Contents

A Computation of $\boldsymbol{\delta}^1_5$

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Abstract. Assuming AD, the axiom of determinacy, the value of the fifth projective ordinal is computed, the result being $\boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$. The strong partition relation on $\boldsymbol{\delta}^1_3$ and the weak partition relation on $\boldsymbol{\delta}^1_5$ are also established, thereby giving the complete first step in the inductive analysis of the projective ordinals.

Key Words: determinacy, projective ordinals.

Introduction

We work throughout this paper in the theory $ZF + AD$. AD, the axiom of determinacy, is the assertion that every two player integer game is determined, that is, one of the players has a winning strategy. By "real" we mean an element of the Baire space ω^ω. The projective ordinals are defined by: $\boldsymbol{\delta}^1_n$ = the supremum of the lengths of the $\boldsymbol{\Delta}^1_n$ prewellorderings of the reals (a prewellordering is a well-ordering of equivalence classes). The projective ordinals are absolute between V and $L(\mathbb{R})$, so all the arguments of this paper may be viewed as taking place in $L(\mathbb{R})$, in which case our determinacy assumption is $AD^{L(\mathbb{R})}$. We mention that results of Woodin [**13**] and Martin-Steel [**9**] show that $AD^{L(\mathbb{R})}$ is a consequence of large cardinal axioms, namely $ZFC + \exists\omega$ many Woodin cardinals with a measurable beyond.

Beginning with the work of Martin, Moschovakis, Solovay in the late 60's, and later others, a structural theory for the projective sets of reals (the $\boldsymbol{\Pi}^1_n$, $\boldsymbol{\Sigma}^1_n$ sets) was developed assuming projective determinacy–the weakening of AD to games with projective payoff. This provided a powerful extension of the ZFC theory of $\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Sigma}^1_1$ sets developed by the "classical" descriptive set theorists (Novikov, Lusin, Sierpinski, Suslin, etc.) from the 20's to the 40's.

This theory was developed largely in terms of the projective ordinals. It was also shown that $\boldsymbol{\delta}^1_1 = \omega_1$, $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ (Martin-Solovay), $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ (Kunen-Martin). Martin established the strong partition relation on $\boldsymbol{\delta}^1_1$ (that is, $\boldsymbol{\delta}^1_1 \to (\boldsymbol{\delta}^1_1)^{\boldsymbol{\delta}^1_1}$, see §1 for the definitions), and Kunen the weak partition relation on $\boldsymbol{\delta}^1_3$ ($\forall\lambda < \boldsymbol{\delta}^1_3$ $\boldsymbol{\delta}^1_3 \to (\boldsymbol{\delta}^1_3)^\lambda$). Kechris and Martin independently formulated the important concept of a homogeneous tree (c.f. [**6**]), which was inspired by earlier particular instances of the construction due to Kunen and Martin-Solovay. Using this concept, Kunen also proposed a program for computing the $\boldsymbol{\delta}^1_n$. The program stalled, however, and the problem of computing $\boldsymbol{\delta}^1_5$ remained open, although its computation was reduced to the computation of the ultrapowers of $\boldsymbol{\delta}^1_3$ by certain measures on $\boldsymbol{\delta}^1_3$. We refer the reader to Moschovakis' book [**10**] and Kechris [**5**] and [**6**] for a more detailed exposition.

In the early 80's Martin [**7**] obtained some important results on the ultrapowers of $\boldsymbol{\delta}^1_3$ by the normal measures on $\boldsymbol{\delta}^1_3$ (see theorem 2.1). Building on this and some joint work with Martin, the author then computed $\boldsymbol{\delta}^1_5 = (\aleph_{\omega(3)})^+$, where $\omega(1) = \omega$, and $\omega(n+1) = \omega^{\omega(n)}$ (ordinal exponentiation). It was subsequently shown that $\boldsymbol{\delta}^1_{2n+1} = (\aleph_{\omega(2n-1)})^+$.

In [**2**] the upper bound for $\boldsymbol{\delta}^1_{2n+1}$ is computed assuming certain inductive hypotheses. A second forthcoming part is necessary to verify the hypotheses at the next level as well as to give the lower bound. The goal of this paper is not only to

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give the complete computation of $\boldsymbol{\delta}^1_5$, but to provide the complete first stage of the inductive analysis of the projective hierarchy.

Our plan is to assume the strong partition relation on $\boldsymbol{\delta}^1_1$ and the weak partition relation on $\boldsymbol{\delta}^1_3$, compute $\boldsymbol{\delta}^1_5$, and then prove the strong partition relation on $\boldsymbol{\delta}^1_3$ and the weak relation on $\boldsymbol{\delta}^1_5$, thereby completing the induction. The methods of this paper are essentially self-contained. We could, in fact, use them to obtain direct proofs of the strong partition relation on $\boldsymbol{\delta}^1_1$ and the weak on $\boldsymbol{\delta}^1_3$ (see for example [**3**] and the forthcoming [**1**]), but to save space we will start from these known results. In addition, to keep our starting point convenient, we assume familiarity with the following two items.

First, we assume that the reader is familiar with the homogeneous tree construction of [**6**]. Actually, some of the arguments of §2 will recall this construction (c.f. the discussion following remark 2.2), so the reader with only a vague recollection of this construction should get by.

Secondly, we assume knowledge of the Kunen result that there is a $\boldsymbol{\Delta}^1_3$ coding of the subsets of ω_ω (c.f. [**11**]). By this we mean precisely as follows. There is a $\boldsymbol{\Delta}^1_3$ prewellordering of the reals with associated norm $\rho\colon \omega^\omega \to \omega_\omega$ (that is, $\rho(x) =$ the rank of x in the prewellordering), and a map $\pi\colon \omega^\omega \overset{onto}{\to} \mathcal{P}(\omega_\omega)$ such that the relation $A(x,y) \leftrightarrow \rho(x) \in \pi(y)$ is $\boldsymbol{\Delta}^1_3$. By Martin's theorem on the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ length unions and intersections, this is equivalent to saying that for all $\alpha < \omega_\omega$ that $\{y\colon \alpha \in \pi(y)\} \in \boldsymbol{\Delta}^1_3$. We will also use small variations of this, such as a $\boldsymbol{\Delta}^1_3$ coding of the subsets of $\omega_\omega \times \omega_\omega$. This assumption is really only for convenience, as we will be establishing higher level analogs of this result in this paper (the reader might also consult [**1**] where the existence of a $\boldsymbol{\Delta}^1_3$ coding of the subsets of ω_ω is established using methods akin to those of this paper).

The methods of this paper can be pushed considerably beyond the projective hierarchy as well. Indeed, the ultimate goal is obtain a complete analysis of the model $L(\mathbb{R})$ assuming AD. The arguments of this paper should represent the successor case for the general $L(\mathbb{R})$ induction. Exactly how far the current methods extend is not clear at the moment. It seems they extend past the first inaccessible cardinal in $L(\mathbb{R})$, but not yet to the first real admissible for $L(\mathbb{R})$ (= the ordinal of the inductive sets). We refer the reader to [**4**] for further discussion.

In §1 we define three canonical families of measures which we denote W^m_1, S^m_1, and W^m_3, measures on ω_1 , ω_ω, and $\boldsymbol{\delta}^1_3$ respectively. We prove they dominate the more general measures from the homogeneous tree construction in a suitable sense. In §2 we prove Martin's theorem on the normal measures, and in §3 we compute the upper bound for $\boldsymbol{\delta}^1_5$. Here the main technical concept of *description* is introduced. §4 introduces the important concept of uniform cofinality, and proves some results which are necessary for converting the analysis of functions in §3 to an analysis of measures on $\boldsymbol{\delta}^1_3$, which is carried out in §5. In §6 we use the results of §5 to prove the strong partition relation on $\boldsymbol{\delta}^1_3$. §7 uses the strong partition relation on $\boldsymbol{\delta}^1_3$ to prove the lower bound for $\boldsymbol{\delta}^1_5$. In §8 we analyze the measures on $(\boldsymbol{\delta}^1_5)^-$, and using this we prove the weak partition relation on $\boldsymbol{\delta}^1_5$ in §9.

We have attempted to make the different sections as independent as possible. The reader interested only in seeing the upper bound for $\boldsymbol{\delta}^1_5$ may read the statements of the theorems in §1, 2, and then read §3.

By a measure μ on a set X, we always mean a countably additive ultrafilter on X (usually X is an ordinal, or product of ordinals). We frequently write $\mu(A) = 1$

for $A \in \mu$, and $\mu(A) = 0$ for $X - A \in \mu$. Recall that (from AD) every ultrafilter on a set is countably additive. If $f : X \to Y$ and μ is a measure on X, then $f(\mu)$ is the measure on Y defined by: $f(\mu)(A) = \mu(f^{-1}(A))$. If μ is a measure, then j_μ denotes the ultrapower embedding corresponding to μ (identifying always the ultrapower with its transitive collapse).

Throughout this paper we adopt the (slightly non-standard) convention that "lexicographic ordering" refers to the reverse lexicographic or Kleene-Brouwer ordering on $ON^{<\omega}$. That is, $\vec{\alpha} <_{lex} \vec{\beta}$ iff there is a least i such that $\alpha_i \neq \beta_i$, in which case $\alpha_i < \beta_i$, or else $\vec{\beta}$ is a proper initial segment of $\vec{\alpha}$.

By a tree on a set X we mean a set $T \subseteq X^{<\omega}$ closed under initial segments. For us, X will always be of the form $\lambda_1 \times \cdots \times \lambda_n$, where $\lambda_1, \dots, \lambda_n \in ON$. As is customary, we identify $(\lambda_1 \times \cdots \times \lambda_n)^{<\omega}$ with the set of $(s_1, \dots, s_n) \in {\lambda_1}^{<\omega} \times \cdots \times {\lambda_n}^{<\omega}$ such that $\mathrm{lh}(s_1) = \cdots = \mathrm{lh}(\lambda_n)$. If T is a tree on $\lambda_1 \times \cdots \times \lambda_n$ and $f \in (\lambda_1)^\omega$, then $T_f = \{(s_2, \dots, s_n) : (f \restriction \mathrm{lh}(s_2), s_2, \dots, s_n) \in T\}$ denotes the section of T along f. Also, $T \restriction \alpha \doteq T \cap (\alpha \times \cdots \times \alpha)^{<\omega}$ denotes the restriction of T to ordinals $< \alpha$. If T is a tree on ON (or, more generally, any set X equipped with a well-ordering) then the tree T is linearly ordered by the Kleene-Brouwer order, which by our convention just means the restriction to T of lexicographic ordering. This is a well-order of T iff T has no infinite branches. In this case, we let $|T|$ denote the rank of T in the Kleene-Brouwer ordering. If, in addition, $s \in T$, then $|T(s)|$ denotes the rank of s in the Kleene-Brouwer ordering on T.

We introduce a notational convention that we will use throughout the paper, and which simplifies the presentation of many arguments. If $\theta \in ON$, $\mu_1, \dots, \mu_n$ are measures, and $P \subseteq ON$, then we write $\forall^*_{\mu_1} \alpha_1 \, \forall^*_{\mu_2} \alpha_2 \dots \forall^*_{\mu_n} \alpha_n \, P(\theta(\alpha_1, \dots, \alpha_n))$ to abbreviate the following: if we fix a function $\alpha_1 \to \theta(\alpha_1)$ representing θ with respect to μ_1, then for μ_1 almost all α_1, if we fix a function $\alpha_2 \to \theta(\alpha_1)(\alpha_2)$ representing $\theta(\alpha_1)$ with respect to μ_2, then for μ_2 almost all α_2, ..., if we fix a function $\alpha_n \to \theta(\alpha_1)(\alpha_2) \cdots (\alpha_n)$ representing $\theta(\alpha_1) \cdots (\alpha_{n-1})$ with respect to μ_n, then for μ_n almost all α_n, $P(\theta(\alpha_1) \cdots (\alpha_n))$. We also extend this convention to pairs of ordinals, etc. For example, given two ordinals θ_1, θ_2, and measures $\mu_1, \dots, \mu_n$, we might write $\forall^*_{\mu_1} \alpha_1 \cdots \forall^*_{\mu_n} \alpha_n \, [\mathrm{cof}(\theta_1(\alpha_1, \dots, \alpha_n)) \leq \theta_2(\alpha_1, \dots, \alpha_n)]$. If the measures are understood, we may suppress writing them, and just write $\forall^* \alpha_1, \dots, \alpha_n \, P(\theta(\vec{\alpha}))$.

Lastly, we catalog some of the known facts about the projective ordinals (mainly about $\boldsymbol{\delta}^1_1$, $\boldsymbol{\delta}^1_3$) that we will use as our starting point. As we mentioned above, this is mainly for convenience, as we will be establishing higher level analogs of these results here. First, all of the $\boldsymbol{\delta}^1_n$ are regular (in fact measurable), $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$, and $\boldsymbol{\delta}^1_{2n+1}$ is the successor of a cardinal of cofinality ω. Also, $\boldsymbol{\delta}^1_1 = \omega_1$, and $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$. Proofs of these facts may be found in [**5**]. Next, $\boldsymbol{\delta}^1_1$ has the strong partition property (defined at the beginning of §1), a result due to Martin. The reader may also consult [**3**] where a different proof (though still using Martin's general method) is given; in fact, this might serve as an introduction to the "analysis of measure" arguments that we will be carrying out (in a more complex form) in §§5, 8. An easy consequence of this (in fact, of the weak partition property) is the existence of a unique normal measure on ω_1, namely the one generated by the c.u.b. filter (see the discussion at the start of §1). Let W^1_1 denote this normal measure on $\boldsymbol{\delta}^1_1$, and W^n_1 the n-fold product of the normal measure (we extend this terminology in §1). There is also a *Kunen tree* $K \subseteq (\omega \times \omega_1)^{<\omega}$ for $\boldsymbol{\delta}^1_1$ (we construct this tree in §2). That is, K has the property that for all $f\colon \omega_1 \to \omega_1$, there is a $x \in \omega^\omega$ with K_x well-founded and such that $|K_x \restriction \alpha| > f(\alpha)$ on a c.u.b. set. From the Kunen tree and the strong partition property on $\boldsymbol{\delta}^1_1$ it follows that $j_{W_1^n}(\omega_1) = \omega_{n+1}$. [The weak partition relation on ω_1 shows that for any $f\colon (\omega_1)^n \to \omega_1$, there is a $g\colon \omega_1 \to \omega_1$ such that $f(\alpha_1, \dots, \alpha_n) < g(\alpha_n)$ for W_1^n almost all $\vec{\alpha}$. Let x be such that $|K \restriction \alpha| > g(\alpha)$ on a c.u.b. set. This defines a map from $[id_n]_{W_1^n}$ onto $[f]_{W_1^n}$, where $id_n(\alpha_1, \dots, \alpha_n) = \alpha_n$. The weak partition property shows that if $h < id_n$ almost everywhere, then almost everywhere h depends only on $(\alpha_1, \dots, \alpha_{n-1})$, that is, $[id_n]_{W_1^n} = j_{W_1^{n-1}}(\omega_1)$. Thus, $j_{W_1^n}(\omega_1) \le (j_{W_1^{n-1}}(\omega_1))^+$ and hence $j_{W_1^n}(\omega_1) \le \omega_{n+1}$ follows. The lower bound can be obtained as a consequence of theorem 7.1.] Kunen (see [**11**]) established the existence of a $\boldsymbol{\Delta}^1_3$ coding for the subsets of ω_ω, mentioned above. From this and Martin's argument, the weak partition property for $\boldsymbol{\delta}^1_3$ follows. Another consequence of this analysis is the fact that for any $\alpha < \boldsymbol{\delta}^1_3$ and measure μ on ω_ω, $j_\mu(\alpha) < \boldsymbol{\delta}^1_3$. We refer the reader to [**11**] for proofs (see also [**1**]). We also recall a theorem of Martin (see [**5**]) that the $\boldsymbol{\Delta}^1_{2n+1}$ sets are closed under $< \boldsymbol{\delta}^1_{2n+1}$ length unions and intersections. We will make frequent use of this fact in our definability computations later.

Finally, the homogeneous tree construction [**6**] produces, assuming the strong partition property of ω_1 and the weak partition property of $\boldsymbol{\delta}^1_3$, a homogeneous tree $T \subseteq (\omega \times \boldsymbol{\delta}^1_3)^{<\omega}$ with $p[T]$ a complete $\boldsymbol{\Pi}^1_3$ complete set. This shows that every $\boldsymbol{\Sigma}^1_4$ set is *weakly homogeneous* (see §2) via a tree on $\omega \times \boldsymbol{\delta}^1_3$, and thus every $\boldsymbol{\Pi}^1_4$ (and hence also every $\boldsymbol{\Sigma}^1_5$) set is λ-Suslin, where λ is the supremum of the ultrapowers $j_\nu(\boldsymbol{\delta}^1_3)$, where ν ranges over the homogeneity measures for T. The Kunen-Martin theorem (see [**5**] or [**10**]) then gives that every $\boldsymbol{\Sigma}^1_5$ well-founded relation on ω^ω has length $< \lambda^+$. Thus, $\boldsymbol{\delta}^1_5 \le \lambda^+$. This, in fact, is the Kunen reduction mentioned earlier, and shows the significance of the homogeneous tree construction. In §1 we will define and analyze these homogeneity measures ν more carefully (we will call them the special homogeneous measures in $\mathcal{W}_3$) and obtain a simpler family of measures (which we call W_3) whose ultrapowers are also cofinal in λ.

CHAPTER 1

Embedding Theorems

We introduce several families of measures. We define three families W_1, S_1, W_3 of *canonical* measures, and three families $\mathcal{W}_1$, $\mathcal{S}_1$, $\mathcal{W}_3$ of more general measures which we call *special homogeneous measures.* The letters W, S refer to the fact that the definitions of these measures use the weak or strong partition relation on $\boldsymbol{\delta}^1_1$ or $\boldsymbol{\delta}^1_3$. We prove two theorems to the effect that the canonical measures dominate the special homogeneous measures. This will allow us to use just the canonical measures at several places in the following analysis, a considerable simplification. Before doing this, we introduce some general notions that we will use frequently in this paper.

Definition 1.1. Let $<_S$ be a well-ordering of a set S. We say $f\colon S \to ON$ having range in the limit ordinals is *of the correct type* if f satisfies:

(1) f is order-preserving.
(2) f is everywhere discontinuous, i.e., $\forall x \in S\ f(x) > \sup_{y <_S x} f(y)$.
(3) f has *uniform cofinality* ω, that is, there is a $g\colon \omega \times S \to ON$ which is increasing in the first argument (i.e., $g(n, x) < g(m, x)$ whenever $n < m$) which induces f. That is, $f(x) = \sup_{n\in\omega} g(n, x)$ for all $x \in S$.

Recall that $\kappa \to (\kappa)^\lambda$ if for every partition $\mathcal{P}\colon (\kappa)^\lambda \to \{0, 1\}$ of the increasing functions from λ to κ into two pieces, there is a homogeneous set $H \subseteq \kappa$ of size κ. That is, $\exists i \in \{0, 1\}\ \forall f \in (H)^\lambda\ \mathcal{P}(f) = i$. We say κ has the *strong partition property* if $\kappa \to (\kappa)^\kappa$, and say κ has the *weak partition property* if $\forall \lambda < \kappa\ \kappa \to (\kappa)^\lambda$.

We (temporarily) define $\kappa \overset{c.u.b.}{\to} (\kappa)^\lambda$ if for every partition $\mathcal{P}$ of the functions $f\colon \lambda \to \kappa$ of the correct type into $\{0, 1\}$, there is a c.u.b. $C \subseteq \kappa$ which is homogeneous for $\mathcal{P}$. That is, $\exists i \in \{0, 1\}\ \forall f\colon \lambda \to C$ of the correct type, $\mathcal{P}(f) = i$.

The essential equivalence of these two formulations of the partition property is well known, and an easy exercise. Specifically, we have:

Fact . For all regular cardinals κ and ordinals $\lambda \leq \kappa$ we have: $\kappa \overset{c.u.b.}{\to} (\kappa)^\lambda \Rightarrow \kappa \to (\kappa)^\lambda$ and $\kappa \to (\kappa)^{\omega\cdot\lambda} \Rightarrow \kappa \overset{c.u.b.}{\to} (\kappa)^\lambda$.

In particular, for all regular cardinals κ, the two notions of strong (or weak) partition property agree. For all purposes in this paper, however, it is the c.u.b. version which is relevant, and we henceforth adopt the c.u.b. version of the partition property as our official definition.

If κ, λ are regular cardinals with $\lambda < \kappa$, then the partition relation $\kappa \to (\kappa)^\lambda$ implies that the λ-c.u.b. filter on κ is a normal measure (i.e., a set $A \subseteq \kappa$ has measure one iff there is a c.u.b. $C \subseteq \kappa$ such that all $\alpha \in C$ of cofinality λ are in A). Since every normal measure on κ gives every c.u.b. set measure one, it also follows that if there are countably many regular $\lambda < \kappa$, then every normal measure on κ is the λ-c.u.b. measure for some regular $\lambda < \kappa$. In particular, there is a unique normal measure on $\boldsymbol{\delta}^1_1 = \omega_1$, and there are exactly three normal measures on $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ (since there are three regular $\lambda < \boldsymbol{\delta}^1_3$, namely, $\lambda = \omega, \omega_1, \omega_2$).

If $(S, <_S)$ is a well-ordering of length κ (or $< \kappa$), and κ has the strong (weak) partition property, then we have a partition property for functions $f\colon S \to \kappa$ of the correct type. This follows simply from identifying $(S, <_S)$ with an ordinal. Suppose, in addition, $\mu_1, \dots, \mu_n$ are measures on S. We define a measure ν on n-tuples of ordinals which we refer to as "the measure induced from the strong (weak) partition relation on κ, functions $f\colon S \to \kappa$ of the correct type, and the measures $\mu_1, \dots, \mu_n$." Specifically, $\nu(A) = 1$ iff $\exists$ c.u.b. $C \subseteq \kappa\, \forall f\colon S \to C$ of the correct type $([f]_{\mu_1}, \dots, [f]_{\mu_n}) \in A$. Later in this paper, we will also consider measures induced from functions having "types" other than the "correct type" (in particular, the functions will have different uniform cofinalities, that is, not necessarily uniform cofinality ω).

