AD, Large Cardinals, and Partition Properties

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We work in the base theory ZF $+$ DC. Most of the the time we will also assume AD.

General Question: Describe the exact large cardinal properties of all of the cardinals $\kappa < \Theta$ assuming AD.

The large cardinal properties must have versions that can be stated in a choice-less context.

Some examples: supercompact, measurable, Rowbottom, Jonsson.

Strength can also be measured in terms of *partition properties*.

General Question: Describe the exact partition properties of all the cardinals below Θ assuming AD.

Large Cardinal Definitions

Definition

κ is *measurable* if there is a κ-complete ultrafilter on κ.

From AD every ultrafilter on a set is countably additive, that is, is a *measure*.

If κ is measurable, then κ has a normal measure. In general, κ may have many normal measures. We say a measure μ on κ is semi-normal if $\mu(C) = 1$ for every c.u.b. set $C \subseteq \kappa$.

If κ is measurable, then κ is regular. However, κ need not be large.

Solovay showed ω_1 is measurable.

Moschovakis, Martin, Kunen showed all the δ_n^1 are regular and then measurable.

There are other regular cardinals below the projective ordinals, and these were shown to be measurable as well.

Using techniques of inner-model theory Steel showed:

Theorem (Steel)

($\mathrm{AD} + V = L(\mathbb{R})$) Every regular $\kappa < \Theta$ is measurable.

The proof also showed $\mathrm{HOD}(x)$ satisfies the GCH below Θ.

So, any wellordered collection of subsets of $\kappa < \Theta$ has size $< \kappa^+$.

Supercompactness

Definition

κ is λ-supercompact if there is a fine, normal measure on $\mathcal{P}_\kappa(\lambda)$.

Solovay showed that ω_1 is $< \Theta$ supercompact from $\mathrm{AD}_{\mathbb{R}}$. This was shown from AD (for $\leq \delta^2_1$) by Harrington Kechris., and Woodin for $< \Theta$.

Becker showed that ω_2 is δ^2_1-supercompact from AD.

We showed all the δ^1_n are δ^2_1-supercompact.

We generalized this too show:

Theorem

All the regular κ which are Suslin or the successor of a Suslin are δ^2_1- supercompact.

Question

How supercompact are the other regular cardinals?

In particular, how supercompact are the other regular cardinals below the projective ordinals?

Many questions about supercompactness measures remain.

A theorem of Woodin says that the supercompactness measures on $\mathcal{P}_{\omega_1}(\lambda)$ are unique. This does not hold for $\kappa > \omega_1$.

Question

How many supercompactness measures are there on $\mathcal{P}_\kappa(\lambda)$?

For example, there are at least two distinct supercompactness measures on $\mathcal{P}_{\delta^1_3}(\delta^1_4)$.

Partition Properties

Recall the Erdős-Rado Partition notation.

Definition

$\kappa \to (\kappa)^\lambda$ if for every partition $\mathcal{P} \colon (\kappa)^\lambda \to \{0, 1\}$ of the increasing functions from λ to κ into two pieces, there is a homogeneous $H \subseteq \kappa$ of size κ.

We say κ has the strong partition property if $\kappa \to (\kappa)^\kappa$, and say κ has the weak partition property if $\kappa \to (\kappa)^\lambda$ for all $\lambda < \kappa$.

In ZFC no infinite exponent partition relations hold. So, ZFC large cardinals properties formulated using partition properties (e.g., Ramsey, Rowbottom, Jonsson) use exponent $< \omega$.

We usually use a c.u.b. reformulation of the partition property. This involves the notion of type of a function.

Definition

We say $f: \alpha \to \text{On}$ has uniform cofinality ω if there is an $f': \alpha \times \omega \to \text{On}$ which is increasing in the second argument and $f(\beta) = \sup_n f'(\beta, n)$ for all $\beta < \alpha$.

Definition

$f: \alpha \to \mathrm{On}$ is of the correct type if f is increasing, everywhere discontinuous, and of uniform cofinality ω.

We can likewise define uniform cofinality ω_1, ω_2, etc.

More generally, for any $g: \alpha \to \mathrm{On}$ we can define f having uniform cofinality g: there is an

$$f' : \{(\beta, \gamma) : \beta < \alpha, \gamma < g(\beta)\}$$

with $f(\beta) = \sup_\gamma f'(\beta, \gamma)$.

We use frequently the “almost everywhere” versions of these notions with respect to some measure μ on $\mathrm{dom}(f)$.

Example

For functions $f\colon \omega_1 \to \omega_1$ there are two uniform cofinalities almost everywhere with respect to the normal measure on ω_1. Namely, $f(\alpha)$ has uniform cofinality ω and $f(\alpha)$ has uniform cofinality α.

For functions $f\colon \lambda \to \kappa$ of type g, let $\lambda' = \text{o.t.}(\text{dom}(g))$.

The partition relation $\kappa \to (\kappa)^{\lambda'}$ induces a partition relation $\kappa \underset{g}{\longrightarrow} (\kappa)^{\lambda}$ of functions $f\colon \lambda \to \kappa$ of type g with c.u.b. homogeneous sets (increasing, discontinuous, and of uniform cofinality g).

