ and $\lim_{(v,t)\to(3/2,1/2)} \rho(v, t) = \infty$ condition i is satisfied at $(x_1, x_2) = (1/2, 1/2)$. Thus it follows from the *Exact-Dimension Theorem* that

$$0 < \mathcal{H}^h(K(\omega)) < \infty \text{ for a.e. } \omega,$$

where $h(t) = t^{\alpha}(\log|\log t|)^{1-(\alpha/m)} = t^{1/2}(\log|\log t|)^{1/2}$.

Example 2 (Mandelbrot's percolation process). In 1974, Mandelbrot introduced a process in $[0, 1]^2$ that he called "canonical curdling" (for discussion, see ref. 6). Fix a positive integer n and a positive number $p < 1$. Partition the unit square into n^2 congruent subsquares: $B_{i,j} = [(i-1)/n, i/n] \times [(j-1)/n, j/n]$, $1 \leq i, j \leq n$.

Each subsquare $B_{i,j}$ "survives" independent of the others with probability p. For each subsquare that survives, rescale and apply the same procedure. This is an n-ary random construction. Clearly, $E(T_1^0 + \ldots + T_{n^2}^0) = n^2 p$ so $K(\omega)$ is nonempty with positive probability if and only if $p > 1/n^2$. Chayes *et al.* (7) have investigated this model in detail and in particular demonstrate that there is a critical probability $p_c < 1$ such that, if $p \geq p_c$, then opposing sides of the unit square are connected with positive probability. Since this process is a random construction, we know that if $K(\omega) \neq \phi$, then the Hausdorff dimension is the solution of $1 = E(T_1^{\alpha} + \ldots + T_{n^2}^{\alpha}) = pn^{2-\alpha}$. Thus, $\alpha = 2 + \log p/\log n$. Chayes *et al.* also calculate this number.

We now show that this construction is covered by our main theorem. First of all, it is trivial that $P(\Sigma_{i=1}^{n^2} T_i^{\alpha} \neq 1) > 0$. Second, for all $\xi > 0$,

$$1/\min\{T_i^{\xi} | T_i > 0\} \leq n^{\xi}.$$

So, for all $\xi > 0$,

$$E[1/\min\{T_i^{\xi} | T_i > 0\}] \leq n^{\xi}.$$

Finally, condition *ii* is satisfied. To see this set $\delta = 1 - (1/n^2)$. Then

$$E\left[\left(\sum_{i=1}^{n^2} T_i^2\right)^t \prod_{i=1}^{n^2} 1_{\{T_i^2 \leq 1-\delta\}}\right] \geq p^{n^2} \geq 1/t^q,$$

for large t and any given q. Thus, if $K(\omega) \neq \phi$, then $0 < \mathcal{H}^h(K(\omega)) < \infty$, where

$$h(t) = t^{\alpha}(\log|\log t|)^{1-(\alpha/2)}.$$

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