We now introduce the families of *canonical measures.* Recall again our starting assumptions that $\boldsymbol{\delta}^1_1 = \omega_1$ has the strong partition property, and $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ has the weak partition property. Again, Note that the $\lambda = 1$ c.u.b. partition property on ω_1 immediately implies that the c.u.b. filter on ω_1 is the unique normal normal measure on ω_1, a well-known consequence of AD.

DEFINITION 1.2. W^n_1 is the measure on increasing n-tuples $(\alpha_1, \dots, \alpha_n)$, $\alpha_1 < \cdots < \alpha_n$, given by the n-fold product of the normal measure on ω_1. We also identify this with a measure on ω_1 by ordering the tuples $(\alpha_1, \dots, \alpha_n)$ first by α_n, then α_{n-1}, etc. We let W_1 denote the collection of measures W^n_1, $n \in \omega$.

The finite (c.u.b.) partition property on ω_1 implies that W^n_1 is generated by taking increasing n-tuples from c.u.b. subsets of ω_1. That is $W^n_1(A) = 1$ iff $\exists$ c.u.b. $C \subseteq \omega_1\ \forall \alpha_1 < \cdots < \alpha_n \in C^n\ (\alpha_1, \dots, \alpha_n) \in A$.

REMARK 1.1. We usually consider W^n_1 as a measure on increasing n-tuples. Whether or not we are identifying it with a measure on ω_1, however, should be clear from context.

Viewing $\mathrm{dom}(W^n_1)$ as the increasing n-tuples permits a useful notational convention; we may write the ordinals in the tuple in any order. For example, if $S = \{(\alpha_1, \alpha_2, \alpha_3)\colon \alpha_2 < \alpha_3 < \alpha_1 < \omega_1\}$, then we may regard W^3_1 as a measure on S as well.

Recall from our discussion in the introduction that $j_{W^n_1}(\omega_1) = \omega_{n+1}$. We use this fact implicitly below.

DEFINITION 1.3. S^n_1 is the measure on ω_{n+1} defined as follows. Let $<_n$ be the ordering on increasing n-tuples of countable ordinals defined by:

$$(\alpha_1, \dots, \alpha_n) <_n (\beta_1, \dots, \beta_n) \leftrightarrow (\alpha_n, \alpha_1, \alpha_2, \dots, \alpha_{n-1}) <_{lex} (\beta_n, \beta_1, \beta_2, \dots, \beta_{n-1}),$$

where $<_{lex}$ denotes lexicographic ordering. Then, S^n_1 is the measure induced from the strong partition relation on ω_1, functions $f\colon\ \mathrm{dom}(<_n) \to \omega_1$ of the correct type, and W^n_1.

Thus, $A \subseteq \omega_{n+1}$ has measure one with respect to S^n_1 iff there is a c.u.b. $C \subseteq \omega_1$ such that for all $f\colon\ \mathrm{dom}(<_n) \to C$ of the correct type, $[f]_{W^n_1} \in A$. We let S_1 denote the collection of measures S^n_1, $n \in \omega$.

Using the weak partition relation on $\boldsymbol{\delta}^1_3$, we define also the canonical family W_3. We use here the fact mentioned in the introduction that for any of the measures $\nu = S^n_1$ (actually, for any measure on ω_ω) we have $\forall \alpha < \boldsymbol{\delta}^1_3\ j_\nu(\alpha) < \boldsymbol{\delta}^1_3$. This is a consequence of the existence of a $\boldsymbol{\Delta}^1_3$ coding of the subsets of ω_ω (c.f. [**11**]; we will carry out some similar computations in §6).

DEFINITION 1.4. W^n_3 is the measure on $\boldsymbol{\delta}^1_3$ induced by the weak partition relation on $\boldsymbol{\delta}^1_3$, functions $f \colon \omega_{n+1} \to \boldsymbol{\delta}^1_3$ of the correct type, and the measure S^n_1 on ω_{n+1}. We let W_3 denote the collection of measures W^n_3, $n \in \omega$.

The following simple lemma is a special case of a "sliding lemma." It shows that functions which are of the correct type almost everywhere with respect to $<_m$ can be extended to functions which are of the correct type everywhere. More general versions will be introduced later.

LEMMA 1.1. *Let $f \colon (\omega_1)^m \to \omega_1$ be of the correct type almost everywhere. That is, suppose $C \subseteq \omega_1$ is c.u.b. and for all pairs $\vec{\alpha}$, $\vec{\beta}$ of increasing m-tuples from C we have $f(\vec{\alpha}) < f(\vec{\beta})$ iff $\vec{\alpha} <_m \vec{\beta}$. Suppose also, $f \restriction C^m$ has uniform cofinality ω and is discontinuous. Then there is an $f' \colon (\omega_1)^m \to \omega_1$ of the correct type such that $\forall^*_{W^m_1} \vec{\alpha}\ f'(\vec{\alpha}) = f(\vec{\alpha})$, and $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$.*

PROOF. Let $l \colon \omega_1 \to \omega_1$ be defined by: $l(\alpha) = \alpha^{th}$ element of C. For $\alpha_1 < \cdots < \alpha_m < \omega_1$, define $f'(\alpha_1, \dots, \alpha_m) = f(l(\alpha_1), \dots, l(\alpha_m))$. This easily works. $\square$

We define now some families of some somewhat more general measures, which we denote by $\mathcal{W}_1$, $\mathcal{S}_1$ and $\mathcal{W}_3$. The reader familiar with the homogeneous tree construction (see [**6**]) should have no trouble verifying that without loss of generality, the measures occurring in the homogeneous tree representations for $\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Pi}^1_2$, $\boldsymbol{\Pi}^1_3$ sets are contained in the families $\mathcal{W}_1$, $\mathcal{S}_1$ and $\mathcal{W}_3$ respectively. For this reason, we refer to the measures in these families as *special homogeneous measures.*

We first define the family $\mathcal{W}_1$. Suppose $\pi = (i_1, i_2, \dots, i_n)$ is a permutation of $(1, 2, \dots, n)$. If $i_1 = n$, we call π a *standard permutation.* We say an n-tuple of ordinals $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ is of type π if it is order isomorphic to π, that is, $\alpha_k < \alpha_l$ iff $i_k < i_l$. If π is a standard permutation, this requires $\alpha_1 > \alpha_2, \dots, \alpha_n$. For π a standard permutation of length n, let $<_\pi$ be the lexicographic ordering on the n-tuples of type π. Note that $<_\pi$ has length ω_1. For such π, let $\mathcal{W}^\pi_1$ be the measure on the tuples of type π induced by W^n_1. Thus, $\mathcal{W}^\pi_1(A) = 1$ iff $\exists$ c.u.b. $C \subseteq \omega_1$ $\forall (\alpha_1, \dots, \alpha_n) \in C^n$ of type π, $(\alpha_1, \dots, \alpha_n) \in A$. Let $\mathcal{W}$ be the family of measures $\mathcal{W}^\pi_1$.

In view of our notational convention, there is little difference between $\mathcal{W}^\pi_1$ and W^n_1, and we will frequently identify the two. For example, if $f \colon \mathrm{dom}(<_\pi) \to \omega_1$ is order-preserving, we will usually write $[f]_{W^n_1}$ in place of $[f]_{\mathcal{W}^\pi_1}$ to denote its equivalence class with respect to the measure $\mathcal{W}^\pi_1$.

If $\pi_1 = (i_1, \dots, i_n)$, $\pi_2 = (j_1, \dots, j_m)$ are permutations and $m > n$, we say π_2 *extends* π_1 if $(j_1, \dots, j_n)$ is order-isomorphic to π_1. If, in addition, $m = n + 1$, then we say π_2 is an *immediate extension* of π_1. For example, $(3, 1, 2)$ is immediately extended by the standard permutations $(4, 2, 3, 1)$, $(4, 1, 3, 2)$ and $(4, 1, 2, 3)$. If π_2 extends π_1 as above, then then is a natural projection $\pi \colon \mathrm{dom}(<_{\pi_2}) \to \mathrm{dom}(<_{\pi_1})$. Namely, $\pi(\alpha_1, \dots, \alpha_n, \dots, \alpha_m) = (\alpha_1, \dots, \alpha_n)$. Clearly $\pi(\mathcal{W}^{\pi_2}_1) = \mathcal{W}^{\pi_1}_1$.

The following notational convention will be used throughout the paper.

DEFINITION 1.5. Let $\pi = (n, i_2, \ldots, i_n)$ be a standard permutation of length n, and $<_\pi$ the associated ordering. Suppose $f\colon \operatorname{dom}(<_\pi) \to \omega_1$ is order-preserving. The k^{th} invariant of f, for $1 \leq k \leq n$, denoted $f(k)$, is defined as follows. Let $\pi \restriction k = (n, i_2, \ldots, i_k)$. For limit ordinals $(\alpha_n, \alpha_{i_2}, \ldots, \alpha_{i_k})$ of type $\pi \restriction k$, define

$$f(k)(\alpha_n, \alpha_{i_2}, \ldots, \alpha_{i_k}) = \sup\{f(\alpha_n, \alpha_{i_2}, \ldots, \alpha_{i_k}, \beta_{i_{k+1}}, \ldots, \beta_{i_n})\colon \\ (\alpha_n, \alpha_{i_2}, \ldots, \alpha_{i_k}, \beta_{i_{k+1}}, \ldots, \beta_{i_n}) \text{ has type } \pi\}.$$

If $\delta < \omega_{n+1}$ is representable by an order-preserving $f\colon \operatorname{dom}(<_\pi) \to \omega_1$, we let $\delta(k) = [f(k)]_{W_1^k}$.

It is easy to check that $\delta(k)$ is well-defined. Also, the permutation π such that δ is representable by an order-preserving $f\colon \operatorname{dom}(<_\pi) \to \omega_1$ is unique, if it exists (for more discussion see §4).

In particular, if $\pi = (n, 1, 2, \ldots, n-1)$ is the permutation defining S_1^n, and $f\colon \operatorname{dom}(<_\pi) \to \omega_1$ is order-preserving, this becomes: for limit ordinals $\alpha_1 < \cdots < \alpha_k < \alpha_n$,

$$f(k+1)(\alpha_1, \ldots, \alpha_k, \alpha_n) = \sup\{f(\alpha_1, \ldots, \alpha_n)\colon \alpha_k < \alpha_{k+1} < \cdots < \alpha_{n-1} < \alpha_n\}.$$

We now define the family $\mathcal{S}_1$.

DEFINITION 1.6. A type-1 tree of measures is a function $\mathcal{C}$ with domain $\operatorname{dom}(\mathcal{C}) = T \cup \{\emptyset\}$, where T is a non-empty finite subtree of $\omega^{<\omega}$, satisfying:

(1) $\mathcal{C}(\emptyset) = (1)$, the unique permutation of length 1.
(2) If t_2 immediately extends t_1 (i.e., $t_2 = t_1^\frown i$ for some $i \in \omega$), then $\mathcal{C}(t_2)$ is a standard permutation which immediately extends the standard permutation $\mathcal{C}(t_1)$.

By the *size* of $\mathcal{C}$ we mean the size of T. We say $\mathcal{C}$ has *depth* n if n is the maximal length of a sequence in $\operatorname{dom}(\mathcal{C})$.

For each such $\mathcal{C}$, we define a corresponding ordering $<_\mathcal{C}$.

DEFINITION 1.7. Let $\mathcal{C}$ be a type-1 tree of measures of depth n. Then $<_\mathcal{C}$ is defined to be lexicographic ordering on sequences $\langle \alpha_1, i_1, \ldots, \alpha_k, i_k \rangle$, $k \leq n$, satisfying:

(1) $(i_1, \ldots, i_k) \in \operatorname{dom}(\mathcal{C})$.
(2) $(\alpha_1, \ldots, \alpha_k)$ has type $\mathcal{C}(i_1, \ldots, i_{k-1})$.

REMARK 1.2. For $\vec{i}$ of maximal length in $\operatorname{dom}(\mathcal{C})$, the value $\mathcal{C}(\vec{i})$ is not used in defining $<_\mathcal{C}$. We require $\mathcal{C}(\vec{i})$ to be defined in this case only for notational convenience.

Note that if $\vec{s} = \langle \alpha_1, i_1, \ldots, \alpha_k, i_k \rangle \in \operatorname{dom}(<_\mathcal{C})$, and $l \leq k$, then $\vec{s} \restriction l = \langle \alpha_1, i_1, \ldots, \alpha_l, i_l \rangle \in \operatorname{dom}(<_\mathcal{C})$ as well. Note also that $\alpha_1 > \alpha_2, \ldots, \alpha_k$, so $<_\mathcal{C}$ has length ω_1.

DEFINITION 1.8. Let $\mathcal{C}$ be a type-1 tree of measures. Then $\nu(\mathcal{C})$ is the measure on tuples $(\ldots, \beta^{\vec{i}}, \ldots)_{\vec{i} \in \operatorname{dom}(\mathcal{C})}$, defined as follows (note: we only consider $\vec{i} \neq \emptyset$ here). A has measure one with respect to $\nu(\mathcal{C})$ if there is a c.u.b. $C \subseteq \omega_1$ such that for all $f\colon \operatorname{dom}(<_\mathcal{C}) \to C$ of the correct type $(\ldots, \beta^{\vec{i}}, \ldots) \in A$, where $\beta^{(i_1, \ldots, i_k)}$

is represented with respect to $\mathcal{W}_1^\pi$, where $\pi = \mathcal{C}(i_1, \ldots, i_{k-1})$, by the function $f^{(i_1,\ldots,i_k)}$ defined by:

$$f^{(i_1,\ldots,i_k)}(\alpha_1, \ldots, \alpha_k) = f(\langle \alpha_1, i_1, \ldots, \alpha_k, i_k \rangle).$$

We let $\mathcal{S}_1$ be the collection of measures of the form $\nu(\mathcal{C})$, for $\mathcal{C}$ a type-1 tree of measures.

Note that if $\mathcal{C}$ has size m, then $\nu(\mathcal{C})$ is a measure on m-tuples. If $f\colon \operatorname{dom}(<_{\mathcal{C}}) \to \omega_1$ is of the correct type, we frequently write $[f]$ for the tuple $(\ldots, \beta^{\vec{i}}, \ldots)_{\vec{i} \in \operatorname{dom}(\mathcal{C})}$ represented by f.

If $\mathcal{C}_1, \mathcal{C}_2$ are type-1 trees of measures with domains $T_1 \cup \{\emptyset\}$, $T_2 \cup \{\emptyset\}$ respectively, and $T_1 \subseteq T_2$, then we say $\mathcal{C}_2$ *extends* $\mathcal{C}_1$ provided $\mathcal{C}_2 \restriction \operatorname{dom}(\mathcal{C}_1) = \mathcal{C}_1$. In this case, $\operatorname{dom}(<_{\mathcal{C}_1}) \subseteq \operatorname{dom}(<_{\mathcal{C}_2})$, and any $f\colon \operatorname{dom}(<_{\mathcal{C}_2}) \to \omega_1$ of the correct type induces (by restriction) an $f'\colon \operatorname{dom}(<_{\mathcal{C}_1}) \to \omega_1$ of the correct type. Thus, if we define $\pi(\ldots, \beta^{\vec{i}}, \ldots)_{\vec{i} \in \operatorname{dom}(\mathcal{C}_2)} = (\ldots, \beta^{\vec{i}}, \ldots)_{\vec{i} \in \operatorname{dom}(\mathcal{C}_1)}$, that is, we delete the appropriate components of the tuple, then $\pi(\nu(\mathcal{C}_2)) = \nu(\mathcal{C}_1)$. If $\operatorname{size}(\mathcal{C}_2) = \operatorname{size}(\mathcal{C}_1) + 1$, then we say $\mathcal{C}_2$ *immediately extends* $\mathcal{C}_1$.

If $\mathcal{C}$ is a type-1 tree of measures, we say a tuple of ordinals $(\ldots, \beta^{\vec{i}}, \ldots)_{\vec{i} \in \operatorname{dom}(\mathcal{C})}$ is *of type* $\mathcal{C}$ if there is an $f\colon \operatorname{dom}(<_{\mathcal{C}}) \to \omega_1$ of the correct type which represents $\vec{\beta}$ (i.e., $\beta^{(i_1,\ldots,i_k)} = [f^{(i_1,\ldots,i_k)}]_{\mathcal{W}_1^\pi}$, where $\pi = \mathcal{C}(i_1, \ldots, i_{k-1})$). Thus, $\nu(\mathcal{C})$ concentrates on tuples $\vec{\beta}$ of type $\mathcal{C}$.

Lemma 1.1 generalizes to orderings $<_{\mathcal{C}}$ for type-1 trees $\mathcal{C}$.

LEMMA 1.2. *Let $\mathcal{C}$ be a type-1 tree of measures, and $C \subseteq \omega_1$ be c.u.b. Let E be the subset of $\operatorname{dom}(<_{\mathcal{C}})$ consisting of sequences $\vec{s} = \langle \alpha_1, i_1, \ldots, \alpha_k, i_k \rangle$ with $\alpha_1, \ldots, \alpha_k \in C$. Suppose $f\colon E \to \omega_1$ is of the correct type. Then there is an $f'\colon \operatorname{dom}(<_{\mathcal{C}}) \to \omega_1$ of the correct type such that $\operatorname{ran}(f') \subseteq \operatorname{ran}(f)$ and $[f'] = [f]$.*

PROOF. Define $l\colon \omega_1 \to \omega_1$ by $l(\alpha) = \alpha^{th}$ element of C. For $\vec{s} = \langle \alpha_1, i_1, \ldots, \alpha_k, i_k \rangle \in \operatorname{dom}(<_{\mathcal{C}})$, define $f'(\vec{s}) = f(\langle l(\alpha_1), i_1, \ldots, l(\alpha_k), i_k \rangle)$. This easily works. $\square$

Finally, we define the family $\mathcal{W}_3$.

DEFINITION 1.9. A type-2 tree of measures $\mathcal{T}$ is a function with domain a non-empty finite subtree of $\omega^{<\omega}$ satisfying:

(1) For $(i_1) \in \operatorname{dom}(\mathcal{T})$, $\mathcal{T}(i_1)$ is a minimal type-1 tree of measures, that is, $\mathcal{T}(i_1)$ has size 1.
(2) If t_2 immediately extends t_1, then $\mathcal{T}(t_2)$ is a type-1 tree of measures which immediately extends the type-1 tree $\mathcal{T}(t_1)$.

We define depth, size, extension, and immediate extension as for type-1 trees.

If $\vec{j} = (j_1, \ldots, j_{l-1}) \in \operatorname{dom}(\mathcal{T})$, then $\nu(\mathcal{T}(\vec{j}))$ is a measure on $l-1$ tuples $(\ldots, \beta^{\vec{i}}, \ldots)_{\vec{i} \in \operatorname{dom}(\mathcal{T}(\vec{j}))}$. Moreover, the sequence $\mathcal{T}(j_1)$, $\mathcal{T}(j_1, j_2)$, …, $\mathcal{T}(j_1, \ldots, j_{l-1})$ determines an order of enumeration of the components $\beta^{\vec{i}}$ of these tuples (namely, $\beta^{\vec{i}}$ is the p^{th} component if $\vec{i} \in \operatorname{dom}(\mathcal{T}(j_1, \ldots, j_p)) - \operatorname{dom}(\mathcal{T}(j_1, \ldots, j_{p-1}))$). We say a sequence $(\beta_1, \ldots, \beta_{l-1})$ is of type $\mathcal{T}(j_1, \ldots, j_{l-1})$ if it enumerates, according to this order, a tuple $\vec{\beta}$ of type $\mathcal{T}(j_1, \ldots, j_{l-1})$.

We also associate to each type-2 tree of measures $\mathcal{T}$ an ordering $<_{\mathcal{T}}$.

DEFINITION 1.10. Let $\mathcal{T}$ be a type-2 tree of measures. Then $<_{\mathcal{T}}$ is lexicographic ordering on sequences $\langle j_1, \gamma_1, \ldots, \gamma_{k-1}, j_k \rangle$ satisfying:

(1) $(j_1, \ldots, j_k) \in \operatorname{dom}(\mathcal{T})$.
(2) $(\gamma_1 \ldots, \gamma_{k-1})$ is of type $\mathcal{T}(j_1, \ldots, j_{k-1})$.

Note that $\gamma_1 < \omega_2$, and all the γ_i are less than ω_ω. Using the weak partition property on $\boldsymbol{\delta}^1_3$, we now associate to the type-2 tree $\mathcal{T}$ a measure $\nu(\mathcal{T})$ on $(\boldsymbol{\delta}^1_3)^{<\omega}$ as follows.

DEFINITION 1.11. Let $\mathcal{T}$ be a type-2 tree of measures, and $<_{\mathcal{T}}$ the corresponding ordering. Then A has $\nu(\mathcal{T})$ measure one iff $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ $\forall f \colon \operatorname{dom}(<_{\mathcal{T}}) \to C$ of the correct type, $\vec{\delta} = (\ldots, \delta^{\vec{j}}, \ldots)_{\vec{j} \in \operatorname{dom}(\mathcal{T})} \in A$. Here, for $\vec{j} = (j_1, \ldots, j_l) \in \operatorname{dom}(\mathcal{T})$, $\delta^{\vec{j}}$ is represented with respect to $\nu(\mathcal{T}(j_1, \ldots, j_{l-1}))$ by the function $f^{\vec{j}}$ where for $(\gamma_1, \ldots, \gamma_{l-1})$ of type $\mathcal{T}(j_1, \ldots, j_{l-1})$,

$$f^{(j_1, \ldots, j_l)}(\gamma_1, \ldots, \gamma_{l-1}) = f(\langle j_1, \gamma_1, \ldots, j_{l-1}, \gamma_{l-1}, j_l \rangle).$$

If $f \colon \operatorname{dom}(<_{\mathcal{T}}) \to ON$, we will usually write $[f]$ for the tuple of ordinals $(\ldots, \beta^{\vec{j}}, \ldots)_{\vec{j} \in \operatorname{dom}(\mathcal{T})}$ represented by f.

Again we have that if $\mathcal{T}_2$ extends $\mathcal{T}_1$, then $\operatorname{dom}(<_{\mathcal{T}_1}) \subseteq \operatorname{dom}(<_{\mathcal{T}_2})$, and there is a natural map π from $\operatorname{dom}(\nu(\mathcal{T}_2))$ to $\operatorname{dom}(\nu(\mathcal{T}_1))$ such that $\pi(\nu(\mathcal{T}_2)) = \nu(\mathcal{T}_1)$.