The simplest type is:

Definition

f has the correct type if f is increasing, discontinuous, and of uniform cofinality ω.

Example

We have the strong partition relation on ω_1 for functions of the correct type, and also for functions of type $g(\alpha) = \alpha$.

The first induces the ω-cofinal normal measure on ω_2, and the second the ω_1-cofinal normal measure on ω_2.

An easy argument shows that

$$\kappa \to (\kappa)^{\lambda'} \Rightarrow \kappa \underset{g}{\longrightarrow} (\kappa)^{\lambda}$$

We say $\kappa \underset{\text{c.u.b.}}{\longrightarrow} (\kappa)^{\lambda}$ if the c.u.b. version of the partition property holds for f of the correct type.

On the other hand, $\kappa \underset{\text{c.u.b.}}{\longrightarrow} (\kappa)^{\lambda}$ implies $\kappa \to (\kappa)^{\lambda}$.

So, for $\lambda = \omega \cdot \lambda$, the usual version $\kappa \to (\kappa)^{\lambda}$ is equivalent to the c.u.b. version $\kappa \underset{\text{c.u.b.}}{\longrightarrow} (\kappa)^{\lambda}$.

We officially adopt the c.u.b. versions of the partition properties.

- $\kappa \to (\kappa)^2$ implies (using c.u.b. version) that κ is measurable.
- In fact, the ω-c.u.b. filter is a normal measure.
- This implies $\kappa \to (\kappa)^{<\omega}$.
- So, κ is measurable iff $\kappa \to (\kappa)^2$ iff $\kappa \to (\kappa)^{<\omega}$.

So, assuming $V = L(\mathbb{R})$, this holds for all regular cardinals $\kappa < \Theta$.

Jonsson and Rowbottom Cardinals

Definition

κ is Jonsson if for every $f : \kappa^{<\omega} \to \kappa$ there is an $A \subseteq \kappa$, $|A| = \kappa$ such that $f''(A^{<\omega}) \neq \kappa$.

Equivalent to the non-existence of a Jonsson algebra.

Definition

κ is Rowbottom if for every $f : \kappa^{<\omega} \to \delta < \kappa$, there is an $A \subseteq \kappa$, $|A| = \kappa$, with $f''(A^{<\omega})$ countable.

ZF facts.

- Measurable $\Rightarrow$ Ramsey $\Rightarrow$ Rowbottom $\Rightarrow$ Jonsson.
- A singular Rowbottom cardinal must have cofinality ω.
- Jonsson cardinals imply $0^\#$ exists.

Some ZFC facts.

- Jonsson and Rowbottom cardinals are equiconsistent.
- It is consistent that every Jonsson cardinal is Ramsey.
- (Tryba, Woodin) If κ is a regular Jonsson cardinal then every stationary $S \subseteq \kappa$ reflects. In particular, successors of regulars are not Jonsson.

Using the cardinal structure given by AD we showed.

Theorem (J, Löwe)

Assume AD*. then every cardinal below* $\aleph_{\omega_1}$ *is Jonsson.*

Woodin then showed using inner-model theory:

Theorem (Woodin)

Assume AD $+ V = L(\mathbb{R})$ *(and so forth). Then every* $\kappa < \Theta$ *is Jonsson.*

We also believe we can show:

Theorem (J, Löwe)

Assume AD*. Every cardinal below* $\aleph_{\omega_1}$ *of cofinality* ω *is Rowbottom.*

Question

Assume AD. Is every cardinal $\kappa < \Theta$ of cofinality ω Rowbottom?

Question

What partition properties do the various cardinals below $\aleph_{\omega_1}$ have assuming AD? What about general cardinals below Θ?

Review of cardinal structure

We assume AD henceforth.

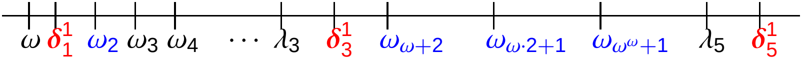
Definition

$\boldsymbol{\delta}^1_n$ is the supremum of the lengths of the $\boldsymbol{\Delta}^1_n$ prewellorderings of the reals.

- $\boldsymbol{\delta}^1_1 = \omega_1$, $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, $\boldsymbol{\delta}^1_5 = \omega_{\omega^{\omega^\omega}+1}$, $\boldsymbol{\delta}^1_7 = \omega_{\omega(5)+1}$, etc. $(\omega(n+1) = \omega^{\omega(n)})$.
- $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$.
- $\boldsymbol{\delta}^1_{2n+1} = (\lambda_{2n+1})^+$ where $\mathrm{cof}(\lambda_{2n+1}) = \omega$. The λ_{2n+1}, $\boldsymbol{\delta}^1_{2n+1}$ are the Suslin cardinals below the projective ordinals.
- The same pattern continues below $\aleph_{\omega_1}$.

- δ^1_{2n+1} has the strong partition property.
- There are $2^{n+1} - 1$ many regular cardinals between δ^1_{2n+1} and δ^1_{2n+3}.
- All the successor cardinals between δ^1_{2n+1} and δ^1_{2n+3} have cofinality $> \delta^1_{2n+1}$.

Picture of the Cardinal Structure



Below δ^1_3

Theorem

All the ω_n are Jonsson and ω_ω is Rowbottom.

We give the proofs using methods that will generalize (which we sketch later).

Definition

W^m_1 is the m-fold product of the ω-cofinal, normal measure on ω_1.

W^m_1 is the measure induced from the weak partition relation on δ^1_1 and the ordered set $\{1, 2, \dots, m\}$

Fact (Martin)

If κ has the strong partition property and μ is a measure on κ, then $j_\mu(\kappa)$ is a cardinal.

Fact

If κ has the strong partition property and μ is a semi-normal measure on κ, then $j_\mu(\kappa)$ is a regular cardinal.

Lemma

$j_{W^m_1}(\omega_1) = \omega_{m+1}$.

Consider the algebra with single generator v_1 and operation $\oplus$. So the terms are $t = \underbrace{\mathrm{v}_1 \oplus \cdots \oplus \mathrm{v}_1}_{m}$

We assign $o(\mathrm{v}_1) = 1$, and $o(s \oplus t) = o(s) + o(t)$, so $o(t) = m$ above.

We have an ordering $<_t$ associated to the term t, namely the order type of m.

Corresponding to this term we associate two measures.

So, $\text{o.t.}(t) = \text{order-type of } <_t = m$.

wlift(t) is the measure defined by the weak partition relation on δ^1_1 and $<_t$. So, **wlift**$(t) = W^m_1$.

slift(t) is the measure on ω_{m+1} induced from the strong partition relation on δ^1_1 and the measure **wlift**(t).

The measure **slift**(t) depends on the way in which we identify $(\omega_1)^m$ with ω_1 (there are $(m-1)!$ ways to do this).

It turns out not to matter, so we take reverse lexicographic order (first by largest, then next largest, etc.).

The fact that the ordering doesn't matter is related to the "global embedding theorem."

Types of Functions on $(\omega_1)^n$

Fact

If $f\colon (\omega_1)^n \to On$, then there is a W_1^n measure one set restricted to which f is ordered by:

$$f(\alpha_1,\dots,\alpha_n) \leq f(\beta_1,\dots,\beta_n) \text{ iff } (\alpha_{\pi(1)},\dots,\alpha_{\pi(j)}) \leq_{lex} (\beta_{\pi(1)},\dots,\beta_{\pi(j)})$$

where $\pi = (\pi(1),\dots,\pi(j))$ ($j \leq n$) is a partial permutation.

If $f\colon (\omega_1)^n \to \omega_1$, then $\pi(1)$ is maximal. If $j = n$ we say f depends on all its arguments.

Fact

If $f\colon(\omega_1)^n \to On$, then almost everywhere $f(\vec{\alpha})$ either has uniform cofinality ω or has uniform cofinality $\alpha_{\pi(k)}$ for some $k \leq j$.

In the latter case there is a partial permutation π' extending π which determines the uniform cofinality.

Example

If $f\colon(\omega_1)^3 \to \omega_1$ has type $\pi = (3,1,2)$, and $f(\alpha_1,\alpha_2,\alpha_3)$ has uniform cofinality α_2, then $\pi' = (4,1,3,2)$.

The possible types are described by (π,ω), (π,π'), or (π,s) (the latter for continuous functions).

Convention: For $f\colon (\omega_1)^n \to \omega_1$, we write the arguments to f in any order.

Definition

Let $f\colon (\omega_1)^n \to \omega_1$ be of type $\pi = (n, i_2, \dots, i_n)$. For $1 \leq j \leq n$, we define the j^{th} invariant, $f(j)$ (of type $\pi \restriction j$), by:

$$f(j)(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_j}) = \sup\{f(\alpha_n, \dots, \alpha_{i_n})\colon \vec{\alpha} \cong \pi\}.$$

Also,

$$f^s(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_j}) = \sup_{\beta < \alpha_{i_j}} f(j)(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_{j-1}}, \beta)$$

Example

Suppose $f\colon (\omega_1)^4 \to \omega_1$ has type $\pi = (4, 1, 3, 2)$. Then

$$f(3)(\alpha, \beta, \gamma) = \sup_{\eta} f(\alpha, \eta, \beta, \gamma)$$

$$f(2)(\alpha, \beta) = \sup_{\eta_1 < \eta_2} f(\alpha, \eta_1, \eta_2, \beta).$$

Given two functions $f, g\colon (\omega_1)^n \to \omega_1$, there is a W^n_1 measure one set on which they are ordered as follows.

Say f has type (π, π'), g has type (σ, σ'). Say $[f] < [g]$.

Let $j_m \leq n$ be least such that $\pi' \restriction (j_m + 1) \not\cong \sigma' \restriction (j_m + 1)$.

$j_m - 1$ is largest j such that $[f(j)]$ could equal $[g(j)]$ for functions of these types.

Then there is a $j \leq j_m$ such that (almost everywhere)

$$f(\alpha_1, \ldots, \alpha_n) < g(\beta_1, \ldots, \beta_m) \text{ iff } (\alpha_n, \alpha_{i_2}, \ldots, \alpha_{i_j}) \leq_{\text{lex}} (\beta_n, \beta_{i_2}, \ldots, \beta_{i_k}).$$

Note that $\pi \restriction j \cong \sigma \restriction j$. Here $(n, i_2, \ldots, i_j) = \pi \restriction j = \sigma \restriction j$.