We prove two embedding theorems; a "local" and a "global" embedding theorem ("local" refers to local or pointwise below $\boldsymbol{\delta}^1_3$ behavior; global refers to a certain behavior at $\boldsymbol{\delta}^1_3$). These theorems have the effect of saying that the canonical families of measures "dominate" the special homogeneous measures introduced above. We consider the local version first.

THEOREM 1.1 (Local Embedding Theorem). *Let $\mathcal{C}$ be a type-1 tree of measures. Then there is a measure S^m_1 in the canonical family S_1 and a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$, $j_{\mathcal{C}}(\alpha) \leq j_m(\alpha)$, where these refer to the embeddings from the ultrapowers by the measures $\nu(\mathcal{C})$, S^m_1 respectively.*

PROOF. We fix a type-1 tree $\mathcal{C}$ and corresponding measure $\nu(\mathcal{C})$. We take $m > \operatorname{depth}(\mathcal{C})$ and claim the theorem is true for $\nu(\mathcal{C})$ and S^m_1. We introduce an auxiliary hypothesis from which the theorem follows easily.

(A): Let $\mathcal{C}$, m be as above. Then there is a measure ν on $(\omega_\omega)^{<\omega}$ and a function $h \colon \omega_{m+1} \to (\omega_\omega)^{<\omega}$ satisfying the following:

(1) If $\nu(\mathcal{C})(A) = 1$ then $\forall^*_{S^m_1} \alpha \, \forall^*_\nu \vec{\beta} \, [h(\alpha)(\vec{\beta}) \in A]$ (recall the notational convention mentioned in the introduction). Written out explicitly, this becomes: if $A \subseteq \operatorname{dom}(\nu(\mathcal{C}))$ has $\nu(\mathcal{C})$ measure one, then for S^m_1 almost all α, if h_α represents $h(\alpha)$ with respect to ν, then for ν almost all $\vec{\beta}$, $h_\alpha(\vec{\beta}) \in A$.
(2) $\forall^*_{S^m_1} \alpha \ \exists \vec{\gamma} \in \operatorname{dom}(\nu(\mathcal{C})) \ \forall^*_\nu \vec{\beta} \ [h(\alpha)(\vec{\beta}) < \vec{\gamma}]$. By this we mean that if $g \colon \operatorname{dom}(<_{\mathcal{C}}) \to \omega_1$ of the correct type represents $\vec{\gamma}$, and if $f \colon \operatorname{dom}(<_{\mathcal{C}}) \to \omega_1$ represents $h(\alpha)(\vec{\beta})$, then for all $(i_1) \in \operatorname{dom}(\mathcal{C})$, $[f^{(i_1)}]_{W^1_1} < [g^{(i_1)}]_{W^1_1}$.

First, we show that the theorem follows from (A). Let $C \subseteq \boldsymbol{\delta}^1_3$ be c.u.b. consisting of limit ordinals and closed under j_ν, the embedding from the ultrapower by ν. For fixed $\delta \in C$, we define an embedding Π from $j_{\mathcal{C}}(\delta)$ into $j_m(\delta)$. Let $\eta < j_{\mathcal{C}}(\delta)$ and fix $f \colon \operatorname{dom}(\nu(\mathcal{C})) \to \delta$ representing η. We set $\Pi(\eta) = [g]_{S^m_1}$ where for $\alpha < \omega_{m+1}$, $g(\alpha)$ is represented with respect to ν by the function $f \circ h_\alpha$, where h_α

is as in (A). More succinctly, we could write $\forall^*_{S_1^m}\alpha\ \forall^*_\nu\vec{\beta}\ (\Pi(\eta))(\alpha,\vec{\beta}) = f(h(\alpha)(\vec{\beta}))$. From (A1), it follows easily that Π is well-defined and order-preserving. From (A2) it follows that $\Pi(\eta) < j_m(\delta)$. We use here the fact that if $h\colon \operatorname{dom}(\nu(\mathcal{C})) \to ON$, then there is a $\nu(\mathcal{C})$ measure one set A such that if $f_1, f_2\colon \operatorname{dom}(<_{\mathcal{C}}) \to \omega_1$ of the correct type represent $\vec{\gamma}_1, \vec{\gamma}_2$ in A, and $\forall (i_1) \in \operatorname{dom}(\mathcal{C})\ [f_1^{(i_1)}]_{W_1^1} < [f_2^{(i_1)}]_{W_1^1}$, then $h(\vec{\gamma}_1) \leq h(\vec{\gamma}_2)$. This latter fact follows from an easy partition argument on ω_1 using lemma 1.2, which we leave to the reader.

We establish (A) by direct construction of the measure ν and function h. Without loss of generality, we may assume that $\operatorname{dom}(\mathcal{C})$ is a *standard tree*, that is, for every non-maximal $\vec{i} \in \operatorname{dom}(\mathcal{C})$, $\{k\colon \vec{i}^\frown k \in \operatorname{dom}(\mathcal{C})\}$ is an initial segment of ω. Let $d = \operatorname{depth}(\mathcal{C})$, $s = \operatorname{size}(\mathcal{C})$, and recall $m > d$. Let

$$\nu = W_1^s \times [S_1^1]^s \times [S_1^2]^s \times \cdots \times [S_1^{m-2}]^s.$$

Here, $[S_1^i]^s$ denotes the s-fold product of the measure S_1^i. Suppose $\alpha = [f]_{W_1^m}$, where $f\colon \operatorname{dom}(<_m) \to \omega_1$ is of the correct type, and we define a function h_α representing $h(\alpha)$ with respect to ν (it is irrelevant what h does off the measure one set of α so representable). Let

$$\vec{\beta} = (\beta_0, \ldots, \beta_s, (\beta_1^1, \ldots, \beta_s^1), \ldots, (\beta_1^{m-2}, \ldots, \beta_s^{m-2})) \in \operatorname{dom}(\nu),$$

where $\beta_0 < \beta_1 < \cdots < \beta_s$, each $\beta_j^i = [g_j^i]_{W_1^i}$ where $g_j^i\colon \operatorname{dom}(<_i) \to \omega_1$ is of the correct type, and whenever g_j^i precedes $g_{j'}^{i'}$ in this enumeration, then $g_j^i(1)(\alpha) < g_{j'}^{i'}(\alpha_1, \ldots, \alpha_{i'-1}, \alpha)$ for all $\alpha_1 < \cdots < \alpha < \omega_1$. Also, assume $\inf(g_1^1) > \beta_s$. Again, the set of $\vec{\beta}$ so representable has ν measure one, and it is irrelevant how we define h_α off this set. We define a function $h_\alpha(\vec{\beta})(-)\colon \operatorname{dom}(<_{\mathcal{C}}) \to \omega_1$ representing $h_\alpha(\vec{\beta})$. Let $D \subseteq \omega_1$ be a c.u.b. set closed under all the functions $g_j^i(1)$. For $\alpha_1 \in D$ and $\vec{s} = \langle \alpha_1, i_1, \ldots, \alpha_k, i_k\rangle \in \operatorname{dom}(<_{\mathcal{C}})$, define

$$(1)\quad h_\alpha(\vec{\beta})(\vec{s}) = f(\beta_{i_1}, g_{i_2}^1(\alpha_2), g_{i_3}^2(\pi(\alpha_2,\alpha_3)), \ldots, g_{i_k}^{k-1}(\pi(\alpha_2,\ldots,\alpha_k)), \delta^k, \ldots, \delta^{m-2}, \alpha_1),$$

where $\delta^l = g_1^l(k-1)(\pi(\alpha_2,\ldots,\alpha_k))$, and $\pi(\vec{\alpha})$ denotes the permutation of $\vec{\alpha}$ which is increasing. Note that $k \leq d < m$, so this makes sense. Also, by assumption on the g_j^i and α_1, the sequence of ordinals being plugged into f is increasing.

First note that if $[g_j^i] = [\tilde{g}_j^i]$ are two sequences of functions both representing $\vec{\beta}$, and $h, \tilde{h}$ are the corresponding functions, then $[h] = [\tilde{h}]$. That is, there is a c.u.b. $C \subseteq \omega_1$ such that if $\vec{s} = \langle \alpha_1, i_1, \ldots, \alpha_k, i_k\rangle \in \operatorname{dom}(<_{\mathcal{C}})$ and $\alpha_1, \ldots, \alpha_k \in C$, then $h(\vec{s}) = \tilde{h}(\vec{s})$. Thus, for fixed f representing α, $h_\alpha(\vec{\beta})$ is well-defined. Suppose next that $[f] = [\tilde{f}] = \alpha$. If C defines a measure one set on which f, $\tilde{f}$ agree, then ν almost all $\vec{\beta}$ are representable by g_j^i having range in C (and $\beta_1, \ldots, \beta_s \in C$). For such $\vec{\beta}$ we clearly have $h_\alpha(\vec{\beta}) = \tilde{h}_\alpha(\vec{\beta})$, where $h, \tilde{h}$ are defined using $f, \tilde{f}$. Thus, $h(\alpha)$ is well-defined, that is, does not depend on the choice of f representing α.

To verify (A1), let $C \subseteq \omega_1$ be c.u.b., and A the set of $\vec{\gamma}$ representable by an $h\colon \operatorname{dom}(<_{\mathcal{C}}) \to C$ of the correct type. Let $\alpha < \omega_{m+1}$ be represented by $f\colon \operatorname{dom}(<_m) \to C$ of the correct type. Let $\vec{\beta}$ be represented by g_j^i satisfying the previously mentioned conditions. Let $D \subseteq C$ be c.u.b. and closed under the $g_j^i(1)$. If $E \subseteq \operatorname{dom}(<_{\mathcal{C}})$ consists of the $\vec{s} = \langle \alpha_1, i_1, \ldots, \alpha_k, i_k\rangle$ with $\alpha_1, \ldots, \alpha_k \in D$, then the function $h\colon E \to \omega_1$ defined by equation (1) is order-preserving, of the correct type, and has range in C. By lemma 1.2, $h_\alpha(\vec{\beta}) = [h] \in A$.

For (A2), note that if $f\colon \operatorname{dom}(<_m) \to \omega_1$ of the correct type represents α, then for any $\vec{\beta}$ and representing functions g^i_j as above, the function h defined by equation (1) representing $h_\alpha(\vec{\beta})$ satisfies the following. For all $\vec{s} = \langle \alpha_1, i_1, \dots, \alpha_k, i_k\rangle \in \operatorname{dom}(<_{\mathcal{C}})$ (with α_1 in a certain c.u.b. set) $h(\vec{s}) < f(1)(\alpha_1)$. $\square$

The next theorem is our "global" embedding theorem. The proof is similar in spirit to that of the local theorem.

THEOREM 1.2 (Global Embedding Theorem). *Let $\mathcal{T}$ be a type-2 tree of measures and $\nu(\mathcal{T})$ the corresponding measure on $(\boldsymbol{\delta}^1_3)^{<\omega}$. Then for some $m \in \omega$, $j_{\nu(\mathcal{T})}(\boldsymbol{\delta}^1_3) \leq j_{W^m_3}(\boldsymbol{\delta}^1_3)$.*

PROOF. As with the local embedding theorem, we proceed through an auxiliary hypothesis:

(B): Let $\mathcal{T}$ be a type-2 tree of measures, and $<_{\mathcal{T}}$ the corresponding ordering. Let $m \geq \operatorname{depth}(\mathcal{T})$. Then there is a measure ν on $(\omega_\omega)^{<\omega}$ and a function h with domain $\operatorname{dom}(\nu)$ and range in tuples of ordinals $(\dots, \beta^{\vec{j}}, \dots)_{\vec{j} \in \operatorname{dom}(\mathcal{T})}$ satisfying the following:

(1) Suppose $A \subseteq \omega_{m+1}$ has S^m_1 measure one. Then for ν almost all $\vec{\alpha}$, if $h(\vec{\alpha}) = (\dots, \beta^{\vec{j}}, \dots)_{\vec{j} \in \operatorname{dom}(\mathcal{T})}$, and for $\vec{j} = (j_1, \dots, j_l) \in \operatorname{dom}(\mathcal{T})$, the function $h^{\vec{j}}_{\vec{\alpha}}\colon \operatorname{dom}(\nu(\mathcal{T}(j_1, \dots, j_{l-1}))) \to ON$ represents $\beta^{\vec{j}}$ with respect to $\nu(\mathcal{T}(j_1, \dots, j_{l-1}))$, then for $\nu(\mathcal{T}(j_1, \dots, j_{l-1}))$ almost all $\vec{\gamma} = (\gamma_1, \dots, \gamma_{l-1})$ we have $h^{\vec{j}}_{\vec{\alpha}}(\vec{\gamma}) \in A$. [If $l = 1$, then we require that $h^{(j_1)}_{\vec{\alpha}} \doteq \beta^{(j_1)} \in A$.]

(2) For almost all $\vec{\alpha}$ with respect to ν, if $h(\vec{\alpha}) = (\dots, \beta^{\vec{j}}, \dots)_{\vec{j} \in \operatorname{dom}(\mathcal{T})}$ and the functions $h^{\vec{j}}_{\vec{\alpha}}$ represent the $\beta^{\vec{j}}$ as above, then the following holds. Define $r\colon <_{\mathcal{T}} \to ON$ by $r(\langle j_1, \gamma_1, \dots, \gamma_{l-1}, j_l\rangle) = h^{(j_1, \dots, j_l)}_{\vec{\alpha}}(\gamma_1, \dots, \gamma_{l-1})$. Then there are measure one sets $A^{(j_1, \dots, j_l)}$ with respect to $\nu(\mathcal{T}(j_1, \dots, j_{l-1}))$ such that r restricted to those tuples $\vec{s} \in \operatorname{dom}(<_{\mathcal{T}})$ with $(\gamma_1, \dots, \gamma_{l-1}) \in A^{(j_1, \dots, j_l)}$ is order-preserving.

We first establish the theorem assuming (B). We produce an embedding Π from the ultrapower of $\boldsymbol{\delta}^1_3$ by $\nu(\mathcal{T})$ into the ultrapower of $\boldsymbol{\delta}^1_3$ by W^m_3, with m, ν chosen as in (B). Let $F\colon \operatorname{dom}(\nu(\mathcal{T})) \to \boldsymbol{\delta}^1_3$ represent an equivalence class with respect to $\nu(\mathcal{T})$. We define $G\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ representing $\Pi([F])$ with respect to W^m_3. Fix $g\colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type representing $[g]_{S^m_1}$. We define $G([g]_{S^m_1})$ to be the ordinal represented with respect to ν by the function $G([g])(-)\colon \operatorname{dom}(\nu) \to \boldsymbol{\delta}^1_3$ defined as follows. Let $\vec{\alpha} \in \operatorname{dom}(\nu)$. Say $h(\vec{\alpha}) = (\dots, \beta^{\vec{j}}, \dots)_{\vec{j} \in \operatorname{dom}(\mathcal{T})}$, and let the functions $h^{\vec{j}}_{\vec{\alpha}}$ represent the $\beta^{\vec{j}}$. If there is a function $t\colon \operatorname{dom}(<_{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of the correct type such that for all $\vec{j} = (j_1, \dots, j_l) \in \operatorname{dom}(\mathcal{T})$ we have $[t^{\vec{j}}]_{\nu(\mathcal{T}(j_1, \dots, j_{l-1}))} = [g \circ h^{\vec{j}}_{\vec{\alpha}}]_{\nu(\mathcal{T}(j_1, \dots, j_{l-1}))}$, then we set $G([g])(\vec{\alpha}) = F([t])$ (recall $t^{\vec{j}}(\gamma_1, \dots, \gamma_{l-1}) = t(\langle j_1, \gamma_1, \dots, \gamma_{l-1}, j_l\rangle)$). Otherwise, set $G([g])(\alpha) = 0$. For fixed g, clearly $G([g])(\vec{\alpha})$ is well-defined, that is, does not depend on the choice of the $h^{\vec{j}}_{\vec{\alpha}}$. From (B1) it follows that if $[g_1]_{S^m_1} = [g_2]_{S^m_1}$, then $G([g_1]) = G([g_2])$. Suppose now that $[F_1]_{\nu(\mathcal{T})} = [F_2]_{\nu(\mathcal{T})}$, and let G_1, G_2 be the functions defined using F_1, F_2 respectively. Let $C \subseteq \boldsymbol{\delta}^1_3$ be c.u.b. such that for all $t\colon \operatorname{dom}(<_{\mathcal{T}}) \to C$ of the correct type we have $F_1([t]) = F_2([t])$. Let $D \subseteq C$ consist of limit points of C, and every $\alpha \in D$ is the α^{th} element of C. Let $g\colon \omega_{m+1} \to D$ be of the correct type. By (B2), for ν almost all $\vec{\alpha}$, if $h(\vec{\alpha}) = (\dots, \beta^{\vec{j}}, \dots)_{\vec{j} \in \mathrm{dom}(\mathcal{T})}$ and the $h^{\vec{j}}_{\vec{\alpha}}$ represent the $\beta^{\vec{j}}$, then there are measure one sets $A^{\vec{j}}$ such that $r\colon E \subseteq \mathrm{dom}(<_{\mathcal{T}}) \to \omega_{m+1}$ is order-preserving. Here $E = \{\langle j_1, \gamma_1, \dots, \gamma_{l-1}, j_l\rangle \colon \vec{\gamma} \in A^{(j_1,\dots,j_l)}\}$. Also, $r(\langle j_1, \gamma_1, \dots, \gamma_{l-1}, j_l\rangle) = h^{(j_1,\dots,j_l)}_{\vec{\alpha}}(\gamma_1, \dots, \gamma_{l-1})$. Thus, $g \circ r$ is also order-preserving, has range in D, and has uniform cofinality ω. Easily, we may extend $g \circ r$ to a $t\colon \mathrm{dom}(<_{\mathcal{T}}) \to C$ of the correct type whence for all $\vec{j} \in \mathrm{dom}(\mathcal{T})$, $[t^{\vec{j}}] = [g \circ r^{\vec{j}}] = [g \circ h^{\vec{j}}_{\vec{\alpha}}]$, these equivalence classes with respect to $\nu(\mathcal{T}(j_1, \dots, j_{l-1}))$. Thus, $G_1([g]) = F_1([t]) = F_2([t]) = G_2([g])$. This shows that $[G_1]_{W_3^m} = [G_2]_{W_3^m}$. Thus Π is well-defined, and the same argument shows Π is order-preserving.

We proceed to establish (B). We fix the type-2 tree of measures $\mathcal{T}$ and corresponding ordering $<_{\mathcal{T}}$. Let $m \geq \mathrm{depth}(\mathcal{T})$. We first define the measure ν.

Fix for the moment $\vec{j}' = \langle j_1, \dots, j_{l-1}\rangle$ which is non-maximal in $\mathrm{dom}(\mathcal{T})$. Recall $\mathcal{T}(\vec{j}')$ is a type-1 tree of measures. We define a measure $\mu^{\vec{j}'} \in \mathcal{W}_1$. Let $a \in \{1, \dots, l-1\}$, and let $\vec{i}_a$ be the unique tuple in $\mathrm{dom}(\mathcal{T}(j_1, \dots, j_a)) - \mathrm{dom}(\mathcal{T}(j_1, \dots, j_{a-1}))$. Say $\vec{i}_a = (i_1, \dots, i_k)$. If $f\colon \mathrm{dom}(<_{\mathcal{T}(\vec{j}')}) \to \omega_1$ is of the correct type, and $\epsilon_1 < \cdots < \epsilon_{m-1} < \omega_1$, we define $\theta^{\vec{j}'}_a(f, \vec{\epsilon}) = f(\langle \delta_1, i_1, \dots, \delta_k, i_k\rangle)$, where $\vec{\delta}$ is the permutation of $(\epsilon_{m-k}, \dots, \epsilon_{m-1})$ of type $\mathcal{T}(\vec{j}')(i_1, \dots, i_{k-1})$. This makes sense since $k \leq \mathrm{depth}(\mathcal{T}(\vec{j}')) < \mathrm{depth}(\mathcal{T}) = m$. Define now a permutation σ of $\{1, \dots, l-1\}$ as follows. For $a, b \in \{1, \dots, l-1\}$, set $a <_\sigma b$ iff $\exists$ c.u.b. $C \subseteq \omega_1$ $\forall f\colon \mathrm{dom}(<_{\mathcal{T}(\vec{j}')}) \to C$ of the correct type $\forall^*_{W_1^{m-1}} \vec{\epsilon}\ [\theta^{\vec{j}'}_a(f, \vec{\epsilon}) < \theta^{\vec{j}'}_b(f, \vec{\epsilon})]$. Let $\sigma = \sigma(\vec{j}')$ be the permutation defined by the ordering $<_\sigma$. Note that if $\vec{j}$ extends $\vec{j}'$, then the corresponding permutation σ extends σ'. Let $\mu^{\vec{j}'} \in \mathcal{W}_1$ be the measure on tuples $(\epsilon_m, \epsilon_1, \dots, \epsilon_{m-1}, \eta_1, \dots, \eta_{l-1})$ of countable ordinals of the following type: $\epsilon_1 < \cdots < \epsilon_{m-1} < \inf\{\eta_i\}$, $\epsilon_m > \sup\{\eta_i\}$, and $(\eta_1, \dots, \eta_{l-1})$ is of type σ. We say such a tuple is of type $\mu^{\vec{j}'}$.

Let $\prec$ be lexicographic ordering on sequences $\langle \epsilon_m, \epsilon_1, \dots, \epsilon_{m-1}, j_1, \eta_1, \dots, \eta_{l-1}, j_l\rangle$ satisfying:

(1) $\vec{j} = (j_1, \dots, j_l) \in \mathrm{dom}(\mathcal{T})$.
(2) $\epsilon_m < \omega_1,\ \epsilon_m > \epsilon_1, \dots, \epsilon_{m-1}, \eta_1, \dots, \eta_{l-1}$.

We let ν be the measure on tuples $\vec{\alpha} = (\dots, \alpha^{\vec{j}}, \dots)_{\vec{j} \in \mathrm{dom}(\mathcal{T})}$ defined as follows. Define $\nu(A) = 1$ iff $\exists$ c.u.b. $C \subseteq \omega_1$ $\forall g\colon \mathrm{dom}(\prec) \to C$ of the correct type, $\vec{\alpha} \in A$. Here, for $\vec{j} = (j_1, \dots, j_l) \in \mathrm{dom}(\mathcal{T})$, and $\vec{j}'$ denoting $(j_1, \dots, j_{l-1})$, $\alpha^{\vec{j}}$ is represented with respect to $\mu^{j'}$ by the function $g^{\vec{j}}$, where for $\vec{s} = (\epsilon_m, \epsilon_1, \dots, \epsilon_{m-1}, \eta_1, \dots, \eta_{l-1})$ of type $\mu^{j'}$ we set $g^{\vec{j}}(\vec{s}) = g(\langle \epsilon_m, \epsilon_1, \dots, \epsilon_{m-1}, j_1, \eta_1, \dots, \eta_{l-1}, j_l\rangle)$.