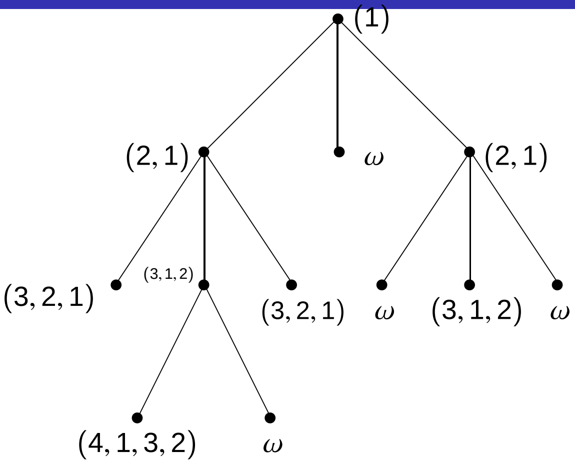
This generalizes immediately to finitely many functions.

An arrangement of finitely many functions can be described by a level-2 tree of uniform cofinalities $\mathcal{R}$.

This is a function with domain a finite subtree of $\omega^{<\omega}$ satisfying:

- $\mathcal{R}(\emptyset) = (1)$ (the unique permutation of length 1).
- $\mathcal{R}(i_1, \ldots, i_k)$ is either the symbol ω, the symbol s, or a permutation extending $\mathcal{R}(i_1, \ldots, i_{k-1})$.
- If $\mathcal{R}(\vec{\imath}) = \omega$ or s, then $\vec{\imath}$ is maximal in dom($\mathcal{R}$).
- If $\mathcal{R}(i_1, \ldots, i_k) = s$, then i_k is least node extending $(i_1, \ldots, i_{k-1})$.

Picture of a Level-2 Tree



Given the level-2 tree $\mathcal{R}$, we define an ordering $<_{\mathcal{R}}$.

dom($<_{\mathcal{R}}$) is the set of tuples $(\alpha_1, i_1, \dots, \alpha_k, i_k)$ where $(i_1, \dots, i_k)$ is a terminal node of dom($\mathcal{R}$) and $(\alpha_1, \dots, \alpha_k) \cong \mathcal{R}(i_1, \dots, i_{k-1})$.

We say $f\colon \text{dom}(<_{\mathcal{R}}) \to \omega_1$ is of type $\mathcal{R}$ if f is order-preserving, discontinuous (except for s case), and $f(\alpha_1, i_1, \dots, \alpha_k, i_k)$ has uniform cofinality as specified by $\mathcal{R}(i_1, \dots, i_k)$. If $\mathcal{R}(i_1, \dots, i_k) = s$, then $f(\alpha_1, i_1, \dots, \alpha_k, i_k)$ is the supremum of smaller values for limit α_k and of uniform cofinality ω otherwise.

Lemma

Suppose $f\colon dom(<_{\mathcal{R}}) \to \omega_1$ and there is a c.u.b. C such that $f \restriction C$ is of type $\mathcal{R}$ ($f \restriction C$ means all the α_j are in C). Then there is an f' satisfying:

- *f has type $\mathcal{R}$.*
- *$f' = f$ almost everywhere (i.e., $f' \restriction D = f \restriction D$ for some c.u.b. D).*
- *$ran(f') \subseteq ran(f)$.*

Proof.

Define $f'(\alpha_1, i_i, \dots, \alpha_k, i_k) = f(\ell_C(\alpha_1), i_1, \dots, \ell_C(\alpha_k), i_k)$, where $\ell_C(\alpha)$ is the α^{th} element of C.

This f' works. □

ω_n is Jonsson

Fix $F\colon (\omega_n)^{<\omega} \to \omega_n$.

Let $\mathcal{R}_1, \mathcal{R}_2, \dots$ enumerate the level-2 trees describing an arrangement of functions $f_1, \dots, f_m, f_{m+1}\colon (\omega_1)^{n-1} \to \omega_1$. We assume each f_i is of type (π, ω) where $\pi = (n-1, 1, 2, \dots, n-2)$.

For each i, consider the partition $\mathcal{P}_i$ of functions $f\colon \mathrm{dom}(<_{\mathcal{R}_i}) \to \omega_1$ of type $\mathcal{R}_i$ according to whether $F([f_1], \dots, [f_m]) \neq [f_{m+1}]$.

Here f induces the functions $f_1, \dots, f_{m+1}$.

Claim

On the homogeneous side of each $\mathcal{P}_i$ the stated property holds.

Proof.

Suppose C were homogeneous for the contrary side. Let $C' =$ limit points of C. Fix $f\colon \text{dom}(<_{\mathcal{R}_i}) \to C'$ of type $\mathcal{R}_i$. Replace f by g where $g(\alpha_1, i_1, \ldots, \alpha_{n-1}, i_{n-1}) = f(\cdots)$ except when $\vec{\imath}$ correspond to f_{m+1} in which case we set

$$g(\alpha_1, i_1, \ldots, \alpha_{n-1}, i_{n-1}) = N_C(f(\alpha_1, i_1, \ldots, \alpha_{n-1}, i_{n-1}).$$

Here $N_C(\alpha)$ is the least element of C greater than α.

Then g is also of type $\mathcal{R}_i$, has range in C, and induces $f_1, \ldots, f_m, g_{m+1}$ with $[g_{m+1}] \neq [f_{m+1}]$. □

Let now C_i be homogeneous for the stated side of $\mathcal{P}_i$, and let $C = \bigcap_i C_i$.

Let $A = \{\alpha < \omega_n \colon \alpha = [f], f \colon (\omega_1)^{n-1} \to C, f \text{ of type } (\pi, \omega)\}$.

Easily A has size ω_n. Fix $g \colon (\omega_1)^{n-1} \to C$ of type (π, ω).

Claim

$[g] \notin F''(A^{<\omega})$.

Proof.

Suppose $[g] = F([f_1], \ldots, [f_m])$. Let i be such that $f_1, \ldots, f_m, f_{m+1} = g$ has type $\mathcal{R}_i$ almost everywhere. By the lemma, get $f'_1, \ldots, f'_m, f'_{m+1}$ in C everywhere. This contradicts the homogeneity of $C \subseteq C_i$. □

ω_ω is Rowbottom

Fix a function $F\colon (\omega_\omega)^{<\omega} \to \omega_n$ for some n.

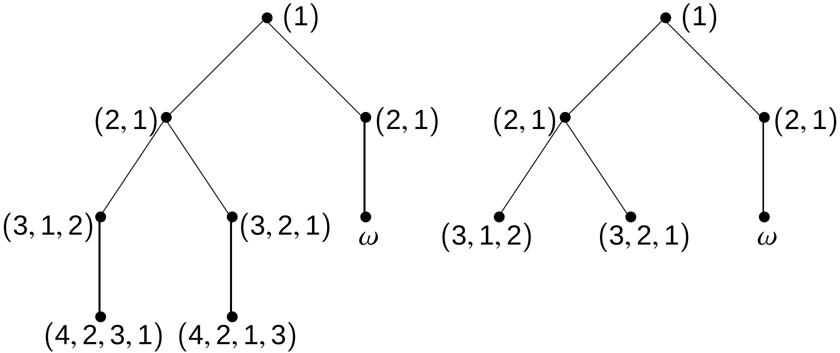
Let $\mathcal{R}_1, \mathcal{R}_2, \dots$ enumerate the possible level-2 trees giving the types of finitely many $f_1, \dots, f_k$ where $[f_j] < \omega_\omega$.

For each i, let $\bar{\mathcal{R}}_i$ denote the type induced by $\mathcal{R}_i$ by restricting to $n - 1$ invariants of the f_j.

This is equivalent to restricting the tree $\mathcal{R}_i$ to nodes of height $\leq n - 1$.

Example

Say $n = 3$, $k = 3$ (i.e., f_1, f_2, f_3), and $\mathcal{R}_i$ and $\bar{\mathcal{R}}_i$ as shown. Here $\mathcal{R}_i$ corresponds to f_1, f_2, f_3 representing $\alpha_1, \alpha_2 < \omega_4$, $\alpha_3 < \omega_2$. Also, $[f_1(1)] = [f_2(1)] < [f_3(1)]$ and $[f_1(2)] < [f_2(2)]$.



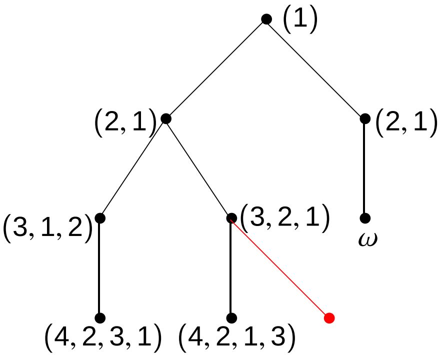
For fixed $\mathcal{R}_i$ corresponding to $f_1, \ldots, f_k$, a non-terminal node $p = (i_1, i_2, \ldots, i_n)$, we define an $\mathcal{R}_{i,p}$ extending $\mathcal{R}_i$.

$\mathcal{R}_{i,p}$ has a copy of the tree $\mathcal{R}_i$ below the node p copied to below p and put completely above the original copy.

This corresponds to adding new function $g_1, \ldots, g_\ell$ of the same type as $f_1, \dot{;} f_\ell$ where we assume $f_1, \ldots, f_\ell$ correspond to nodes below p. Note that $g_a(n-1) = f_b(n-1)$ for '$1 \leq a, b \leq \ell$.

Example

If j of the previous example corresponds to the node labeled $(3, 2, 1)$ then we have the following.



The Partition: We partition f of type $\mathcal{R}_{i,p}$ according whether

$$F([f_1],\dots,[f_\ell],[f_{\ell+1}],\dots,[f_k]) = F([g_1],\dots,[g_\ell],[f_{\ell+1}],\dots,[f_k]).$$

Here $f\colon \mathrm{dom}(<_{\mathcal{R}_{i,p}}) \to \omega_1$ induces $f_1,\dots,f_k$ and $g_1,\dots,g_\ell$.