We define the function h as in (B). Fix $\vec{\alpha} \in \mathrm{dom}(\nu)$, and fix $g\colon \mathrm{dom}(\prec) \to \omega_1$ of the correct type representing $\vec{\alpha}$ as in the previous paragraph. We define the tuple $h(\vec{\alpha}) = (\dots, \beta^{\vec{j}}, \dots)_{\vec{j} \in \mathrm{dom}(\mathcal{T})}$. Fix $\vec{j} = (j_1, \dots, j_l) \in \mathrm{dom}(\mathcal{T})$, and we define a function $h^{\vec{j}}_{\vec{\alpha}}$ representing $\beta^{\vec{j}}$ with respect to $\nu(\mathcal{T}(j'))$, where again $\vec{j}' = (j_1, \dots, j_{l-1})$. Let $(\gamma_1, \dots, \gamma_{l-1}) \in \mathrm{dom}\, \nu(\mathcal{T}(\vec{j}'))$, and suppose $f\colon \mathrm{dom}(<_{\mathcal{T}(\vec{j}')}) \to \omega_1$ of the correct type represents $\vec{\gamma}$. Set $h^{\vec{j}}_{\vec{\alpha}}(\vec{\gamma}) = [H]_{W_1^m}$, where

$$H(\epsilon_1, \dots, \epsilon_m) = g(\langle \epsilon_m, \epsilon_1, \dots, \epsilon_{m-1}, j_1, \eta_1, \dots, \eta_{l-1}, j_l\rangle) \tag{2}$$

and $\eta_a = \theta_a^{\vec{j}'}(f,(\epsilon_1,\ldots,\epsilon_{m-1}))$. Clearly this does not depend on the choice of f representing $\vec{\gamma}$.

We first show that this is well-defined. Suppose $g_1, g_2\colon \operatorname{dom}(\prec) \to \omega_1$ are of the correct type and both represent $\vec{\alpha}$. That is, for all $\vec{j} = (j_1,\ldots,j_l) \in \operatorname{dom}(\mathcal{T})$ we have that $g_1^{\vec{j}}$ and $g_2^{\vec{j}}$ agree $\mu^{\vec{j}'}$ almost everywhere, where again $\vec{j}' = (j_1,\ldots,j_{l-1})$. If $h_1(\vec{\alpha})$, $h_2(\vec{\alpha})$ are defined using g_1, g_2 respectively, then we must show that for all $\vec{j} = (j_1,\ldots,j_l) \in \operatorname{dom}(\mathcal{T})$ that $(h_1)_{\vec{\alpha}}^{\vec{j}}(\vec{\gamma}) = (h_2)_{\vec{\alpha}}^{\vec{j}}(\vec{\gamma})$ for $\nu(\mathcal{T}(\vec{j}'))$ almost all $(\gamma_1,\ldots,\gamma_{l-1})$, where as usual, $\vec{j}' = (j_1,\ldots,j_{l-1})$. Let $C \subseteq \omega_1$ define measure one sets with respect to the $\mu^{\vec{j}'}$, $\vec{j} \in \operatorname{dom}(\mathcal{T})$, on which $g_1^{\vec{j}}$, $g_2^{\vec{j}}$ agree. Fix $\vec{j} = (j_1,\ldots,j_l) \in \operatorname{dom}(\mathcal{T})$, and let $\vec{\gamma} = (\gamma_1,\ldots,\gamma_{l-1})$ be represented by $f\colon \operatorname{dom}(<_{\mathcal{T}(\vec{j}')}) \to C$ of the correct type. We may also assume f is such that $\forall^*_{W_1^{m-1}}\vec{\epsilon}\ (\theta_1^{\vec{j}'}(f,\vec{\epsilon}),\ldots,\theta_{l-1}^{\vec{j}'}(f,\vec{\epsilon}))$ is order-isomorphic to $\sigma^{\vec{j}'}$. Then for almost all $\epsilon_1,\ldots,\epsilon_m$ in C we have $(\epsilon_m,\epsilon_1,\ldots,\epsilon_{m-1},\theta_1^{\vec{j}'}(f,\vec{\epsilon}\restriction m-1),\ldots,\theta_{l-1}^{\vec{j}'}(f,\vec{\epsilon}\restriction m-1))$ is of type $\mu^{\vec{j}'}$, and all the ordinals of this tuple lie in C. Thus, $g_1^{\vec{j}}$, $g_2^{\vec{j}}$ agree on this tuple, and this shows that $(h_1)_{\vec{\alpha}}^{\vec{j}}(\vec{\gamma}) = (h_2)_{\vec{\alpha}}^{\vec{j}}(\vec{\gamma})$.

To verify (B1), let $C \subseteq \omega_1$ be c.u.b. defining a S_1^m measure one set A. Fix $g\colon \operatorname{dom}(\prec) \to C$ of the correct type representing $\vec{\alpha}$, and let $h(\vec{\alpha}) = (\ldots,\beta^{\vec{j}},\ldots)_{\vec{j}\in\operatorname{dom}(\mathcal{T})}$. Fix $\vec{j} = (j_1,\ldots,j_l) \in \operatorname{dom}(\mathcal{T})$. Then $\beta^{\vec{j}}$ is represented with respect to $\nu(\mathcal{T}(j'))$ by $h_{\vec{\alpha}}^{\vec{j}}(\vec{\gamma})$, where for almost all $\vec{\gamma}$ represented by $f\colon \operatorname{dom}(<_{\mathcal{T}(\vec{j}')}) \to \omega_1$ of the correct type we have $h_{\vec{\alpha}}^{\vec{j}}(\vec{\gamma}) = [H]_{W_1^m}$, and for W_1^m almost all $\vec{\epsilon}$, $H(\vec{\epsilon})$ is given by equation (2). Since g has range in C, so does H. Since g has uniform cofinality ω and is discontinuous, the same also holds for H (almost everywhere). From the definition of $\prec$ and equation (2) we also see that H is almost everywhere order-preserving with respect to $<_m$. From lemma 1.1, there is an $H'\colon \operatorname{dom}(<_m) \to C$ of the correct type with $[H']_{W_1^m} = [H]_{W_1^m}$. Thus, $[H]_{W_1^m} \in A$.

To verify (B2), Fix again $g\colon \operatorname{dom}(\prec) \to C$ of the correct type representing $\vec{\alpha}$, and let $h(\vec{\alpha}) = (\ldots,\beta^{\vec{j}},\ldots)_{\vec{j}\in\operatorname{dom}(\mathcal{T})}$. Let $\vec{j} = (j_1,\ldots,j_l) \in \operatorname{dom}(\mathcal{T})$, and suppose $(\gamma_1,\ldots,\gamma_{l-1})$, $(\gamma_1',\ldots,\gamma_{l-1}')$ are represented by $f_1, f_2\colon \operatorname{dom}(<_{\mathcal{T}(\vec{j}')}) \to \omega_1$ of the correct type, with $\vec{j}' = (j_1,\ldots,j_{l-1})$. Suppose also that $\gamma_1 = \gamma_1'$, …, $\gamma_{l-1} < \gamma_{l-1}'$, and we show that $h_{\vec{\alpha}}^{\vec{j}}(\vec{\gamma}) < h_{\vec{\alpha}}^{\vec{j}}(\vec{\gamma}')$, the other cases being similar. Then for W_1^{m-1} almost all $\epsilon_1,\ldots,\epsilon_{m-1}$, $\theta_1^{\vec{j}'}(f_1,\vec{\epsilon}) = \theta_1^{\vec{j}'}(f_2,\vec{\epsilon})$, …, $\theta_{l-1}^{\vec{j}'}(f_1,\vec{\epsilon}) < \theta_{l-1}^{\vec{j}'}(f_2,\vec{\epsilon})$. Since g is order-preserving, if H_1, H_2 are defined using f_1, f_2 respectively, then $[H_1]_{W_1^m} < [H_2]_{W_1^m}$, and we are done. This establishes (B) and the theorem. $\square$

Recall now that the homogeneous tree construction (see [**6**]) gives that $\boldsymbol{\delta}_5^1 \leq (\sup j_\nu(\boldsymbol{\delta}_3^1))^+$ where ν ranges over the measures on $\boldsymbol{\delta}_3^1$ occurring in the homogeneous tree construction for a complete $\boldsymbol{\Pi}_3^1$ set. Hence, we obtain as an immediate corollary to the global embedding theorem:

COROLLARY 1.1. $\boldsymbol{\delta}_5^1 \leq \left(\sup_m j_{W_3^m}(\boldsymbol{\delta}_3^1)\right)^+$.

CHAPTER 2

Martin's Theorem

This chapter is devoted to proving the following theorem of Martin [**7**] referred to in the introduction.

THEOREM 2.1. *Let* $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ *be given, and* $\kappa < \boldsymbol{\delta}^1_3$ *a regular cardinal, that is,* $\kappa = \omega, \omega_1,$ *or* ω_2. *Then there is a tree* T *on* $\boldsymbol{\delta}^1_3$ *such that for almost all* α *with respect to the* κ*-cofinal normal measure on* $\boldsymbol{\delta}^1_3$, $F(\alpha) < |T \restriction \sup_j j(\alpha)|$, *where the supremum ranges over embeddings corresponding to ultrapowers by measures occurring in the homogeneous tree construction on a complete* $\boldsymbol{\Pi}^1_1$ *or* $\boldsymbol{\Pi}^1_2$ *set, depending on whether* $\kappa = \omega_1$ *or* $\kappa = \omega_2$ *(if* $\kappa = \omega$, *we use* $|T \restriction \alpha|$ *in the formula above).*

REMARK 2.1. For $\kappa = \omega$ the result is due to Kunen and is easier – a proof appears in [**5**]. In fact, the argument is essentially identical to that for the corresponding result on ω_1 which is sketched below.

We develop first some facts pertaining to the coding of ordinals $< \kappa$, where κ is either ω_1 or ω_2. Most of these facts, including the Kunen tree analysis, are well-known, but we include them for the sake of completeness.

If $T \subseteq (\omega \times \lambda)^{<\omega}$ is a tree, and $x \in \omega^\omega$, recall $T_x = \{f \in \lambda^\omega\colon \forall n\ (x \restriction n, f \restriction n) \in T\}$ denotes the section of the tree at x. Similarly, if $s \in \omega^{<\omega}$, then $T_s = \{\vec{\alpha}\colon (s, \vec{\alpha}) \in T\}$. Both of these notions extend in an obvious manner to trees on $\omega \times \omega \times \lambda$, etc. Recall $[T]$ denotes the set of infinite branches through T and $p[T]$ its projection on ω^ω.

Recall a tree $T \subseteq (\omega \times \lambda)^{<\omega}$ is *homogeneous* if there are measures μ_s for $s \in \omega^{<\omega}$ with $T_s \neq \emptyset$, such that $\mu_s(T_s) = 1$, if t extends s then $\pi^t_s(\mu_t) = \mu_s$ where $\pi^t_s\colon \lambda^{\mathrm{lh}(t)} \to \lambda^{\mathrm{lh}(s)}$ is the natural projection $\pi^t_s(\vec{\alpha}) = (\vec{\alpha} \restriction \mathrm{lh}(s))$, and finally, the following homogeneity property is satisfied: for all $x \in \omega^\omega$, if $x \in p[T]$ and $A_1, A_2, \ldots$ are sets with $\mu_{x \restriction n}(A_n) = 1$ for all n, then $\exists f \in \lambda^\omega\ [(x, f) \in [T] \wedge \forall n\ f \restriction n \in A_n]$. We refer the reader to [**6**] for more information on homogeneous trees and their construction.

Let $LO \subseteq 2^\omega$ be the set of codes of characteristic functions for linear orderings of ω, i.e., $x \in LO$ iff the relation defined by $n <_x m \Leftrightarrow x(\langle n, m\rangle) = 1$ is a linear ordering. Likewise, let WO be the set of codes of characteristic functions for well-orderings of ω. LO is closed, and $WO \in \boldsymbol{\Pi}^1_1$. For $x \in WO$, let $|x| < \omega_1$ be the rank of the well-ordering $<_x$. The standard Shoenfield tree construction produces a homogeneous tree T on $\omega \times \omega_1$ projecting to WO. Let us recall one way of doing this. Let $a_1 = 1$ and for $i > 1$, $a_i = \sup_{j<i} \langle j, i\rangle + 1$. If $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of countable ordinals, let $\beta_1 = \alpha_{a_1}$, $\beta_2 = \alpha_{a_2}$, …, $\beta_k = \alpha_{a_k}$ be the subsequence of $\vec{\alpha}$ at coordinates $a_1, a_2, \ldots$. Let $\gamma_1, \ldots, \gamma_l$ denote the remaining components of $\vec{\alpha}$ (so $k + l = n$). If $s \in 2^{n+1}$, put $(s, (\alpha_0, \ldots, \alpha_n))$ in T iff $\exists x \in LO$ extending s and the following conditions are met (where $\vec{\beta}$, $\vec{\gamma}$ are defined relative to $(\alpha_1, \ldots, \alpha_n)$). We

require $\alpha_0 > \alpha_1, \ldots, \alpha_n$, $\gamma_1 < \cdots < \gamma_l$, $\gamma_l < \inf_{1 \leq j \leq k} \beta_j$, and for all $1 \leq j_1 < j_2 \leq k$ we have $\beta_{j_1} < \beta_{j_2}$ iff $s(\langle j_1, j_2 \rangle) = 1$. Easily, $WO = p[T]$. Also, T is homogeneous with measures in $\mathcal{W}_1$, as $\vec{\alpha} \in T_s$ iff $\vec{\alpha}$ is order-isomorphic to a certain permutation. Also, if $(x, f) \in [T]$, then $|x| \leq f(0)$.

We abstract these properties into the following definition.

Definition 2.1. Let κ be a regular cardinal, $P \subseteq \omega^\omega$, and $x \to |x|$ a norm on P with range cofinal in κ. If T is a homogeneous tree on $\omega \times \lambda$ for some $\lambda \in ON$, then we say $P, |\ |, T$ satisfy $(\star)$ provided $P = p[T]$ and $\forall (x, f) \in [T]\ |x| \leq f(0) < \kappa$.

Thus, the Shoenfield tree T constructed above along with WO and the usual norm on WO satisfy $(\star)$ for $\kappa = \omega_1$.

We consider now $\kappa = \omega_2$. The Shoenfield tree T above satisfies also the following property. If $\omega^2 \leq \alpha < \omega_1$ is a limit ordinal, $x \in WO$, and $|x| = \alpha$, then $T_x \restriction \alpha + 1$ is ill-founded. To see this, choose f so that the $\vec{\gamma}$ subsequence of f consists of integers, and $\beta_i = \omega + |\ i\ |_{<_x}$. We now construct the Kunen tree. Let $S \subseteq (\omega \times \omega)^{<\omega}$ be a tree with $p[S]$ a $\mathbf{\Sigma}^1_1$-complete set. Thus, $\sup\{|S_y| \colon S_y$ is well-founded $\} = \omega_1$. Define the tree $K \subseteq (\omega \times \omega \times \omega_1 \times \omega \times \omega)^{<\omega}$ as follows. Put $(s, t, \vec{\alpha}, u, v)$ in K iff $\exists z, x, y$ extending s, t, u $[z(x) = y \wedge (t, \vec{\alpha}) \in T \wedge (u, v) \in S]$. Here we are viewing every $z \in \omega^\omega$ as a strategy for II in an integer game. K is our Kunen tree, and has the following property. If $f \colon \omega_1 \to \omega_1$, then there is a z with K_z well-founded and such that for all limit $\alpha > \omega^2$, $|K_z \restriction (\alpha + 1)| > f(\alpha)$. [Play the game where I plays x, II plays y, and II wins iff $x \in WO \to (S_y$ is well-founded $\wedge |S_y| > f(|x|)$. II wins by boundedness, and a winning strategy for II, coded as a real z, produces the result.]

Remark 2.2. If we use the variant of T in which we delete the initial ordinals α_0 from the tree, and then construct K as above, we improve the inequality slightly to $|K_z \restriction \alpha| > f(\alpha)$. It is convenient to use our definition of T for the analysis below, however.

We now apply the homogeneous tree construction to the tree K to produce a tree $V \subseteq (\omega \times \omega_\omega)^{<\omega}$. We recall briefly this construction. Let $i \to (b_i, c_i, d_i)$ be a bijection from ω to the 3-tuples from $\omega^{<\omega} - \{\emptyset\}$ with $\mathrm{lh}(b_i) = \mathrm{lh}(c_i) = \mathrm{lh}(d_i) \doteq l_i + 1$. We may assume that if (b_j, c_j, d_j) properly extends (b_i, c_i, d_i), then $j > i$. Order the 4-tuples (b, α, c, d) from $(\omega \times \omega_1 \times \omega \times \omega)$ by: $(b, \alpha, c, d) < (b', \alpha', c', d')$ iff $(\alpha, b, c, d) <_{lex} (\alpha', b', c', d')$. For every $s \in \omega^{<\omega}$, this ordering extends to the Kleene-Brouwer order $\prec_s$ on K_s. If $(b, \alpha, c, d) \in K_s$, it is convenient to also regard α as being in the domain of $\prec_s$ (thus, $\alpha \succ_s (b, \alpha, c, d)$ for any b, c, d). For $s \in \omega^{<\omega}$, let μ_s be the homogeneity measure on T_s. Put $(s, (\beta_0, \ldots, \beta_{n-1})) \in V$ iff $\exists f \colon \mathrm{dom}(\prec_s) \to \omega_1$ of the correct type such that $\forall 1 \leq i < n\ \beta_i = [f^i]_{\mu_{b_i}}$. Here, for $i > 0$ and $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{l_i}) \in \mathrm{dom}(\mu_{b_i})$, $f^i(\vec{\alpha}) = f(b_i, \vec{\alpha}, c_i, d_i)$. Also, we require that $\beta_0 = [f^0]_{W^1_1}$, where $f^0(\alpha_0) = f(\alpha_0)$. Then for all $z \in \omega^\omega$, K_z is well-founded iff V_z is ill-founded. [If K_z is well-founded, then $(z, \vec{\beta}) \in [V]$, where the β_i are represented by an f of the correct type from the Kleene-Brouwer order on K_z to ω_1. For the other direction, see the following paragraph.]

Suppose now $(z, \vec{\beta}) \in [V]$. We claim that K_z is well-founded and if $g \colon \omega_1 \to \omega_1$ is defined by $g(\alpha) = |K_z \restriction (\alpha + 1)|$, then $[g]_{W^1_1} \leq \beta_0$. To see this, fix for each $n \geq 1$ an f_n of the correct type from the Kleene-Brouwer ordering on $K_{z \restriction n}$ to ω_1 such that $\forall 1 \leq i < n\ [f^i_n]_{\mu_{b_i}} = \beta_i$, and also $[f^0_n]_{W^1_1} = \beta_0$. For each $i > 0$, let $C_i \subseteq \omega_1$ be c.u.b.

defining a μ_{b_i} measure one set A_i such that $\forall n,m > i\ \forall \vec{\alpha} \in A_i\ f^i_n(\vec{\alpha}) = f^i_m(\vec{\alpha})$. Likewise, let C_0 be such that $\forall n,m > 0\ f^0_n(\alpha_0) = f^0_m(\alpha_0)$ for all $\alpha_0 \in C_0$. For $\alpha_0 \in C_0$, let $h(\alpha_0)$ be the common value of the $f^0_n(\alpha_0)$. Let $C = \bigcap_i C_i$. Let $D \subseteq C$ consist of limit points of C, and every $\alpha \in D$ is the α^{th} element of C. Fix $\delta \in D$. For $(b_i, \vec{\alpha}, c_i, d_i) \in K_z$ with $\alpha_0 \leq \delta$, define $F(b_i, \vec{\alpha}, c_i, d_i) = f^i_n(b_i, \pi(\vec{\alpha}), c_i, d_i)$ for any (all) $n > i$. Here $\pi(\alpha_0, \ldots, \alpha_k) = (l(\alpha_0), \ldots, l(\alpha_k))$, where $l(\alpha) =$ the α^{th} element of C. Then F is order-preserving from the Kleene-Brouwer ordering on $K_z \restriction (\delta+1)$ to $h(\delta)$. Thus, $|K_z \restriction (\delta+1)| \leq h(\delta)$. Since $[h]_{W^1_1} = \beta_0$, this establishes the claim.

We now define our coding for $\kappa = \omega_2$. Let $P = \{z \colon K_z \text{ is well-founded }\}$. For $z \in P$, let $|z| = [g]_{W^1_1}$, where $g(\alpha) \doteq |K_z \restriction (\alpha+1)|$. Let $V \subseteq (\omega \times \omega_\omega)^{<\omega}$ be the homogeneous tree constructed above. The claim of the previous paragraph shows that P, $|\ |$, V satisfy $(\star)$ for $\kappa = \omega_2$. An easy computation shows that $P \in \Pi^1_2$. Note also that V is homogeneous, with all of the homogeneity measures in $\mathcal{S}_1$ (these measures are induced from the strong partition relation on ω_1 and functions f of the correct type from a subset of the Kleene-Brouwer ordering on K_s to ω_1).

The following lemma summarizes the discussion to this point.

LEMMA 2.1. *Let $\kappa = \omega_1$ (respectively ω_2). Then there is a $\boldsymbol{\Pi}^1_1$ ($\boldsymbol{\Pi}^1_2$) set P, a norm $x \to |x|_P < \kappa$ on P with range cofinal in κ, and a homogeneous tree $T \subseteq (\omega \times \omega_1)^{<\omega}$ ($T \subseteq (\omega \times \omega_\omega)^{<\omega}$) with homogeneity measures in $\mathcal{W}_1$ ($\mathcal{S}_1$) which satisfy $(\star)$.*

We note also that for $\alpha < \kappa$ that $P_\alpha \doteq \{x \in P \colon |x|_P = \alpha\} \in \boldsymbol{\Delta}^1_3$. This is clear if $\kappa = \omega_1$, as then $P_\alpha \in \boldsymbol{\Delta}^1_1$. For the $\kappa = \omega_2$ case, let $[g]_{W^1_1} = \alpha$, so $x \in P_\alpha$ iff $\forall^*_{W^1_1}\gamma$ $|K_x \restriction \gamma + 1| = g(\gamma)$. Since for fixed $\gamma, \delta < \omega_1$ the set $\{x \colon K_x \restriction \gamma + 1 \text{ is well-founded of rank } \delta\}$ is $\boldsymbol{\Delta}^1_1$, it suffices to observe the following well-known claim.

CLAIM 1. If $\{A_\gamma\}_{\gamma < \omega_1}$, is a sequence of $\boldsymbol{\Delta}^1_1$ sets, then $A = \{x \colon \forall^*_{W^1_1}\gamma\ (x \in A_\gamma)\}$ is $\boldsymbol{\Delta}^1_3$.

PROOF. Let K again be the Kunen tree constructed above. Let C_0 be the set of limits of limit ordinals greater than ω^2. If K_z is well-founded, let $C_z = C_0 \cap \{\gamma \colon \forall \beta < \gamma\ |K_z \restriction \beta| < \gamma\}$. Then for any c.u.b. $C \subseteq \omega_1$ there is a z with K_z well-founded such that $C_z \subseteq C$, since we may choose z so that for all limit $\beta > \omega^2$ we have $|K_z \restriction \beta + 1| > N_C(\beta) \doteq$ the least element of C greater than β. Then

$$A(x) \leftrightarrow \exists z[K_z \text{ is well-founded} \wedge \forall \gamma \in C_0(\gamma \in C_z \to x \in A_\gamma)].$$

For all $\gamma \in C_0$, $\{z \colon K_z \text{ is well-founded } \wedge \gamma \in C_z\} \in \boldsymbol{\Delta}^1_3$, so by the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ intersections, the expression inside the square brackets defines a $\boldsymbol{\Delta}^1_3$ relation. Thus, $A \in \boldsymbol{\Sigma}^1_3$. A similar computation shows $\neg A \in \boldsymbol{\Sigma}^1_3$. $\square$

We will not need anything about our coding of the (cofinally many) ordinals below κ than that contained in lemma 2.1 and the following paragraph. We fix P, the norm $x \to |x|_P$, and the homogeneous tree T as in lemma 2.1.

We next need to introduce a coding for functions $f : \kappa \to \boldsymbol{\delta}^1_3$. Let $\rho \colon \omega^\omega \overset{onto}{\to} \omega_\omega$ be the norm from a $\boldsymbol{\Delta}^1_3$ prewellordering, so for all $\beta < \omega_\omega$, $\rho^{-1}(\beta) \in \boldsymbol{\Delta}^1_3$. We may assume without loss of generality that $\forall s \in \omega^{<\omega}\ \forall \alpha < \omega_\omega\ \exists x$ extending s with $\rho(x) = \alpha$. Recall there is a $\boldsymbol{\Delta}^1_3$ coding of the subsets of $\kappa \times \omega_\omega \times \omega_\omega$. That is, there is a map $\pi \colon \omega^\omega \overset{onto}{\to} \mathcal{P}(\kappa \times \omega_\omega \times \omega_\omega)$ such that for all $(\alpha, \beta, \gamma) \in \kappa \times \omega_\omega \times \omega_\omega$,

$\{x\colon (\alpha,\beta,\gamma)\in\pi(x)\}\in\mathbf{\Delta}^1_3$. We fix ρ,π for the rest of this proof. For $z\in\omega^\omega$ and $\alpha<\kappa$ let (with a slight abuse of notation) $\pi(z)(\alpha)=\{(\beta,\gamma)\colon(\alpha,\beta,\gamma)\in\pi(z)\}$. For $\alpha<\kappa$ and $\beta<\boldsymbol{\delta}^1_3$, define $z\in E_{\alpha,\beta}$ iff $\forall\alpha'\le\alpha\ \pi(z)(\alpha')$ is well-founded of rank $\le\beta$. Using the closure of $\mathbf{\Delta}^1_3$ under $<\boldsymbol{\delta}^1_3$ length unions and intersections, it is straightforward to see that $E_{\alpha,\beta}\in\mathbf{\Delta}^1_3$. Define the relation R by:

$$\begin{aligned}R(z,x)&\Leftrightarrow x\in P\wedge\forall\alpha\le|x|_P\ \pi(z)(\alpha)\text{ is well-founded}\\&\Leftrightarrow\neg\exists w\exists v\ \forall n\ [\rho(v)\le|x|_P\wedge(\rho(v),\rho(w_n),\rho(w_{n+1}))\in\pi(z)].\end{aligned}$$

Using claim 1 and the closure of $\mathbf{\Delta}^1_3$ under $<\boldsymbol{\delta}^1_3$ unions and intersections, it is easy to see that the relation defined by the expression in square brackets is $\mathbf{\Delta}^1_3$, and thus R is $\mathbf{\Pi}^1_3$. Let $U\subseteq(\omega\times\omega\times\boldsymbol{\delta}^1_3)^{<\omega}$ be a homogeneous tree with $R=p[U]$, and with the homogeneity measures for U being $\boldsymbol{\delta}^1_3$-complete (which the homogeneous tree construction gives). For $r_n,s_n\in\omega^{<\omega}$ of the same length, let ν_{r_n,s_n} be the corresponding homogeneity measure for U_{r_n,s_n}.

For $z\in\omega^\omega$, we define a partial function $f_z\colon\kappa\to\boldsymbol{\delta}^1_3$ as follows. For $\alpha<\kappa$, we have $\alpha\in\mathrm{dom}(f_z)$ iff $\pi(z)(\alpha)$ is wellfounded, in which case $f_z(\alpha)$ is the rank of $\pi(z)(\alpha)$.

Definition 2.2. We define the set of *codes* CO for ordinals below $\boldsymbol{\delta}^1_3$ by: $z\in CO$ iff $\forall\alpha<\kappa\ f_z(\alpha)$ is defined (i.e., $\pi(z)(\alpha)$ is well-founded). For $z\in CO$, let $||z||=\sup\limits_{\alpha<\kappa}f_z(\alpha)$.

If $\beta<\boldsymbol{\delta}^1_3$ and $\mathrm{cof}(\beta)=\kappa$, then $\exists z\in CO\ ||z||=\beta$. Also, this coding has the following boundedness property. If $S\subseteq CO$ is $\mathbf{\Sigma}^1_3$, then $\sup\{||z||\colon z\in S\}<\boldsymbol{\delta}^1_3$. This follows since this supremum is bounded by the length of the following $\mathbf{\Sigma}^1_3$ well-founded relation: $(z,x,y)\prec(z',x',y')$ iff $z=z'\in S$, $\rho(x)=\rho(x')<\kappa$, and $(\rho(x),\rho(y),\rho(y'))\in\pi(z)$.

We now "integrate" the tree U to produce a tree V. Let s_n be an enumeration of $\omega^{<\omega}-\{\emptyset\}$ with any sequence preceding its extensions. Let μ_n denote the homogeneity measure for T corresponding to s_n (thus μ_n is a measure on T_{s_n}). Let l_n be the length of s_n, so μ_n is a measure on l_n-tuples.

Define the tree $V\subseteq(\omega\times\boldsymbol{\delta}^1_3)^{<\omega}$ by: $((a_0,\dots,a_{n-1}),(\beta_0,\dots,\beta_{n-1}))\in V$ provided the following conditions are satisfied.

(1) $((a_0,\dots,a_{n-2}),(\beta_0,\dots,\beta_{n-2}))\in V$.
(2) Let $q_1<q_2<\dots<q_{l_n}=n$ be the integers such that $s_{q_i}=s_n\restriction i$. For $1\le i\le l_n$, let $f_i\colon\mathrm{dom}(\mu_{q_i})\to\boldsymbol{\delta}^1_3$ represent β_{q_i}. Then we require that for μ_n almost all $(\alpha_0,\dots,\alpha_{l_n-1})$ that $((a_0,\dots,a_{l_n-1}),s_n,(\gamma_0,\dots,\gamma_{l_n-1}))\in U$, where $\gamma_i=f_i(\alpha_0,\dots,\alpha_{i-1})$.

Claim 2. V has the following properties.

(1) $\forall z\in\omega^\omega$ (V_z is ill-founded iff $z\in CO$).
(2) There is a c.u.b. set $D\subseteq\boldsymbol{\delta}^1_3$ such that for all $\delta\in D$, if $z\in CO$ and $||z||=\delta$, then $V_z\restriction\sup\limits_j j(\delta)$ is ill-founded, where the supremum ranges over embeddings j corresponding to the measures μ_i occurring in T.

To verify (1) of the claim, suppose that V_z is ill-founded. Let $x\in P$, and we show that $R(z,x)$. Let $q_1<q_2<\cdots$ be such that $s_{q_i}=x\restriction i$. Then, from the definition of V it follows that for all $k\in\omega$ there are representing functions

$f_k\colon \operatorname{dom}(\mu_{q_k}) \to \boldsymbol{\delta}^1_3$ and measure one sets A^k with respect to μ_{q_k} such that for all k and $(\alpha_0, \dots, \alpha_{k-1}) \in A_k$ we have $(z \restriction k, x \restriction k, (f_1(\alpha_0), \dots, f_k(\alpha_0, \dots, \alpha_{k-1}))) \in A$. Since $x \in P$, by homogeneity there is a sequence $\vec{\alpha}$ such that $\forall k\ \vec{\alpha} \restriction k \in A_k$. Let $\gamma_i = f_i(\alpha_0, \dots, \alpha_{i-1})$. Then $(z, x, \vec{\gamma}) \in [U]$, so $R(z, x)$.

Secondly, we verify property (2) of V. We first define the c.u.b. set $D \subseteq \boldsymbol{\delta}^1_3$ as required. For fixed $z \in \omega^\omega$, consider the following game G_z :

I.	$x(0)$, $x(1)$,...	x
	$\alpha(0)$, $\alpha(1)$,...	$\vec{\alpha}$
II.	$\gamma(0)$, $\gamma(1)$,...	$\vec{\gamma}$

Here, $x(i) \in \omega$, $\alpha(i) < \omega_\omega$, and $\gamma(i) < \boldsymbol{\delta}^1_3$. In addition, I must play $\alpha(0)$ such that $\forall \alpha \le \alpha(0)\ f_z(\alpha)$ is defined, or I loses. II then wins provided $\forall n[(x \restriction n, \vec{\alpha} \restriction n) \in T \to (z \restriction n, x \restriction n, \vec{\gamma} \restriction n) \in U$. This game is closed for II, hence determined. Now, I cannot win this game, for suppose I had a winning strategy, say σ. Let σ call for I to initially play $x(0), \alpha(0)$. Let $x(1), \alpha(1)$ be such that for some $\nu_{z(0),x(0)}$ measure one set A_1 and all $\gamma(0) \in A_1$, $\sigma(\gamma(0)) = x(1), \alpha(1)$. Continuing, define $x(n), \alpha(n)$ so that for some $\nu_{z\restriction n, x\restriction n}$ measure one set A_n, and all $(\gamma(0), \dots, \gamma(n-1)) \in A_n$, we have $\sigma(\vec{\gamma} \restriction n) = x(n), \alpha(n)$. We use here the $\boldsymbol{\delta}^1_3$-completeness of the measures in U (note also that each $U_{z\restriction n, x\restriction n}$ is non-empty). We have thus defined a real x and an ordinal sequence $\vec{\alpha}$. Clearly $(x, \vec{\alpha}) \in [T]$. By $(\star)$, $|x|_P \le \alpha(0)$. Thus, $R(z, x)$. By homogeneity, we may fix $\vec{\gamma} \in (\boldsymbol{\delta}^1_3)^\omega$ such that $\forall n\ \vec{\gamma} \restriction n \in A_n$. If II plays $\vec{\gamma}$, then II defeats I, a contradiction. Hence, II must win the game G_z. Since G_z is closed for II and II is playing ordinals, II has a canonical winning strategy σ_z for G_z (obtained by playing the least ordinal winning from that point on).

For $z \in \omega^\omega$ and $\alpha < \kappa$, let G^α_z be the game defined as G_z except I has the extra requirement that $\alpha(0) \le \alpha$. Of course, II still wins these games as well. Moreover, we claim that for all $\alpha < \kappa$ and $\beta < \boldsymbol{\delta}^1_3$ that there is a least ordinal $l(\alpha, \beta) < \boldsymbol{\delta}^1_3$ such that for all $z \in E_{\alpha,\beta}$, II can win the game G^α_z playing only ordinals $\gamma(i) < l(\alpha, \beta)$. To see this, fix α, β, and let $Q \subseteq (\omega \times \omega_\omega)^{<\omega}$ be a tree such that $p[Q] = E_{\alpha,\beta}$. We may assume that if $(s, \vec{\zeta}) \in Q$ then $\exists z$ extending s such that $z \in E_{\alpha,\beta}$. Define the auxiliary game $\bar{G}$ as follows:

I.	$z(0)$, $z(1)$,...	z
	$\zeta(0)$, $\zeta(1)$,...	$\vec{\zeta}$
	$x(0)$, $x(1)$,...	x
	$\alpha(0)$, $\alpha(1)$,...	$\vec{\alpha}$
II.	$\gamma(0)$, $\gamma(1)$,...	$\vec{\gamma}$

Here, $z(i), x(i) \in \omega$, $\alpha(i), \zeta(i) < \omega_\omega$, $\gamma(i) < \boldsymbol{\delta}^1_3$, and $\alpha(0) \le \alpha$. II wins provided for all n, if $(z \restriction n, \vec{\zeta} \restriction n) \in Q$ and $(x \restriction n, \vec{\alpha} \restriction n) \in T$, then $(z \restriction n, x \restriction n, \vec{\gamma} \restriction n) \in U$. This game is closed for II hence determined. Again using the $\boldsymbol{\delta}^1_3$-completeness of the measures in U, it follows that II wins this game, say by σ. Let $\Xi = \sup\{\sigma(s, \vec{\zeta}, t, \vec{\alpha}) \colon (s, \vec{\zeta}) \in Q \wedge (t, \vec{\alpha}) \in T\}$. Then clearly for any $z \in E_{\alpha,\beta}$, II can win the game G_z^α playing ordinals $< \Xi$, since we may fix $\vec{\zeta}$ such that $(z, \vec{\zeta}) \in [Q]$, and then follow II's strategy σ in $\bar{G}$. Hence, $l(\alpha, \beta) < \boldsymbol{\delta}_3^1$. Let $D_0 \subseteq \boldsymbol{\delta}_3^1$ be a c.u.b. set closed under l, and let D be the set of limit points of D_0. D will be our required c.u.b. set as in property (2) for V.

So, suppose $z \in CO$, and f_z has range in D_0 with $\sup(f_z) = \delta$. We show that $V_z \restriction \sup_j j(\delta)$ is ill-founded. Consider again the game G_z. If I makes first move $(x(0), \alpha(0))$, then II can win the run of the game playing ordinals $< l(\alpha(0), f_z(\alpha(0))) < \delta$. In fact, II has a canonical winning strategy to do so. This defines a winning strategy σ for II in G_z with the property that σ plays only ordinals below δ. For $n \in \omega$, define $\gamma(n)$ to be the ordinal represented with respect to μ_n by the function $\vec{\alpha} = (\alpha(0), \ldots, \alpha(l_n - 1)) \to \sigma(s_n, \vec{\alpha})$. Thus, $\gamma(n) < j_{\mu_n}(\delta)$. From the definition of V, $(z, \vec{\gamma}) \in [V]$. This completes the proof of (2) of the claim.

Using the tree V it is now easy to complete the proof of Martin's theorem. Let $W \subseteq (\omega \times \omega_\omega)^{<\omega}$ be a tree with $p[W]$ a $\boldsymbol{\Sigma}_3^1$-complete set. Then $\sup\{|W_y| \colon R_y$ is well-founded $\} = \boldsymbol{\delta}_3^1$. Define the tree $M \subseteq (\omega \times \omega \times \boldsymbol{\delta}_3^1 \times \omega \times \omega_\omega)^{<\omega}$ by: $(s, t, \vec{\gamma}, u, \vec{\delta}) \in M$ iff $\exists w, z, y$ extending s, t, u such that $w(z) = y$, $(t, \vec{\gamma}) \in V$, and $(u, \vec{\delta}) \in W$. Here we view w as a strategy for II in an integer game. Suppose now $F \colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ is given. Play the integer game where I, II play out $z, y \in \omega^\omega$. II wins provided $(z \in CO) \Rightarrow [W_y$ is well founded $\wedge |W_y| > F(||z||)]$. By boundedness, II wins, say by strategy τ. From property (1) of V and the definition of τ it follows that M_τ is well-founded. From property (2) of V we have that if $\delta \in D$ and $\mathrm{cof}(\delta) = \kappa$, then $|M_\tau \restriction \sup_j j(\delta)| > F(\delta)$, the supremum ranging over embeddings j corresponding to the homogeneity measures μ_n for T. This completes the proof of theorem 2.1.

Remark 2.3. It is possible to give a variation of the proof of theorem 2.1 which avoids the use of the (closed) ordinal games, and uses instead the proof of the Kunen-Martin theorem. Briefly, one first defines the tree U now to be a tree on $\omega \times \omega \times \omega_\omega \times \omega \times \omega_\omega$ with $(s, t, \vec{\alpha}, u, \vec{\beta}) \in U$ iff $(t, \vec{\alpha}) \in T$ and $(u, \vec{\beta})$ builds, as in the Kunen-Martin theorem, reals $w_0, w_1, \ldots$, and ordinal witnesses that for all n, $A(z, x, w_n, w_{n+1})$, where z, x are the reals s and t are building, and $A(z, x, w, v) \leftrightarrow (\rho(x), \rho(w), \rho(v)) \in \pi(z)$, so $A \in \boldsymbol{\Sigma}_3^1$. The homogeneous tree construction applied to U then produces a tree V as in claim 2, and the proof finishes as before.

CHAPTER 3

The Upper Bound for $\boldsymbol{\delta}^1_5$

Descriptions

Our first goal is to introduce a set of "finitary" objects which we call *descriptions*–they "describe" how to build equivalence classes of functions $f : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to the measures W^m_3. We will then introduce a "lowering operator" $\mathcal{L}$ on the set $\mathcal{D}$ of descriptions, and prove our main theorem which analyzes equivalence classes $[f]_{W^m_3}$ in terms of $\mathcal{D}$ and $\mathcal{L}$. We get that the ultrapower of $\boldsymbol{\delta}^1_3$ by W^m_3 is bounded by $\aleph_\Xi$, where Ξ is a countable ordinal computed from $\mathcal{D}$ and $\mathcal{L}$–essentially the rank of $\mathcal{L}$ on $\mathcal{D}$. From the global embedding theorem, this then gives an upper bound for $\boldsymbol{\delta}^1_5$.

We fix an $m \in \omega$ and a sequence of measures $K_1, \ldots, K_t$ for some $t \in \omega$, where each $K_i = W^{r(i)}_1$ or $S^{r(i)}_1$ for some $r(i) \in \omega$. Relative to this fixed sequence of measures, we will define a set $\mathcal{D}_m = \mathcal{D}_m(K_1, \ldots, K_t)$ of descriptions. We will then set $\mathcal{D}(K_1, \ldots, K_t) = \underset{m}{\cup} \mathcal{D}_m(K_1, \ldots, K_t)$ to be the set of descriptions relative to $K_1, \ldots, K_t$. We will usually suppress the background sequence of measures when writing, and refer simply to $\mathcal{D}$ and $\mathcal{D}_m$. We call $\mathcal{D}_m$ the set of m-descriptions. The elements of $\mathcal{D}$ as we define them will involve indices, certain symbols and other parameters. We leave it to the reader to give an interpretation of our definition into formal set theory, if desired.

In general, descriptions will have *indices* associated with them, which will be of one of two types (3 types occur in the general projective case–see [**2**]). An index will be of the form (f_m) or $()$, and written as a superscript on the description. Thus, a description will be of the form $d^{(f_m)}$ or $d^{()}$. The notation f_m for the index is meant to suggest a function, in this case a function from m into ω_1. Roughly speaking, the purpose of the description $d^{(f_m)}$ is to describe, given functions $h_1, \ldots, h_t$ representing ordinals $[h_i] \in \mathrm{dom}(K_i)$, and a function $f_m : m \to \omega_1$, how to produce an ordinal $< \omega_1$ (if $d = d^{()}$, then the ordinal is produced from just $h_1, \ldots, h_t$). The form of the indices could be notationally simplified here a bit, but we have chosen to retain the notation which coincides with that of the general projective case, where the indexing becomes somewhat more involved.

Descriptions indexed as $d^{(f_m)}$ will be called *type 0 descriptions*, and those of the form $d^{()}$, *type -1 descriptions*. We will frequently suppress writing the index when it is understood or irrelevant. Descriptions will also fall into one of two types–*basic* or *non-basic* (the basic descriptions will be defined outright, while the non-basic descriptions will be defined in terms of other descriptions).

By definition, $\mathcal{D}_m(K_1, \ldots, K_t)$ consists of those descriptions of index (f_m) for $m \geq 1$, and $\mathcal{D}_0(K_1, \ldots, K_t)$ consists of those of index $()$.

Before giving the general definitions, it might be useful to consider a simple example for motivation. Suppose $K_1 = K_2 = S^1_1$, and $m = 1$. If $h_1, h_2 : \omega_1 \to \omega_1$

are given, consider $h_1 \circ h_2 \colon \omega_1 \to \omega_1$. The map $(h_1, h_2) \to h_1 \circ h_2$ is not well-defined on the equivalence classes, and thus does not induce a map from $\omega_2 \times \omega_2 \to \omega_2$. However, for K_1 almost all $\beta_1 < \omega_2$, if $[h_1] = [h_1'] = \beta_1$, then for K_2 almost all β_2 if $[h_2] = [h_2'] = \beta_2$, then $[h_1 \circ h_2] = [h_1' \circ h_2']$ (all equivalence classes with respect to W_1^1). This is because almost all β_2 will be represented by an h_2 having range in a c.u.b. set on which h_1, h_1' agree. Thus, composition does define an ordinal θ with respect to the *iteration* of the measures K_1, K_2. Namely, θ is represented with respect to K_1 by the function which assigns to $[h_1] \in \mathrm{dom}(K_1)$ the value $\theta(h_1)$ which in turn is represented with respect to K_2 by the function which assigns to $[h_2] \in \mathrm{dom}(K_2)$ the value $\theta(h_1, h_2) = [h_1 \circ h_2]$. The ordinal θ is well-defined. Note that order of composition is important here; if we use $h_2 \circ h_1$, we do not produce an ordinal with respect to the iteration of K_1, K_2.