Claim

On the homogeneous side of the partition the stated property holds.

The claim follows easily from a sliding argument and the following.

Fact

For any c.u.b. $C \subset \omega_1$ there is an $f\, dom(<_{\mathcal{R}_i}) \to C$ such that there are $\theta \geq \omega_n$ many $f_\alpha \colon dom(<_{\mathcal{R}_i}) \to C$ such that all pairs (f_α, f_β) have the same $n-1$ invariants and are of type $\mathcal{R}_{i,p}$.

Proof.

Given $\alpha = [h]$, $h \colon (\omega_1)^{n-1} \to \omega_1$, define f_α using

$$f_\alpha(\alpha_1, i_1, \dots, \alpha_{n-1}, i_{n-1}, \alpha_n, \dots, i_m) = \\ f(\alpha_1, i_1, \dots, \alpha_{n-1}, i_{n-1}, h(\alpha_2, \dots, \alpha_n), \dots, i_m).$$

□

The argument here uses the special types $\pi = (n, 1, 2, 3, \dots, n-1)$ we are considering, but the proof can be made to work for other types.

From the fact the claim follows since F has range in $\omega_{n-1} < \theta$.

Let $C \subseteq \omega_1$ be homogeneous for all the partitions.

It follows that for $\alpha_1 = [f_1], \ldots, \alpha_k = [f_k]$ with the f's having range in C of type π, the value $F(\alpha_1, \ldots, \alpha_k)$ depends only on the type $\mathcal{R}_i$ and the $n-1$ invariants $f_j(n-1)$.

Let $D \subseteq C$ be the set of closure points of C

Fix f of type π into D. Let $A \subseteq \omega_\omega$ be the set of g of type $(m, 1, 2, \ldots, m-1)$ for some $m \geq n$ with range in C.

Then $|A| = \omega_\omega$ and $F \restriction A^{<\omega}$ takes only countably many values (depends only on the type $\mathcal{R}_i$).

Below δ^1_5

Most of the arguments given above for the cardinals below δ^1_3 are general.

We need a representation of the cardinals below δ^1_5 which will allow us to use these arguments.

Such a notational system was described (and proved) below δ^1_5 by J. and Khafizov, and later by J. and Löwe for the general case δ^1_{2n+1}.

Recall that before we had the algebra generated by a single generator v_1. This algebra $\mathcal{A}_1$ has height ω.

We now extend this algebra.

The General Algebra

Have generators $v_1, v_2, v_3, \dots, v_\beta, \dots$ and operations $\oplus$, $\otimes$.

Let $\mathcal{A}_\alpha$ be the free associative, left-distributive algebra over $\{v_\beta \colon \beta < \alpha\}$.

For the cardinals below δ^1_5 we only need the first ω many generators.

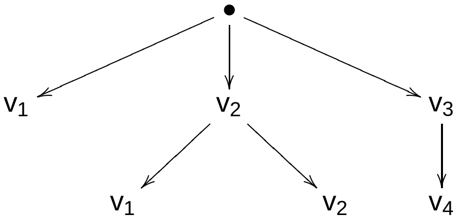
Let s, t, etc. denote terms in this algebra.

For example $t = v_1 \oplus ((v_1 \oplus v_2) \otimes v_2) \oplus (v_4 \otimes v_3) \in \mathcal{A}_5$.

Each term can be viewed as a tree, exactly as in the case of ordinal arithmetic.

Example

For $t = \mathrm{v}_1 \oplus ((\mathrm{v}_1 \oplus \mathrm{v}_2) \otimes \mathrm{v}_2) \oplus (\mathrm{v}_4 \otimes \mathrm{v}_3)$ we have the tree:



We inductively assign to each generator v_α an ordinal height $o(v_\alpha)$ and a measure $g(v_\alpha)$ which lives on an order-type o.t.(v_α).

We then extend these assignments to general terms $t \in \mathcal{A}$: $o(t) \in$ On is the height of t, $g(t)$ is the germ of t (a collection of measures), and an order-type o.t.(t).

Fix an ordering on the n-tuples, say the Gödel ordering (order first by largest element, then next largest, etc.).

We define $o(t)$ first.

Definition

We define $o(t)$ inductively through the following.

- $o(\mathrm{v}_1) = 1$
- $o(s \oplus t) = o(s) + o(t)$
- $o(s \otimes t) = o(s) \cdot o(t)$.
- $o(\mathrm{v}_\alpha) = \sup\{o(t) \colon t \in \mathcal{A}_\alpha\}$

This gives: $o(\mathrm{v}_1) = 1$, $o(\mathrm{v}_2) = \omega$, $o(\mathrm{v}_3) = \omega^\omega$, $o(\mathrm{v}_4) = \omega^{\omega^2}, \dots, o(\mathrm{v}_n) = \omega^{\omega^{n-2}}$.

For $\alpha \geq \omega$, $o(\mathrm{v}_\alpha) = \omega^{\omega^\alpha}$.

Definition

S^n_1 is the measure on ω_{n+1} induced by the strong partition relation on ω_1, functions $f\colon (\omega_1)^n \to \omega_1$ of type $\pi = (n, 1, 2, \dots, n-1)$ and the measure W^n_1 on $(\omega_1)^n$.