In general, the purpose of the description $d^{(f_m)}$ is to tell us how to compose the functions $h_1, \dots, h_t$ to build a function $h \colon (\omega_1)^m \to \omega_1$ representing an ordinal $\theta < \omega_{m+1}$. θ will be represented with respect to the iteration of the measures $K_1, \dots, K_t$, as in the example above. The basic descriptions will correspond to functions defined outright, while the non-basic descriptions describe functions obtained by composing functions obtained from sub-descriptions. In the example above, the underlying basic description describes the identity function $f(\alpha) = \alpha$. In the formal notation below, the description for this example is $d^{(1)} = (1; (2; (1)))$. Here (1) describes the basic (identity) function, $(2; (1))$ describes plugging this basic function into h_2 (to yield h_2), and $(1; (2; (1)))$ describes plugging h_2 into h_1 to yield $h_1 \circ h_2$.

We proceed with the formal definition of $\mathcal{D} = \mathcal{D}(K_1, \dots, K_t)$. Along with $\mathcal{D}$ we simultaneously define a numerical function $k : \mathcal{D} \to \{1, \dots, t\} \cup \{\infty\}$. $k(d)$ will be the least k such that h_k is mentioned in d (or $k = \infty$ if no such h_k is mentioned). The definition of $\mathcal{D}$, k is by reverse induction on $k(d)$.

Definition 3.1 (Definition of $\mathcal{D}$). The set $\mathcal{D} = \mathcal{D}(K_1, \dots, K_t)$ and the function $k \colon \mathcal{D} \to \{1, \dots, t\} \cup \{\infty\}$ are defined by reverse induction on $k(d)$ through the following cases:

Basic Type -1: We allow descriptions of the form $d^{()} = (k; p)^{()}$ where $1 \leq k \leq t$, $K_k = W_1^r$, and $1 \leq p \leq r$. We set $k(d) = k$.

Basic Type 0: We allow:

(1) $d^{(f_m)} = (k; p)^{(f_m)}$ where $1 \leq k \leq t$, $K_k = W_1^r$ and $1 \leq p \leq r$. We set $k(d) = k$.

(2) $d^{(f_m)} = (p)^{(f_m)}$ where $1 \leq p \leq m$. We set $k(d) = \infty$.

Non-Basic Descriptions: We allow:

(1) $d^{(f_m)} = (k; d_r^{(f_m)}, d_1^{(f_m)}, d_2^{(f_m)}, \dots, d_l^{(f_m)})^{(f_m)}$, where $1 \leq k \leq t$, $K_k = S_1^r$, $l \leq r-1$. Also, $d_r, d_1, \dots, d_l \in \mathcal{D}(K_1, \dots, K_t)$ and $k(d_1)$, …, $k(d_l)$, $k(d_r) > k$. We set $k(d) = k$.

(2) $d^{(f_m)} = (k; d_r{}^{(f_m)}, d_1{}^{(f_m)}, \dots, d_l{}^{(f_m)})^{s(f_m)}$. Here, s is a formal symbol (which stands for "sup"), $1 \leq l \leq r-1$, $K_k = S_1^r$, $d_r, d_1, \dots, d_l \in \mathcal{D}(K_1, \dots, K_t)$, and $k(d_1)$, …, $k(d_l)$, $k(d_r) > k$. We set $k(d) = k$. We must have $r \geq 2$ in this case.

(3) & (4) The same as (1) and (2) above except with indices $()$ everywhere replacing (f_m).

Throughout the rest of the paper, we use the notational convention (s) to denote that the symbol s may or may not appear. For example, $(k; d_r, d_1, \ldots, d_l)^{(s)}$ denotes either $(k; d_r, d_1, \ldots, d_l)$ or $(k; d_r, d_1, \ldots, d_l)^s$.

We next define the function h which interprets descriptions. That is, h builds the ordinal "θ" of the example, given the functions $h_1, \ldots, h_t$. Fix $d \in \mathcal{D}$ and $h_1, \ldots, h_t$, where if $K_i = W_1^r$ then $h_i\colon r \to \omega_1$, and if $K_i = S_1^r$ then $h_i\colon \mathrm{dom}(<_r) \to \omega_1$ of the correct type. We define the ordinal $h(d; h_1, \ldots, h_t)$. If $d = d^{()}$ then $h(d; h_1, \ldots, h_t) < \omega_1$ and if $d = d^{(f_m)}$ then $h(d; h_1, \ldots, h_t) < \omega_{m+1}$ and is represented with respect to W_1^m by a function which we also denote by $h(d; h_1, \ldots, h_t)(\alpha_1, \ldots, \alpha_m)$. If the sequence $h_1, \ldots, h_t$ is understood, we frequently employ vector notation, e.g., writing $h(d, \vec{h})$ for $h(d; h_1, \ldots, h_t)$. Also, we suppress writing indices when they are understood. We define h through cases by reverse induction on $k(d)$ as follows. Recall that for $h\colon \mathrm{dom}(<_r) \to \omega_1$ of the correct type, and $l + 1 \leq r$, $h(l+1)\colon\ <_{l+1} \to \omega_1$ is defined by $h(l+1)(\alpha_1, \ldots, \alpha_l, \alpha_k) = \sup\{h(\alpha_1, \ldots, \alpha_l, \beta_{l+1}, \ldots, \beta_{r-1}, \alpha_r)\colon \alpha_l < \beta_{l+1} < \cdots < \beta_{r-1} < \alpha_r\}$.

Definition 3.2 (Definition of $h(d; h_1, \ldots, h_t)$).

Basic Type -1: $d^{()} = (k; p)$. Then $h(d; \vec{h}) = h_k(p)$.

Basic Type 0:

(1) $d^{(f_m)} = (k; p)$. Then $h(d; \vec{h})(\vec{\alpha}) = h_k(p)$.

(2) $d^{(f_m)} = (p)$. (So $1 \leq p \leq m$). Then $h(d; \vec{h})(\vec{\alpha}) = \alpha_p$.

Non-basic:

(1) $d^{(f_m)} = (k; d_r, d_1, \ldots, d_l)$, where $K_k = S_1^r$.

(a) $l = r - 1$. Then $h(d; \vec{h})(\vec{\alpha}) = h_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_r; \vec{h})(\vec{\alpha}))$.

(b) $l < r - 1$. Then

$$h(d; \vec{h})(\vec{\alpha}) = h_k(l+1)(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_l; \vec{h})(\vec{\alpha}), h(d_r, \vec{h})(\vec{\alpha})).$$

(2) $d^{(f_m)} = (k; d_r, d_1, \ldots, d_l)^s$, where $K_k = S_1^r$ and $l \geq 1$ here. Then

$$h(d; \vec{h})(\vec{\alpha}) = \sup\{h_k(l+1)(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_{l-1}; \vec{h})(\vec{\alpha}), \beta_l, h(d_r; \vec{h})(\vec{\alpha}))\colon \beta_l < h(d_l; \vec{h})(\vec{\alpha})\}.$$

3. & 4. Same as (1) and (2) where now $d = d^{()}$. The definitions are the same as in (1), (2), except now $h(d; \vec{h})$ is a single ordinal $< \omega_1$, that is, the ordinals $\vec{\alpha}$ do not appear in the formulas.

We adopt the following notational convention for the rest of the paper. If $K_1, \ldots, K_t \in W_1 \cup S_1$, so each K_i is of the form $W_1^{r(i)}$ or $S_1^{r(i)}$, then "for almost all $h_1, \ldots, h_t$" abbreviates the statement "there is a c.u.b. $C_1 \subseteq \omega_1$ such that for all $h_1\colon \mathrm{dom}(<_{r(1)}) \to C_1$ of the correct type (if $K_1 = S_1^{r(1)}$, and for $K_1 = W_1^{r(1)}$, $h_1\colon r(1) \to C_1$) there is a c.u.b. $C_2 \subseteq \omega_1$ such that for all $h_2\colon \mathrm{dom}(<_{r(2)}) \to C_2$ (or $h_2\colon r(2) \to C_2$), …, there is a c.u.b. $C_t \subseteq \omega_1$ such that for all $h_t\colon \mathrm{dom}(<_{r(t)}) \to C_t$ (or $h_t\colon r(t) \to C_t$)".

We now define an ordering $<$ on the set $\mathcal{D} = \mathcal{D}(K_1, \ldots, K_t)$. Descriptions d_1, d_2 will be comparable only if they are indexed identically.

Definition 3.3 (Definition of $<$). Let $d_1, d_2 \in \mathcal{D}$ have the same index. Then we set $d_1 < d_2$ iff for almost all $h_1, \ldots, h_t$, $h(d_1; \vec{h}) < h(d_2; \vec{h})$.

We leave it to the reader to check the fact that if $d_1 \neq d_2$ (and have the same index), then either $d_1 < d_2$ or $d_2 < d_1$.

The definition of $<$ can be reformulated in a purely combinatorial fashion, which we will need in §3. We state this reformulation in the following fact, whose proof is straightforward and left to the reader.

FACT . Let d_1, d_2 be defined with respect to $K_1, \dots, K_t \in W_1 \cup S_1$, and have the same index. Then $d_1 < d_2$ iff one of the following cases holds. Let $k = \min(k(d_1), k(d_2))$.

(1) $k = \infty$, so d_1, d_2 are basic type-0, say $d_1 = (i)$, $d_2 = (j)$. Then we require $i < j$.
(2) $k < \infty$, and $k = k(d_1) < k(d_2)$. Then either d_1 is basic type -1 (i.e., $K_k = W_1^r$), or else $K_k = S_1^r$, so d_1 is of the form $d_1 = (k; \bar{d}_r, \bar{d}_1, \dots, \bar{d}_l)^{(s)}$, and we have $\bar{d}_r < d_2$.
(3) $k < \infty$, and $k(d_1) > k(d_2) = k$. Then $K_k = S_1^r$, so d_2 is of the form $d_2 = (k; d'_r, d'_1, \dots, d'_l)^{(s)}$, and we have $d_1 \leq d'_r$.
(4) $k < \infty$, and $k = k(d_1) = k(d_2)$. If $K_k = W_1^r$, then d_1, d_2 are basic type -1 of the form $d_1 = (k; i)$, $d_2 = (k; j)$. We then require $i < j$. Suppose then that $K_k = S_1^r$, so that d_1, d_2 have the forms $d_1 = (k; \bar{d}_r, \bar{d}_1, \dots, \bar{d}_{l_1})^{(s)}$, $d_2 = (k; d'_r, d'_1, \dots, d'_{l_2})^{(s)}$. Then, we require one of the following to hold.
  (a) $\bar{d}_r < d'_r$.
  (b) $\bar{d}_r = d'_r$, and $\exists i \leq \min(l_1, l_2)\ \bar{d}_i < d'_i$ and $\forall 1 \leq j < i\ \bar{d}_j = d'_j$. In the remaining cases, we assume $\bar{d}_r = d'_r$ and $\forall 1 \leq i \leq \min(l_1, l_2)\ \bar{d}_i = d'_i$.
  (c) $l_1 < l_2$, and s appears in d_1.
  (d) $l_1 > l_2$, and s does not appear in d_2.
  (e) $l_1 = l_2$, s appears in d_1 and not in d_2.

We now introduce two "well-definedness" conditions for descriptions, which we call conditions C and D. Condition C guarantees that the equivalence class $[h(d; \vec{h})]_{W_1^m}$ is well-defined with respect to the iteration $K_1, \dots, K_t$. Condition D guarantees, in addition, that $[h(d; \vec{h})]$ is either representable by an $h\colon \mathrm{dom}(<_m) \to \omega_1$ of the correct type (where $d = d^{(f_m)}$), or is a supremum of such ordinals.

DEFINITION 3.4 (Definition of C). We say $d \in \mathcal{D}$ satisfies condition C if either d is basic or else d is non-basic, say of the form $d = (k; d_r, d_1, \dots, d_l)^{(s)}$, all $d_r, d_1, \dots, d_l$ satisfy C, and $d_1 < d_2 < \dots < d_l < d_r$.

Recall that in the definition of $\mathcal{D}$, we also required that $k < k(d_1), \dots, k(d_l), k(d_r)$.

We define a set $\bar{\mathcal{D}}$ of "extended descriptions." For every $d \in \mathcal{D}$, there will two extended descriptions $\bar{d} \in \bar{\mathcal{D}}$.

DEFINITION 3.5 (Definition of $\bar{\mathcal{D}}$). $\bar{\mathcal{D}}_m(K_1, \dots, K_t)$ consists of objects of the form (d) or $(d)^s$, where $d \in \mathcal{D}_m(K_1, \dots, K_t)$, together with the distinguished object $()^s$. We let $\bar{\mathcal{D}}(K_1, \dots, K_t) = \bigcup_m \bar{\mathcal{D}}(K_1, \dots, K_t)$. By the index of $\bar{d} = (d)$ or $(d)^s$, we mean the index of the description d.

Formally, the sets $\mathcal{D}$ and $\bar{\mathcal{D}}$ are disjoint, though we can view $\mathcal{D} \subseteq \bar{\mathcal{D}}$ by identifying $d \in \mathcal{D}$ with $(d) \in \bar{\mathcal{D}}$. For $\bar{d} \in \bar{\mathcal{D}}$, we will write $\bar{d}^{(f_m)}$ when we want to emphasize the index of $\bar{d}$ is (f_m).

DEFINITION 3.6 (Definition of D). Suppose $\bar{d} = (d)^{(s)} \in \bar{\mathcal{D}}$, and $d = d^{(f_m)} \in \mathcal{D}_m$. Then we say $(d)^{(s)}$ satisfies condition D provided d satisfies C and:

(1) If s does not appear (i.e., $\bar{d} = (d)$) then for almost all $h_1, \ldots, h_t$, $h(d; \vec{h}) \colon \omega_1^m \to \omega_1$ is of the correct type almost everywhere (i.e. restricted to a measure one set with respect to W_1^m).

(2) If s appears (i.e., $\bar{d} = (d)^s$) then for almost all $h_1, \ldots, h_t$, $h(d; \vec{h})$ is the supremum of ordinals representable by functions $h \colon \omega_1^m \to \omega_1$ of the correct type almost everywhere. We also define $()^s$ to satisfy D.

We next prove a lemma which shows the significance of condition C. As above, we take "for almost all h_i" to mean with respect to the measure on h_i coming from the strong partition relation on ω_1 and $h_i \colon \operatorname{dom}(<_{r(i)}) \to \omega_1$ of the correct type for $K_i = S_1^{r(i)}$, and the measure $W_1^{r(i)}$ on $h_i \colon r(i) \to \omega_1$ for $K_i = W_1^{r(i)}$. Also, we use the convention that "$h_i = h_i'$ almost everywhere" means $h_i = h_i'$ in the case $K_i = W_1^{r(i)}$.

LEMMA 3.1. *Suppose d satisfies condition C. Then for almost all h_1, if $h_1' = h_1$ almost everywhere, then for almost all h_2, if $h_2' = h_2$ almost everywhere, …, for almost all h_t, if $h_t' = h_t$ almost everywhere, then $h(d; h_1, \ldots, h_t) = h(d; h_1', \ldots, h_t')$.*

PROOF. If d is basic, the result is immediate. Suppose then that $d = d^{(f_m)} = (k; d_r, d_1, \ldots, d_l)^{(s)}$. By induction (reverse induction on $k(d)$), we may assume the lemma for $d_1, \ldots, d_l, d_r$. Also, from condition C and the definition of $<$, for almost all $h_1, \ldots, h_t$, $h(d_1; \vec{h}) < h(d_2; \vec{h}) < \ldots < h(d_l; \vec{h}) < h(d_r; \vec{h})$. If the lemma fails, then, we may fix $h_1, h_1', h_2, h_2', \ldots, h_t, h_t'$ with $h(d; \vec{h}) \neq h(d; \vec{h'})$, $h(d_i; \vec{h}) = h(d_i, \vec{h'})$ for $1 \leq i \leq l$ or $i = r$, and $h(d_1; \vec{h}) < \ldots < h(d_l; \vec{h}) < h(d_r; \vec{h})$. We may further assume that for $i \geq k$, h_i has range in a c.u.b. set $C_k \subseteq \omega_1$ defining a measure one set on which h_k and h_k' agree. Suppose to be specific that $l = r - 1$ and s does not appear (the other cases being essentially the same). Then, $h(d; \vec{h})(\alpha_1, \ldots, \alpha_m) = h_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_l; \vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha}))$, with a similar expression holding with $\vec{h'}$ everywhere. Now, for almost $\vec{\alpha}$, $h(d_i; \vec{h})(\vec{\alpha}) = h(d_i; \vec{h'})(\vec{\alpha})$ for all $1 \leq i \leq r - 1$, $h(d_1; \vec{h})(\vec{\alpha}) < \ldots < h(d_r; \vec{h})(\vec{\alpha})$ and also $h(d_i; \vec{h})(\vec{\alpha}) \in C_k$ as $k(d_i) > k$ for $1 \leq i \leq r$, and it easy to check that $h(d; \vec{h})(\vec{\alpha}) \in \operatorname{ran}(h_k)$ for $k = k(d)$, or is a sup of ordinals in the range of h_k. This is enough to guarantee that $h(d; \vec{h})(\vec{\alpha}) = h(d; \vec{h'})(\vec{\alpha})$, and to complete the proof of the lemma. □

If d satisfies C with respect to the sequence $K_1, \ldots, K_t$, we define an ordinal $h(d; K_1, \ldots, K_t)$. We represent this with respect to K_1 by the function which assigns to $[h_1] \in \operatorname{dom}(K_1)$ the value $h(d; h_1, K_2, \ldots, K_t)$. This in turn is represented with respect to K_2 by the function which assign to $[h_2] \in \operatorname{dom}(K_2)$ the value $h(d; h_1, h_2, K_3, \ldots, K_t)$, etc., and where $h(d; h_1, \ldots, h_t)$ is defined in definition 3.2. From lemma 3.1 it follows that $h(d; K_1, \ldots, K_t)$ is well-defined.

Recall our notational convention concerning iterated measures mentioned in the introduction. When using this convention with respect to a sequence $K_1, \ldots, K_t$ of canonical measures, we usually employ the following abbreviation. For $\theta \in ON$ and $P \subseteq ON$, we abbreviate $\forall^*_{K_1}[h_1]\ \forall^*_{K_2}[h_2] \ldots \forall^*_{K_t}[h_t]\ P(\theta([h_1], \ldots, [h_t]))$ by writing just $\forall^* h_1, \ldots, h_t\ P(\theta(h_1, \ldots, h_t))$. For example, if $\theta \in ON$ and d satisfies C, then $\forall^* h_1, \ldots, h_t\ \theta(h_1, \ldots, h_t) < h(d; h_1, \ldots, h_t)$ abbreviates the statement: if we fix a representing function $\alpha_1 \to \theta(\alpha_1)$ for θ with respect to K_1, then for K_1 almost all α_1 and h_1 with $[h_1] = \alpha_1$, if we fix a representing function $\alpha_2 \to \theta(\alpha_1, \alpha_2)$ for $\theta(\alpha_1)$ with respect to K_2, then for K_2 almost all α_2 and $[h_2] = \alpha_2$, ..., if we fix a representing function $\alpha_t \to \theta(\alpha_1, \dots, \alpha_t)$ for $\theta(\alpha_1, \dots, \alpha_{t-1})$ with respect to K_t, then for K_t almost all α_t and $[h_t] = \alpha_t$ we have $\theta(\alpha_1, \dots, \alpha_t) = h(d; h_1, \dots, h_t)$. For this example, this statement is equivalent to $\theta < h(d; K_1, \dots, K_t)$.

We next show how to use extended descriptions to generate equivalence classes of functions from $\boldsymbol{\delta}^1_3$ to $\boldsymbol{\delta}^1_3$ with respect to the measures W^m_3. Fix $m \in \omega$, measures $K_1, \dots, K_t$, and let $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \dots, K_t)$, where $(d)^{(s)}$ satisfies condition D. Let $g\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ be given (g could be the identity function id). We define an ordinal which we denote by $(g; (d)^{(s)}; K_1, \dots, K_t)$. We represent this ordinal with respect to W^m_3 by the function which assigns to $[f]_{S^m_1}$ the ordinal $(g; f; (d)^{(s)}; K_1, \dots, K_t)$, where $f\colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ is of the correct type. We represent $(g; f; (d)^{(s)}; K_1, \dots, K_t)$ with respect to K_1 by the function which assigns to $[h_1]$ the ordinal $(g; f; (d)^{(s)}; h_1, K_2, \dots, K_t)$, and in general, represent $(g; f; (d)^{(s)}; h_1, \dots, h_{i-1}, K_i, \dots, K_t)$ with respect to K_i by the function which assigns to $[h_i]$ the ordinal $(g; f; (d)^s; h_1, \dots, h_i, K_{i+1}, \dots, K_t)$. Finally, we define $(g; f; (d)^{(s)}, h_1, \dots, h_t)$ by cases as follows:

(1) If s does not appear, then $(g; f; (d); h_1, \dots, h_t) = g(f(h(d; \vec{h})))$.
(2) If s appears, then $(g; f; (d)^s; h_1, \dots, h_t) = g(\sup_{\beta < h(d;\vec{h})} f(\beta))$.
(3) For the object $()^s$, we define $(g; f; ()^s; h_1, \dots, h_t) = g(\sup_{\beta < \omega_{m+1}} f(\beta))$.

Using D, and the fact that if $[f]_{S^m_1} = [f']_{S^m_1}$ then the h_i may be chosen to have range in a c.u.b. set defining a measure one set with respect to S^m_1 on which f, f' agree, it follows that $(g; (d)^{(s)}; K_1, \dots, K_t)$ is well-defined.

Our next goal is to define the lowering operator $\mathcal{L}$ on $\mathcal{D}$. To define $\mathcal{L}$, we define, more generally, operators $\mathcal{L}^k$ for $1 \le k \le t$ or $k = \infty$ (for fixed $K_1, \dots, K_t$). $\mathcal{L}^k$ will be defined on those d satisfying $k(d) \ge k$, except for a unique $d = d(k)$ which we call the *minimal* description with respect to $\mathcal{L}^k$ (and $K_1, \dots, K_t$). We will then take $\mathcal{L} = \mathcal{L}^1$. We define $\mathcal{L}^k$ (by reverse induction on k) through the following cases.