Note that S^1_1 is the ω-cofinal normal measure on ω_2. Also, $S^n_1 = \textbf{slift}(\mathrm{v}_1 \cdot n)$ (using $(n, 1, 2, \dots, n-1)$ in identifying $(\omega_1)^n$ with ω_1).

Definition

We define $g(\mathrm{v}_n)$ for $n < \omega$ as follows.

- $g(\mathrm{v}_1) =$ the principal measure on 1 (a single point). o.t.$(\mathrm{v}_1) = 1$.
- $g(\mathrm{v}_2) = W^1_1$. o.t.$(\mathrm{v}_2) = \omega_1$.
- $g(\mathrm{v}_3) = S^1_1$. o.t.$(\mathrm{v}_3) = \omega_2$.
- $g(\mathrm{v}_n) = S^{n-2}_1$. o.t.$(\mathrm{v}_n) = \omega_{n-1}$.

For $t \in \mathcal{A}$, let T_t be the finite tree with nodes labeled by generators corresponding to t. Then $g(t)$ is the collection of measures $\{\nu^{i_1,\dots,i_k}\}$ for $(i_1, \dots, i_k) \in T_t$.

Let $<_t$ be lexicographic ordering on tuples $(i_1, \beta_1, \dots, i_k, \beta_k)$ where $(i_1, \dots, i_k) \in \mathrm{dom}(T_t)$ and $\beta_j <$ o.t.$(g(\mathrm{v}^{\vec{i}}))$.

Then o.t.(t) is the length of $<_t$. This is obtained from the o.t.$(\mathrm{v}^{\vec{i}})$ using $+, \cdot$ as in the case of $o(t)$.

Example

For $t = \mathrm{v}_1 \oplus ((\mathrm{v}_1 \oplus \mathrm{v}_2) \otimes \mathrm{v}_2) \oplus (\mathrm{v}_4 \otimes \mathrm{v}_3)$ we have o.t.$(t) = 1 + (1+\omega)\cdot\omega + \omega^{\omega^2}\cdot\omega^{\omega} = \omega^{\omega^2}\cdot\omega^{\omega}$.

Definition

wlift(t) is the measure on δ^1_3 induced by the weak partition relation on δ^1_3, function $f\colon \mathrm{dom}(<_t) \to \delta^1_3$ continuous (uniform cofinality ω at successors), and the measures $g(t)$.

slift(t) is naturally a measure on tuples $\gamma^{\vec{\imath}}$, $\vec{\imath} \in T_t$.

Definition

slift(t) is the measure on $j_\mu(\delta^1_3)$, $\mu = \textbf{slift}(t)$, induced by the strong partition relation on δ^1_3, functions $F\colon \delta^1_3 \to \delta^1_3$ of the correct type, and the measure **slift**(t) on δ^1_3.

As before, **slift**(t) depends on the bijection between $(\delta^1_3)^n$ and δ^1_3 used.

Theorem

For any $t \in \mathcal{A}_\omega$ and $\mu = \mathbf{wlift}(t)$, $j_\mu(\delta^1_3) = \aleph_{\omega+o(t)+1}$.

So, we represent the successor cardinals $\delta^1_3 < \kappa < \delta^1_5$ as $\kappa = j_{\mathbf{wlift}}(t)(\delta^1_3)$ for $t \in \mathcal{A}_\omega$.

Recall the successor cardinals $\delta^1_1 < \kappa < \delta^1_3$ were represented as $k = j_{\mathbf{wlift}}(y)(\delta^1_1)$ for $t \in \mathcal{A}_1$.

This representation has other applications. For example, we can easily read off the cofinalities of the successor cardinals.

Theorem

Suppose $\boldsymbol{\delta}_3^1 < \aleph_{\alpha+1} < \boldsymbol{\delta}_5^1$*. Let* $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$*, where* $\omega^\omega > \beta_1 \geq \cdots \geq \beta_n$*, be the normal form for* α*. Then:*

- *If* $\beta_n = 0$*, then cof*$(\kappa) = \boldsymbol{\delta}_4^1 = \aleph_{\omega+2}$*.*
- *If* $\beta_n > 0$*, and is a successor ordinal, then cof*$(\kappa) = \aleph_{\omega \cdot 2+1}$*.*
- *If* $\beta_n > 0$ *and is a limit ordinal, then cof*$(\kappa) = \aleph_{\omega^\omega+1}$*.*

A Rowbottom Example

The proofs of Jonsson and Rowbottom are now similar to the arguments given earlier below δ^1_3.

Example

We sketch the proof that $\kappa = \aleph_{\omega^{\omega \cdot 2}}$ is Rowbottom.

$\aleph_{\omega^{\omega \cdot 2}} = \sup_n \aleph_{\omega^{\omega} \cdot \omega^n}$, so we consider the terms

$$t_n = \mathrm{v}_3 \otimes \underbrace{\mathrm{v}_2 \otimes \cdots \otimes \mathrm{v}_2}_{n}.$$

Analysis of types of functions and the sliding lemmas are proved as before.