Definition 3.7 (Definition of $\mathcal{L}^k$).

I: $k = \infty$. So, d is basic type 0 with $d = d^{(f_m)} = (i)$ where $1 \le i \le m$. If $i > 1$, then $\mathcal{L}^\infty(d) = (i-1)$. For $i = 1$, d is minimal with respect to $\mathcal{L}^\infty$.

II: $1 \le k \le t$.

(1) $k = k(d)$.

(a) d basic type -1, so $d = (k; p)$. For $p > 1$, we set $\mathcal{L}^k(d) = (k; p-1)$, and for $p = 1$ we define d to be minimal.

(b) $d = d^{(f_m)} = (k; d_r, d_1, \dots, d_l)$, where $l = r - 1$, and $K_k = S^r_1$. We set $\mathcal{L}^k(d) = (k; d_r, d_1, \dots, d_l)^s$ if $l \ge 1$, and if $d = (k; d_r)$, we set $\mathcal{L}^k(d) = d_r$.

(c) d non-basic (as in b), $l < r - 1$, and s does not appear. First assume $l \ge 1$. If $\mathcal{L}^{k+1}(d_r)$ is defined and $> d_l$, we set $\mathcal{L}^k(d) = (k; d_r, d_1, \dots, d_l, \mathcal{L}^{k+1}(d_r))$. If $\mathcal{L}^{k+1}(d_r)$ is not defined or is $\le d_l$, we set $\mathcal{L}^k(d) = (k; d_r, d_1, \dots, d_l)^s$. If $d = (k; d_r)$, we set $\mathcal{L}^k(d) = (d_r; \mathcal{L}^{k+1}(d_r))$ if $\mathcal{L}^{k+1}(d_r)$ is defined, and otherwise $\mathcal{L}^k(d) = d_r$.

(d) d non-basic, and s appears (so $l \geq 1$). We set $\mathcal{L}^k(d) = (k; d_r, d_1, \ldots, d_{l-1}, \mathcal{L}^{k+1}(d_l))$ if $\mathcal{L}^{k+1}(d_l)$ is defined and satisfies $\mathcal{L}^{k+1}(d_l) > d_{l-1}$ if $l \geq 2$. Otherwise, we set $\mathcal{L}^k(d) = (k; d_r, d_1, \ldots, d_{l-1})^s$ if $l \geq 2$ and for $l = 1$, $\mathcal{L}^k(d) = d_r$.

(2) $k < k(d)$, $K_k = W_1^{r(k)}$.
   (a) d not minimal with respect to $\mathcal{L}^{k+1}$. We set $\mathcal{L}^k(d) = \mathcal{L}^{k+1}(d)$.
   (b) d minimal with respect to $\mathcal{L}^{k+1}(d)$. We set $\mathcal{L}^k(d) = (k; r(k))$.

(3) $k < k(d)$, $K_k = S_1^{r(k)}$.
   (a) d not minimal with respect to $\mathcal{L}^{k+1}$. Set $\mathcal{L}^k(d) = (k; \mathcal{L}^{k+1}(d))$.
   (b) d minimal with respect to $\mathcal{L}^{k+1}$. Then d is minimal with respect to $\mathcal{L}^k$.

One can easily check that for each k there is a unique d minimal with respect to $\mathcal{L}^k$, and this d is also minimal with respect to the ordering $<$ on those d for which $k(d) \geq k$. Among such d, for each fixed index and sequence of measures, there is also a description which is maximal in the ordering $<$, which we call the *maximal description*, and denote by $\tilde{d}^k$ (except for one case below). This is defined by reverse induction on k as follows. For $k = \infty$ (so $d = d^{(f_m)}$), we set $\tilde{d}^\infty = (m)$. If $1 \leq k \leq t$ and $K_k = S_1^{r(k)}$, we set $\tilde{d}^k = (k; \tilde{d}^{k+1})$ if $\tilde{d}^{k+1}$ is defined, and otherwise we do not define $\tilde{d}^k$. If $K_k = W_1^{r(k)}$ we set $\tilde{d}^k = \tilde{d}^{k+1}$ if $\tilde{d}^{k+1}$ is defined, and otherwise set $\tilde{d}^k = (k; r(k))$. Finally, for a given index and sequence of measures, we set $\tilde{d} = \tilde{d}^1$. It is easy to check that the one case where $\tilde{d}^k$ is not defined is when $d = d^{()}$, and $\forall k' \geq k\ K_{k'} = S_1^{r(k')}$.

We wish to extend the $\mathcal{L}$ operation to an operation defined on $\bar{\mathcal{D}}$, which we also denote by $\mathcal{L}$. Again, we assume an index (f_m) and measures $K_1, \ldots, K_t$ are fixed. We set $\mathcal{L}((d)) = (d)^s$ and $\mathcal{L}((d)^s) = (\mathcal{L}(d))$ if $\mathcal{L}(d)$ is defined, and otherwise $(d)^s$ will be minimal with respect to $\mathcal{L}$. Also, we set $\mathcal{L}(()^s) = (\tilde{d})$ if $\tilde{d}$ is defined, and otherwise $()^s$ is minimal.

Finally, we define the operation $\bar{\mathcal{L}}$ on $\bar{\mathcal{D}}$ by $\bar{\mathcal{L}}((d)^{(s)}) = \mathcal{L}^{(l)}((d)^{(s)})$, where l is least such that $\mathcal{L}^{(l)}((d)^{(s)})$ satisfies D, where $\mathcal{L}^{(l)}$ denotes the l^{th} iterate of $\mathcal{L}$, if such an l exists. If not, then we define $(d)^{(s)}$ to be minimal with respect to $\bar{\mathcal{L}}$.

The Main Lemma

We now state the central result of this chapter.

THEOREM 3.1 (Main Lemma). *Let $(d)^{(s)} \in \bar{\mathcal{D}}(K_1, \ldots, K_t)$ satisfy D. Suppose $\theta < (id; (d)^{(s)}; K_1, \ldots, K_t)$. Then:*

(1) *If $(d)^{(s)}$ is not minimal with respect to $\bar{\mathcal{L}}$, then there is a $g\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ such that $\theta < (g; \bar{\mathcal{L}}((d)^{(s)}); K_1, \ldots, K_t)$.*
(2) *If d is minimal with respect to $\bar{\mathcal{L}}$, then $\theta < \boldsymbol{\delta}_3^1$.*

The main result necessary for proving the main lemma is a result we call the auxiliary lemma, from which the main lemma will follow rather easily.

If $C \subseteq \omega_1$ is unbounded, let $N_C\colon \omega_1 \to \omega_1$ be defined by: $N_C(\alpha) =$ the least element of C greater than α. If $f\colon \omega_1 \to \omega_1$ has unbounded range, let N_f denote $N_{\mathrm{ran}(f)}$. Finally, if $d = d^{(f_m)}$, let $h(d; h_1, \ldots, h_t; N_C)$ be the function from $\omega_1^m \to \omega_1$ defined by $h(d; \vec{h}; N_C)(\vec{\alpha}) = N_C(h(d; \vec{h})(\vec{\alpha}))$. Likewise we define $h(d; \vec{h}; N_f)$.

We now state the auxiliary lemma. We fix $K_1, \dots, K_t$. Then for $1 \leq k \leq t$ or $k = \infty$ we have:

THEOREM 3.2 (Auxiliary Lemma H_k). *Let $d \in \mathcal{D}$ satisfy C relative to the sequence $K_1, \dots, K_t$ and let $1 \leq k \leq k(d)$. Suppose $\theta < h(d; K_1, \dots, K_t)$. Then:*

1. *If d is non-minimal with respect to $\mathcal{L}^k$, then for almost all $h_1, \dots, h_{k-1}$, there is a $C_k \subseteq \omega_1$ such that for almost all $h_k, \dots, h_t$, $\theta(h_1, \dots, h_t) < h(\mathcal{L}^k(d); h_1, \dots, h_t; N_C)$. Here, and in (2), "$k-1$" is interpreted as equal to t if $k = \infty$.*
2. *If d is minimal with respect to $\mathcal{L}^k$, then for almost all $h_1, \dots, h_{k-1}$, there is an $\alpha < \omega_1$ such that for almost all $h_k, \dots, h_t$, $\theta(h_1, \dots, h_t) < \alpha$.*

Before turning to the proof of the auxiliary lemma, we first present a "sliding" lemma which generalizes slightly lemma 1.1. It says we can change a function on a measure zero set to make it have certain properties. Although simple, the sliding lemma and later generalizations play an important role in the arguments of this paper.

LEMMA 3.2. *Let $f\colon \operatorname{dom}(<_m) \to \omega_1$ be order preserving almost everywhere (i.e., restricted to a measure one set with respect to W_1^m). Let $g\colon \operatorname{dom}(<_j) \to \omega_1$ be order-preserving almost everywhere for some $j \leq m$. Suppose $[f(j)] < [g]$ but $[f(j-1)] = [g(j-1)]$ (if $j = 1$, we omit the later requirement). Then there is an order-preserving $f'\colon \operatorname{dom}(<_m) \to \omega_1$ with $[f'] = [f]$ and an order-preserving $g'\colon \operatorname{dom}(<_j) \to \omega_1$ with $[g'] = [g]$ such that for all limit $\vec{\alpha} = \alpha_1 < \cdots < \alpha_j$, $f'(j)(\vec{\alpha}) < g'(\vec{\alpha}) < N_{f'}(f'(j)(\vec{\alpha}))$. Also, $\operatorname{ran}(f') \subseteq \operatorname{ran}(f)$.*

PROOF. Fix f, g as in the lemma, and let $C_0 \subseteq \omega_1$ define measure one sets restricted to which f, g are order-preserving. Let $C_1 \subseteq C_0$ be c.u.b. and such that for all $\alpha_1 < \cdots < \alpha_j$ in C_1 we have:

$$\begin{aligned} &f(j)(\alpha_1, \dots, \alpha_j) < g(\alpha_1, \dots, \alpha_j) < \\ &\qquad \inf\{f(\alpha_1, \dots, \alpha_{j-2}, N_{C_1}(\alpha_{j-1}), \beta_j, \dots, \beta_{m-1}, \alpha_j) : \beta_j, \dots, \beta_{m-1} \in C_0)\}. \end{aligned}$$

Let $D \subseteq C_1$ be the set of closure points of C_1, that is, $\alpha \in D$ is the α^{th} element of C_1. For $\alpha < \omega_1$, let $l(\alpha) =$ the $2 \cdot \alpha^{th}$ element of C_1. Thus, $l(\alpha) = \alpha$ for $\alpha \in D$. Define $f'(\alpha_1, \dots, \alpha_m) = f(l(\alpha_1), \dots, l(\alpha_m))$. Clearly f' is order-preserving, as $f \restriction (C_1)^m$ is, and has range a subset of the range of f. Also, $f' \restriction D^m = f \restriction D^m$. Likewise, define $g'(\alpha_1, \dots, \alpha_j) = g(l(\alpha_1), \dots, l(\alpha_j))$. Clearly $[g'] = [g]$ and for all limit $\alpha_1 < \cdots < \alpha_j$, $g'(\vec{\alpha}) > f'(j)(\vec{\alpha})$. Finally, note that for all limit $\alpha_1 < \cdots < \alpha_j$

$$\begin{aligned} &g'(\alpha_1, \dots, \alpha_j) = g(l(\alpha_1), \dots, l(\alpha_j)) < \\ &\inf\{f(l(\alpha_1), \dots, l(\alpha_{j-2}), N_{C_1}(l(\alpha_{j-1})), \beta_j, \dots, \beta_{m-1}, l(\alpha_j)) : \beta_j, \dots, \beta_{m-1} \in C_0)\} \\ &\qquad \leq N_{f'}(f'(j)(\alpha_1, \dots, \alpha_j)), \end{aligned}$$

since $N_{f'}(f'(j)(\alpha_1, \dots, \alpha_j))$ is of the form $f'(\alpha_1, \dots, \alpha_{j-1}+1, \gamma_j, \dots, \gamma_{m-1}, \alpha_j) = f(l(\alpha_1), \dots, l(\alpha_{j-1}+1), \beta_j, \dots, \beta_{m-1}, l(\alpha_j))$ for some $\beta_j, \dots, \beta_{m-1} \in C_1$, and $l(\alpha_{j-1}+1) > N_{C_1}(l(\alpha_{j-1}))$. $\square$

REMARK 3.1. The restriction to limit $\vec{\alpha}$ is only to ensure that $f(j)(\vec{\alpha})$ is defined. If $j = m$ (so $f(j) = f$), for example, then the conclusion of the lemma holds for all $\vec{\alpha} = \alpha_1 < \cdots < \alpha_j$. In fact, if $\alpha_j > \alpha_{j-1} + m - j$ and we define $f(j)(\vec{\alpha})$ as in the limit case, then the lemma remains valid.

COROLLARY 3.1. *Let $f\colon \mathrm{dom}(<_m) \to \omega_1$ be order-preserving almost everywhere, $r\colon \omega_1 \to \omega_1$, and $j \le m$. Then there is an order-preserving $f'\colon \mathrm{dom}(<_m) \to \omega_1$ such that $[f'] = [f]$, $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$, and $\forall^*\alpha_1, \ldots, \alpha_j\ N_{f'}(f'(j)(\alpha_1, \ldots, \alpha_j)) > f(j)(\alpha_1, \ldots, r(\alpha_{j-1}), \alpha_j)$.*

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PROOF OF THEOREM 3.2. We proceed by reverse induction on k. We fix d, k, θ as in H_k, and assume $H_{k'}$, for all $k' > k$. We consider the necessary cases. As before, we employ vector notation when there is no danger of confusion; thus we write $\forall^* h_1, \ldots, h_t\ \theta(\vec{h}) < h(d; \vec{h})$ in place of $\forall^* h_1, \ldots, h_t$ $\theta(h_1, \ldots, h_t) < h(d; h_1, \ldots, h_t)$. If $\theta(h_1, \ldots, h_k) \in ON$, we write $\forall^* h_{k+1}, \ldots, h_t$ $\theta(h_1, \ldots, h_t)$ in place of $\forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_k)(h_{k+1}, \ldots, h_t)$.

case (1): $k = \infty$.

Hence, $d = d^{(f_m)} = (p)$ where $1 \le p \le m$. By assumption, $\forall^* h_1, \ldots, h_t, \theta(\vec{h}) < h(d; \vec{h})$. If we let $\theta(\vec{h})(-)$ represent $\theta(\vec{h})$ with respect to W_1^m, then for almost all $\alpha_1, \ldots, \alpha_m$ we have $\theta(\vec{h})(\vec{\alpha}) < h(d; \vec{h})(\vec{\alpha}) = \alpha_p$ from the definition of h. Hence, if $p > 1$ we have $\forall^* h_1, \ldots, h_t$, $\exists C \subseteq \omega_1$ such that for almost all $\alpha_1, \ldots, \alpha_m$, $\theta(\vec{h})(\vec{\alpha}) < N_C(\alpha_{p-1}) = h(\mathcal{L}^\infty(d); \vec{h}; N_C)$. If $p = 1$, then $\forall^* h_1, \ldots, h_t\ \exists \alpha < \omega_1$ such that for almost all $\alpha_1, \ldots, \alpha_m$, $\theta(\vec{h})(\vec{\alpha}) < \alpha$, so $\theta(\vec{h}) < \omega_1$.

For the remaining cases, we have $1 \le k \le t$.

case (2): $k < k(d)$, $K_k = W_1^r$.

First, suppose that d is non-minimal with respect to $\mathcal{L}^{k+1}$. Then, by induction $\forall^* h_1, \ldots, h_k\ \exists$ a c.u.b. $C \subseteq \omega_1$ such that $\forall^* h_{k+1}, \ldots, h_t$, $\theta(\vec{h}) < h(\mathcal{L}^{k+1}(d); \vec{h}; N_C)$. Note that if $C' \subseteq C$, then $N_{C'} \ge N_C$, and thus the above equation holds with $N_{C'}$ as well. Since the ω-cofinal normal measure on ω_2 is ω_2-additive, it follows that $\forall^* h_1, \ldots, h_{k-1}$, $\exists$ c.u.b. $C \subseteq \omega_1\ \forall^* h_k, \ldots, h_t$, $\theta(\vec{h}) < h(\mathcal{L}^{k+1}(d); \vec{h}; N_C)$. As $\mathcal{L}^k(d) = \mathcal{L}^{k+1}(d)$, we are done.

Next, suppose that d is minimal with respect to $\mathcal{L}^{k+1}$. Then, by induction, $\forall^* h_1, \ldots, h_k$, $\exists \alpha < \omega_1\ \forall^* h_{k+1}, \ldots, h_t$, $\theta(\vec{h}) < \alpha$. Now, for fixed $h_1, \ldots, h_{k-1}$ and $\theta(h_1, \ldots, h_{k-1})$, such that $\forall^* h_k\ \exists \alpha < \omega_1\ \forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_t) < \alpha$ we consider the partition $\mathcal{P}$, where we partition functions $h_k\colon r \to \omega_1$ with the extra value $\alpha > \sup h_k$ according to whether or not $\forall^* h_{k+1}, \ldots, h_t$, $\theta(h_1, \ldots, h_t) < \alpha$. Clearly on the homogeneous side of the partition the property stated in $\mathcal{P}$ holds. If we let $C \subseteq \omega_1$ be homogeneous for $\mathcal{P}$, then clearly $\forall^* h_k, \ldots, h_t$, $\theta(\vec{h}) < N_C(h_k(r))$ $= h(\mathcal{L}^k(d); \vec{h}; N_C)$ as $\mathcal{L}^k(d) = (k; r)$ in this case. This establishes H_k in this case.

case (3): $k < k(d)$, $K_k = S_1^r$.

First suppose that d is minimal with respect to $\mathcal{L}^k$, so with respect to $\mathcal{L}^{k+1}$ as well. So, by induction, $\forall^* h_1, \ldots, h_k\ \exists \alpha < \omega_1\ \forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_t) < \alpha$. Since K_k is ω_2-additive in this case, it follows immediately that $\forall^* h_1, \ldots, h_{k-1}$, $\exists \alpha < \omega_1\ \forall^* h_k, \ldots, h_t$, $\theta(h_1, \ldots, h_t) < \alpha$, and we are done.

Next, suppose d is non-minimal with respect to $\mathcal{L}^k$, hence non-minimal with respect to $\mathcal{L}^{k+1}$ (from the definition of $\mathcal{L}^k$ in this case). By induction, $\forall^* h_1, \ldots,$ $h_k\ \exists C \subseteq \omega_1\ \forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_t) < h(\mathcal{L}^{k+1}(d); h_1, \ldots, h_t; N_C)$. Now, for fixed $h_1, \ldots, h_{k-1}$ and $\theta(h_1, \ldots, h_{k-1})$ such that $\forall^* h_k\ \exists C \subseteq \omega_1\ \forall^* h_{k+1}, \ldots, h_t$ $\theta(h_1, \ldots, h_t) < h(\mathcal{L}^{k+1}(d); h_1, \ldots, h_t; N_C)$ we consider the following partition:

$\mathcal{P}(h_1, \ldots, h_{k-1})$: We partition functions $h_k\colon \mathrm{dom}(<_r) \to \omega_1$ of the correct type with the "extra value" $g(\alpha)$ of uniform cofinality ω inserted between $h_k(1)(\alpha) \doteq$

$\sup_{\beta_1<\ldots<\beta_{r-1}<\alpha} h_k(\beta_1, \ldots, \beta_{r-1}, \alpha)$ and $N_{h_k}(h_k(1)(\alpha))$ (more precisely, we partition functions from the corresponding order type into ω_1 of the correct type) according to whether or not $\forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_t) < h(\mathcal{L}^{k+1}(d); h_1, \ldots, h_t; N_g)$.

We claim that on the homogeneous side of the partition, the property stated in $\mathcal{P}(h_1, \ldots, h_{k-1})$ holds. We suppose not, and fix a c.u.b. $C \subseteq \omega_1$ homogeneous for the contrary side, and such that for $h_k\colon \mathrm{dom}(<_r) \to C$ of the correct type $\exists g$ $\forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_t) < h(\mathcal{L}^{k+1}(d);\, h_1, \ldots, h_t,\, N_g)$. Let C' = the set of closure points of C. Fix $h_k\colon \mathrm{dom}(<_r) \to C'$ of the correct type and fix $g\colon \omega_1 \to \omega_1$ such that $\forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_t) < h(\mathcal{L}^{k+1}(d); h_1, \ldots, h_t; N_g)$. Note that this inequality depends only on $[g]_{W^1_1}$. From lemma 3.2 with $j = 1$, there is an order-preserving $h'_k\colon \mathrm{dom}(<_r) \to \mathrm{ran}(h_k) \subseteq C'$ with $[h'_k] = [h_k]$ and $[g] < [N_{h'_k}(h'_k(1))]$. Note that h'_k has uniform cofinality ω as h_k does. Easily, there is a $g'\colon \omega_1 \to C$ with $[g'] = [g]$ and h'_k, g' ordered as in $\mathcal{P}(h_1, \ldots, k_{k-1})$. This contradicts the assumed homogeneity of C.

Hence, on the homogeneous side of the partition the property stated in $\mathcal{P}$ holds. Let $C \subseteq \omega_1$ be homogeneous for $\mathcal{P}(h_1, \ldots, h_{k-1})$. Then, for almost all h_k, the function $g \doteq N_C \circ h_k(1)$ satisfies $\forall^* h_{k+1}, \ldots, h_t\ \theta(\vec{h}) < h(\mathcal{L}^{k+1}(d); \vec{h}; N_g)$. However, $h(\mathcal{L}^{k+1}(d); \vec{h}; N_g) = h(\mathcal{L}^k(d); \vec{h}; N_C)$, as $\mathcal{L}^k(d) = (k; \mathcal{L}^{k+1}(d))$ in this case. This establishes H_k in this case.

<u>case (4)</u>: $k = k(d)$, $K_k = W^r_1$.

Here $d = (k;p)$ where $1 \leq p \leq r$, and $\forall^* h_1, \ldots, h_t\ \theta(\vec{h}) < h_k(p)$. For $p = 1$ (so d is minimal with respect to $\mathcal{L}^k$) it follows that $\forall^* h_1, \ldots, h_k\ \exists \alpha < h_k(1)\ \forall^* h_{k+1}, \ldots, h_t\ \theta(\vec{h}) < \alpha$. From this it easily follows that $\forall^* h_1, \ldots, h_{k-1}\ \exists \alpha < \omega_1\ \forall^* h_k, \ldots, h_t\ \theta(\vec{h}) < \alpha$ by normality. The $p > 1$ case is similar.