If $f\colon \text{dom}(<_{t_n}) \to \delta^1_3$, then we define the invariants $f(0)$, $f(1), \dots, f(n) = f$ (note: we regard f as being from $(\omega_1)^n$ to δ^1_3 having uniform cofinality ω_2).

f represents $(\alpha_0, \alpha_1, \dots, \alpha_n)$, where $\alpha_0 = \sup(f)$, $\alpha_1 = [f(1)]$, etc.

We identify $\vec{\alpha}$ with an ordinal by ordering first by α_0, then say by α_1, etc.

We consider $F\colon \delta^1_3 \to \delta^1_3$ of the correct type. We define the invariants $F(1), \dots, F(n+1) = F$.

$$F(j)(\alpha_0, \dots, \alpha_{j-1}) = \sup_{\alpha_j, \dots, \alpha_n} F(\alpha_0, \dots, \alpha_n).$$

We sketch a proof of the key point:

Lemma

Let $C \subseteq \boldsymbol{\delta}^1_3$ be c.u.b., and $\delta < \aleph_{\omega^{\omega \cdot 2}}$. Let $n < \omega$. Then there is an $F\colon dom(\mu_n) \to C$ such that there are $\theta \geq \delta$ many $[F']$ for $F'\colon dom(\mu_m) \to C$ with $F'(n+1) = F$.

Proof.

Consider the ordering $<'$ on tuples $(\alpha_0, \dots, \alpha_n, \beta)$ where $\vec{\alpha} \in \text{dom}(\mu_n)$ and $\beta < h(\alpha_0)$ for some fixed function $h \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$.

Choose $m > n$ and h so that the function $G(\alpha_0, \dots, \alpha_m) = h(\alpha_0)$ satisfies $[G]_{\mu_m} > \delta$.
Fix $F \colon \text{dom}(<') \to C'$ of uniform cofinality ω_2. .

For $\gamma = [H]_{\mu_m} < [G]_{\mu_m}$, set

$$F_\gamma(\alpha_0, \dots, \alpha_n) = F'(\alpha_0, \dots, \alpha_n, H(\alpha_0, \dots, \alpha_m)).$$

□

A Cardinal Computation

Example

We consider the cardinal $\kappa = \aleph_{\omega^{\omega\cdot 2}+\omega^3+1}$. We identify a description giving this cardinal and the corresponding term in the algebra A_ω. We sketch a proof that these actually represent this cardinal.

Let $t = (v_3 \otimes v_3) \oplus (v_2 \oplus v_2 \oplus v_2)$.

So, $o(t) = \omega^{\omega\cdot 2} + \omega \cdot 3$.

Also, o.t.$(t) = \omega_2 \cdot \omega_2 + \omega_1 \cdot 3$.

We define a description relative to the measure sequence

$$W_3, S^4_1, W^3_1, S^5_1, W^2_1.$$

We denote elements of these measure spaces by: f, h_1, (η_1, η_2, η_3), h_2, $(\eta_4, \eta_5) = f, \vec{h}$.

d is the object which defines the ordinal in the iterated ultrapower assigning to f, $h_1, \dots$, the ordinal $(d; f; h_1, (\eta_1, \eta_2, \eta_3), h_2, (\eta_4, \eta_5)) < \omega_3$ which is represented w.r.t. W^2_1 by $(\alpha, \beta) \mapsto (d; f; \vec{h}; (\alpha, \beta))$ where:

$$d(f; \vec{h}; \alpha, \beta) = h^s_1(\eta_2, \eta_3, h^s_3(2)(\eta_5, \alpha), \beta).$$

Let $\mu = \textbf{slift}(t)$. So, μ is a measure on δ^1_3.

Let $\kappa_1 = j_\mu(\delta^1_3)$. So, $\kappa_1 \geq \kappa$.

Let $\kappa_2 = (d; f; \vec{h})$ be the ordinal represented by the description. From the description analysis, $\kappa_2 \leq \kappa$.

To finish, it suffices to show that $\kappa_1 \leq \kappa_2$. It then follows that $\kappa = j_\mu(\delta^1_3) = (d; f; \vec{h})$.

We define an embedding $\pi\colon j_\mu(\boldsymbol{\delta}^1_3) \to (d; W_3; S^4_1, W^3_1, S^5_1, W^2_1)$. For $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ representing $[F]_\mu$, let $\pi([F])$ be represented in the iterated ultrapower by:

$$\pi([F])(f; \vec{h}) = F([\theta]),$$

where $\theta\colon\ <_t \to \boldsymbol{\delta}^1_3$ is defined as follows.

For $\delta_1 < \delta_2 < \delta_3 < \omega_1$, let

$$\theta(\delta_1, \delta_2, \delta_3) = f([(\alpha, \beta) \mapsto h_1(\eta_2, \eta_3, h_3(\eta_4, \delta_1, \delta_2, \delta_3), \beta)]).$$

For $\rho_1 = [\ell_1] < [\ell_2] = \rho_2 < \omega_2$, let

$$\theta(\rho_1, \rho_2) = f([(\alpha, \beta) \mapsto h_1(\eta_1, \ell_1(\alpha), \ell_2(\alpha), \beta)]).$$