<u>case (5)</u>: $k = k(d)$, $K_k = S^r_1$, $d = (k; d_r, d_1, \ldots, d_l)$, where $l = r - 1$.

For fixed $h_1, \ldots, h_{k-1}$ and $\theta(h_1, \ldots, h_{k-1})$ such that $\forall^* h_k, \ldots, h_t\ \theta(h_1, \ldots, h_t) < h(d; h_1, \ldots, h_t)$ we consider the partition $\mathcal{P}_k(h_1, \ldots, h_{k-1})$: we partition $h_k\colon \mathrm{dom}(<_r) \to \omega_1$ of the correct type with the extra value $g(\alpha_1, \ldots, \alpha_r)$ of uniform cofinality ω inserted between $\sup_{\beta<\alpha_{r-1}} h_k(\alpha_1, \alpha_2, \ldots, \beta, \alpha_r)$ and $h_k(\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, \alpha_r)$, according to whether or not $\forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_t) < h(\mathcal{L}^k(d); h_1, \ldots, h_t; N_g)$. We claim that on the homogeneous side of the partition the property stated in $\mathcal{P}_k(h_1, \ldots, h_{k-1})$ holds. We suppose not, and let $C \subseteq \omega_1$ be homogeneous for the contrary side. We fix an $\bar{h}_k\colon (\omega\times <_r) \to C$ of the correct type inducing $h_k\colon \mathrm{dom}(<_r) \to C$ of the correct type, and such that $\forall^* h_{k+1}, \ldots, h_t\ \theta(\vec{h}) < h(d; \vec{h})$. Now, if $d = d^{(f_m)}$, then $\forall^* h_{k+1}, \ldots, h_t\ \forall^* \alpha_1, \ldots, \alpha_m\ \theta(\vec{h})(\vec{\alpha}) < h(d; \vec{h})(\vec{\alpha}) = h_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_{r-1}; \vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha}))$. Hence, $\forall^* h_{k+1}, \ldots, h_t\ \forall^* \alpha_1, \ldots, \alpha_m\ \exists p < \omega\ \theta(\vec{h})(\vec{\alpha}) < \bar{h}_k(p, h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_{r-1}; \vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha}))$. Hence, by countable additivity of the measures involved, there is a $p \in \omega$ such that $\forall^* h_{k+1}, \ldots, h_t\ \forall^* \alpha_1, \ldots, \alpha_m$ the above holds. Fixing such a p, we then let $g(\alpha_1, \ldots, \alpha_{r-1}, \alpha_r) = \bar{h}_k(p, \alpha_1, \ldots, \alpha_{r-1}, \alpha_r)$. Then, $\forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_t) < h(\mathcal{L}^k(d); h_1, \ldots, h_t; N_g)$, as $\mathcal{L}^k(d) = (k; d_r, \ldots, d_l)^s$. This contradicts the definition of C. Hence, on the homogeneous side of the partition, the property stated in $\mathcal{P}_k(h_1, \ldots, h_{k-1})$ holds. We fix $C \subseteq \omega_1$ homogeneous for $\mathcal{P}_k$, and it now follows easily that $\forall^* h_k, \ldots, h_t\ \theta(\vec{h}) < h(\mathcal{L}^k(d); \vec{h}; N_C)$, which establishes H_k.

<u>case (6)</u>: $k = k(d)$, $K_k = S^r_1$, $d = (k; d_r, d_1, \ldots, d_l)^s$, where $l = r - 1$.

We suppose first that $\mathcal{L}^{k+1}(d_l)$ is defined and is $> d_{l-1}$ (if $l > 1$). Suppose $d = d^{(f_m)}$ (if $d = d^{()}$, omit the ordinals $\vec{\alpha}$ in the argument below). In this case $\mathcal{L}^k(d) = (k; d_r, \ldots, d_{l-1}, \mathcal{L}^{k+1}(d_l))$. Now, $\forall^* h_1, \ldots, h_t\ \forall^* \alpha_1, \ldots, \alpha_m\ \theta(\vec{h})(\vec{\alpha}) < h(d; \vec{h})(\vec{\alpha}) = \sup_{\beta < h(d_l;\vec{h})(\vec{\alpha})} h_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_{l-1}; \vec{h})(\vec{\alpha}), \beta, h(d_r; \vec{h})(\vec{\alpha}))$.

Hence, $\forall^* \vec{h}\ \forall^* \vec{\alpha}$ there is a $\bar{\theta}(\vec{h})(\vec{\alpha}) < h(d_l; \vec{h})(\vec{\alpha})$ such that $\theta(\vec{h})(\vec{\alpha}) < h_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_{l-1}; \vec{h})(\vec{\alpha}), \bar{\theta}(\vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha}))$. So, there is an ordinal $\bar{\theta}$ such that $\forall^* \vec{h}\ \bar{\theta}(\vec{h}) < h(d_l; \vec{h})$ and such that $\forall^* \vec{h}\ \forall^* \vec{\alpha}\ \theta(\vec{h})(\vec{\alpha}) < h_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, \bar{\theta}(\vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha}))$. By induction (since $k(d_l) > k$) $\forall^* h_1, \ldots, h_k\ \exists C \subseteq \omega_1\ \forall^* h_{k+1}, \ldots, h_t\ \bar{\theta}(\vec{h}) < h(\mathcal{L}^{k+1}(d_l); \vec{h}; N_C)$.

Now, for fixed $h_1, \ldots, h_{k-1}$, and $\theta(h_1, \ldots, h_{k-1})$, $\bar{\theta}(h_1, \ldots, h_{k-1}) \in ON$ such that (a) $\forall^* h_k, \ldots, h_t\ \forall^* \alpha\ \theta(h_1, \ldots, h_t)(\vec{\alpha}) < h_k(\ldots, \bar{\theta}(\vec{h})(\vec{\alpha}), \ldots)$ and (b) $\forall^* h_k\ \exists C_1 \subseteq \omega_1\ \forall^* h_{k+1}, \ldots, h_t\ \bar{\theta}(\vec{h}) < h(\mathcal{L}^{k+1}(d_l); \vec{h}; N_{C_1})$, we consider the partition $\mathcal{P}_k(h_1, \ldots, h_{k-1})$: we partition $h_k\colon \mathrm{dom}(<_r) \to \omega_1$ of the correct type with the extra value $g(\alpha_1, \ldots, \alpha_r)$ of uniform cofinality ω inserted between $h_k(\alpha_1, \ldots, \alpha_r)$ and $N_{h_k}(h_k(\alpha_1, \ldots, \alpha_r))$, according to whether $\forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_t) < h(\mathcal{L}^k(d); h_1, \ldots, h_t; N_g)$. We claim again that on the homogeneous side of the partition the property stated in $\mathcal{P}_k(h_1, \ldots, h_{k-1})$ holds. We suppose not, and fix a c.u.b. $C \subseteq \omega_1$ homogeneous for the contrary side. Fix $h_k\colon \mathrm{dom}(<_r) \to C$ of the correct type such that (a) is satisfied and $\exists C_1 \subseteq \omega_1$ with (b) satisfied (for almost all $h_{k+1}, \ldots, h_t$). Fix such a C_1. In particular,

$$(1)\quad \forall^* h_{k+1}, \ldots, h_t\ \forall^* \vec{\alpha}\ \theta(\vec{h})(\vec{\alpha}) < h_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_{l-1}; \vec{h})(\vec{\alpha}), N_{C_1}(h(\mathcal{L}^{k+1}(d_l); \vec{h})(\vec{\alpha})), h(d_r; \vec{h})(\vec{\alpha})).$$

By corollary 3.1, there is an order-preserving $h'_k\colon \mathrm{dom}(<_r) \to \mathrm{ran}(h_k) \subseteq C$ (thus h'_k is of the correct type) with $[h'_k] = [h_k]$ and such that

$$\forall^* \beta_1, \ldots, \beta_r\ N_{h'_k}(h'_k(\beta_1, \ldots, \beta_r)) > h_k(\beta_1, \ldots, N_{C_1}(\beta_{r-1}), \beta_r).$$

Let g be defined by $g(\beta_1, \ldots, \beta_r) = h_k(\beta_1, \ldots, N_{C_1}(\beta_{r-1}), \beta_r)$. Then there is a g' with $[g'] = [g]$ and h'_k, g' ordered as in $\mathcal{P}(h_1, \ldots, h_{k-1})$.

Note that $\forall^* h_{k+1}, \ldots, h_t\ h(d_i; h_1, \ldots, h_k, \ldots, h_t) = h(d_i; h_1, \ldots, h'_k, \ldots, h_t)$, for $1 \le i \le r$. Also, $\forall^* h_{k+1}, \ldots, h_t\ \theta(h_1, \ldots, h_k, \ldots, h_t) = \theta(h_1, \ldots, h'_k, \ldots, h_t)$, as $[h_k] = [h'_k]$. From the homogeneity of C for the contrary side of the partition we thus have:

$$\forall^* h_{k+1}, \ldots, h_t\ \forall^* \vec{\alpha}\ \theta(\vec{h})(\vec{\alpha}) \ge h(\mathcal{L}^k(d); h_1, \ldots, h'_k, \ldots, h_t; N_g)(\vec{\alpha}) = N_g(h'_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(\mathcal{L}^{k+1}(d_{r-1}); \vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha}))).$$

We may also assume that $h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(d_r; \vec{h})(\vec{\alpha})$ lie in a c.u.b. set on which h'_k, h_k agree, as $k(d_1), \ldots, k(d_r) > k$. Thus,

$$(2)\quad \forall^* h_{k+1}, \ldots, h_t\ \forall^* \vec{\alpha}\ \theta(\vec{h})(\vec{\alpha}) \ge N_g(h_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, h(\mathcal{L}^{k+1}(d_{r-1}); \vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha}))) = h_k(h(d_1; \vec{h})(\vec{\alpha}), \ldots, N_{C_1}(h(\mathcal{L}^{k+1}(d_l); \vec{h})(\vec{\alpha})), h(d_r; \vec{h})(\vec{\alpha})).$$

Equations (1), (2) give a contradiction. Hence, on the homogeneous side of the partition, the property stated in $\mathcal{P}_k(h_1, \dots, h_{k-1})$ holds. Fixing a c.u.b. $C \subseteq \omega_1$ homogeneous for $\mathcal{P}_k$ now witnesses H_k.

If $\mathcal{L}^{k+1}(d_l) \le d_{l-1}$, or $\mathcal{L}^{k+1}(d_l)$ is not defined, we proceed along the above lines, using instead the partition $\mathcal{P}_k(h_1, \dots, h_{k-1})$ where the extra value $g(\alpha_1, \dots, \alpha_{r-2}, \alpha_r)$ is inserted between $\gamma \doteq \sup_{\beta_1 < \alpha_{r-2}, \beta_2 < \alpha_r} h_k(\alpha_1, \dots, \beta_1, \beta_2, \alpha_r)$ and $N_{h_k}(\gamma)$, where if $r = 2$ we use $\gamma = \sup_{\beta_1 < \beta_2 < \alpha_r} h_k(\beta_1, \beta_2)$. It follows by an argument similar to that above that on the homogeneous side of the partition the desired property holds. Again, this establishes H_k (for the case $r = 2$, we use also the fact that for almost all α, $\sup_{\beta_1 < \beta_2 < \alpha} h_k(\beta_1, \beta_2) = \alpha$).

The remaining cases are similar to the previous one, so we merely sketch the differences.

case (7): $k = k(d)$, $K_k = S_1^r$, $l < r - 1$, $d = (k; d_r, d_1, \dots, d_l)$.

We consider the case where $\mathcal{L}^{k+1}(d_r)$ is defined and $> d_l$, the other cases again following along similar but somewhat easier lines. From the definition of $<_r$ and the fact that h_k may assumed to be order-preserving we have:

$$\forall^* \vec{h}\ \forall^* \vec{\alpha}\ \theta(\vec{h})(\vec{\alpha}) < h(d; \vec{h})(\vec{\alpha}) = \sup_{\beta_{l+1} < h(d_r; \vec{h})(\vec{\alpha})} h_k(l+2)(h(d_1; \vec{h})(\vec{\alpha}), \dots, h(d_l; \vec{h})(\vec{\alpha}), \beta_{l+1}, h(d_r; \vec{h})(\vec{\alpha})).$$

It follows that there is a $\bar{\theta}$ such that $\forall^* \vec{h}\ \bar{\theta}(\vec{h}) < h(d_r; \vec{h})$ and

$$\forall^* \vec{h}\ \forall^* \vec{\alpha}\ \theta(\vec{h})(\vec{\alpha}) < h_k(l+2)(h(d_1; \vec{h})(\vec{\alpha}), \dots, h(d_l; \vec{h})(\vec{\alpha}), \bar{\theta}(\vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha})).$$

We apply induction to $\bar{\theta}$, and then proceed as in case (6), where we use instead the partition $\mathcal{P}_k(h_1, \dots, h_{k-1})$: we partition $h_k\colon \operatorname{dom}(<_r) \to \omega_1$ of the correct type with the extra value $g(\alpha_1, \dots, \alpha_{l+1}, \alpha_r)$ inserted between $\gamma \doteq h_k(l+2)(\alpha_1, \dots, \alpha_{l+1}, \alpha_r)$ and $N_{h_k}(\gamma)$ according to whether or not

$$\forall^* h_{k+1}, \dots, h_t\ \theta(h_1, \dots, h_t) < h(\mathcal{L}^k(d); h_1, \dots, h_t; N_g).$$

As in case (6), we establish that on the homogeneous side of the partition the property stated in $\mathcal{P}_k(h_1, \dots, h_{k-1})$ holds, this time using corollary 3.1 with $j = l + 2$. A c.u.b. $C \subseteq \omega_1$ homogeneous for $\mathcal{P}_k$ again witnesses H_k.

case (8): $k = k(d)$, $K_k = S_1^r$, $l < r - 1$, $d = (k; d_r, d_1, \dots, d_l)^s$.

We again consider the case where $\mathcal{L}^{k+1}(d_l)$ is defined and $> d_{l-1}$ (if $l > 1$), the other cases following by similar but easier arguments. We get that there is a $\bar{\theta}$ such that $\forall^* \vec{h}\ \bar{\theta}(\vec{h}) < h(d_l; \vec{h})$ and

$$\forall^* \vec{h}\ \forall^* \vec{\alpha}\ \theta(\vec{h})(\vec{\alpha}) < h_k(l+1)(h(d_1; \vec{h})(\vec{\alpha}), \dots, h(d_{l-1}; \vec{h})(\vec{\alpha}), \bar{\theta}(\vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha})).$$

We apply induction to $\bar{\theta}$, and for appropriate fixed $h_1, \dots, h_{k-1}$ and ordinals $\theta(h_1, \dots, h_{k-1})$, $\bar{\theta}(h_1, \dots, h_{k-1})$ consider the partition $\mathcal{P}_k(h_1, \dots, h_{k-1})$: we partition $h_k\colon \operatorname{dom}(<_r) \to \omega_1$ of the correct type with the extra value $g(\alpha_1, \dots, \alpha_l, \alpha_r)$ inserted between $h_k(l+1)(\alpha_1, \dots, \alpha_l, \alpha_r)$ and $N_{h_k}(h_k(l+1)(\alpha_1, \dots, \alpha_l, \alpha_r))$ according to whether

$$\forall^* h_{k+1}, \dots, h_t\ \forall^* \vec{\alpha}\ \theta(\vec{h})(\vec{\alpha}) < h(\mathcal{L}^k(d); h_1, \dots, h_t; N_g) = N_g(h_k(l+1)(h(d_1; \vec{h})(\vec{\alpha}), \dots, h(d_{l-1}; \vec{h})(\vec{\alpha}), h(\mathcal{L}^{k+1}(d_l); \vec{h})(\vec{\alpha}), h(d_r; \vec{h})(\vec{\alpha}))).$$

Using corollary 3.1 with $j = l+1$, it follows that on the homogeneous side of the partition the property stated in $\mathcal{P}_k$ holds, and a c.u.b. $C \subseteq \omega_1$ homogeneous for $\mathcal{P}_k$ again witnesses H_k.

This completes the proof of the auxiliary lemma H_k in all cases. $\square$

We now return to the proof of the main lemma. The arguments are similar to those of the auxiliary lemma, so we will be somewhat abbreviated.

We first consider the case where s does not appear, so $\theta < (id; (d); K_1, \dots, K_t)$, and $\mathcal{L}((d)) = (d)^s$, where $d \in \mathcal{D}_m$. We fix $F \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ representing θ with respect to W^m_3, and let $F(f)$ denote $F([f]_{S^m_1})$ for $f \colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$. Hence, for almost all $f \colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type, $F(f) < (id; f; (d); K_1, \dots, K_t)$. We consider the partition $\mathcal{P}$: we partition functions $f \colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type, with the extra value $g(\alpha)$ of uniform cofinality ω inserted between $\sup_{\beta<\alpha} f(\beta)$ and $f(\alpha)$, according to whether or not $F(f) < (N_g; f; (d)^s; K_1, \dots, K_t)$. We claim that on the homogeneous side of the partition, the property stated in $\mathcal{P}$ holds. If not, we fix a $C \subseteq \boldsymbol{\delta}^1_3$ homogeneous for the contrary side, and fix $\bar{f} \colon \omega \times \omega_{m+1} \to C$ witnessing $f \colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ is of uniform cofinality ω, and such that $F(f) < (id; f; (d); K_1, \dots, K_t)$. Hence, $\forall^* \vec{h}\ \exists n \in \omega\ F(f)(h_1, \dots, h_t) < \bar{f}(n, h(d; \vec{h}))$, and so by countable additivity $\exists n \in \omega\ \forall^* h_1, \dots, h_t\ F(f)(\vec{h}) < \bar{f}(n, h(d; \vec{h}))$. Letting $g(\alpha) = \bar{f}(n, \alpha)$, we now contradict the homogeneity of C for the contrary side. Fixing now a $C \subseteq \boldsymbol{\delta}^1_3$ homogeneous for $\mathcal{P}$, and letting $g = N_C$, we have $\theta < (g; (d)^s; K_1, \dots, K_t)$.

We next suppose s appears, so $\theta < (id; (d)^s; K_1, \dots, K_t)$. We assume d is non-minimal with respect to $\mathcal{L}$, the minimal case following easily from the auxiliary lemma. We first show that for some $g \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$, $\theta < (g; \mathcal{L}((d)^s); K_1, \dots, K_t)$. We fix F as above and consider the partition $\mathcal{P}$: we partition $f : \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type with the extra value $g(\alpha)$ of uniform cofinality ω inserted between $f(\alpha)$ and $f(\alpha+1)$ according to whether or not $F(f) < (N_g; f; \mathcal{L}((d)^s); K_1, \dots, K_t)$. We again claim that on the homogeneous side of the partition the property stated in $\mathcal{P}$ holds. If not, we fix $C \subseteq \boldsymbol{\delta}^1_3$ homogeneous for the contrary side and let C' = the set of closure points of C. We fix $f \colon \omega_{m+1} \to C'$ of the correct type with $F(f) < (id; f; (d)^s; K_1, \dots, K_t)$. So, $\forall^* \vec{h}\ F(f)(\vec{h}) < (id; f; (d)^s; \vec{h}) = \sup_{\beta < h(d;\vec{h})} f(\beta)$. Hence there is a θ with $\theta < h(d; K_1, \dots, K_t)$ and $\forall^* \vec{h}\ F(f)(\vec{h}) < f(\theta(\vec{h}))$. By the auxiliary lemma, there is a c.u.b. $C_0 \subseteq \omega_1$ such that $\forall^* h_1, \dots, h_t\ \theta(\vec{h}) < h(\mathcal{L}(d); \vec{h}; N_{C_0})$. We fix such a C_0. We now claim that there are f', g satisfying:

(1) $f', g \colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ are of the correct type, ordered as in $\mathcal{P}$, $[f'] = [f]$, f', g have range in C, and $f' \geq f$ everywhere.

(2) For all $\eta < \omega_{m+1}$, $g(\eta) \geq f(p(\eta))$, where if $\vec{\alpha} \to \eta(\vec{\alpha})$ represents η with respect to W^m_1, then $p(\eta)$ is represented by $\vec{\alpha} \to N_{C_0}(\eta(\vec{\alpha}))$.

In fact, we may define f', g as follows. Let C'_0 = the set of closure points of C_0. For $\alpha < \omega_1$, let $l(\alpha) = \alpha^{th}$ element of C'_0. Define $\bar{l} \colon \omega_{m+1} \to \omega_{m+1}$ by: if $\vec{\alpha} \to \eta(\vec{\alpha})$ represents $\eta < \omega_{m+1}$ with respect to W^m_1, then $\bar{l}(\eta)$ is represented by $\vec{\alpha} \to l(\eta(\vec{\alpha}))$. For $\eta < \omega_{m+1}$, define $f'(\eta) = f(\bar{l}(\eta))$. Similarly, for $\eta < \omega_{m+1}$, define $g(\eta) = f(\bar{\eta})$, where $\bar{\eta}$ is represented by $\vec{\alpha} \to \bar{\eta}(\vec{\alpha}) = \omega^{th}$ element of C_0 after $l(\eta(\vec{\alpha}))$. It is not difficult to verify that f', g have the desired properties.

From the homogeneity of C for the contrary side of the partition we have:

$$\forall^* h_1, \dots, h_t\ F(f'; \vec{h}) \geq (id; f'; \mathcal{L}((d)^s); \vec{h}; N_g) = N_g(f'(h(\mathcal{L}(d); \vec{h}))).$$

However, we also have:

$$\forall^* h_1, \dots, h_t\ F(f'; \vec{h}) = F(f, \vec{h}) < f(\theta(\vec{h})) \leq \\ f(p(h(\mathcal{L}(d); \vec{h}))) \leq N_g(f'(h(\mathcal{L}(d); \vec{h}))).$$

This contradiction establishes that on the homogeneous side of the partition, the property stated in $\mathcal{P}$ holds. If we let $C \subseteq \boldsymbol{\delta}^1_3$ be homogeneous for $\mathcal{P}$ and $g = N_C$, it now follows easily that $\theta < (g; \mathcal{L}((d)^s); K_1, \dots, K_t)$.

If $\mathcal{L}((d)^s)$ satisfies D, we are done. Otherwise, it is not difficult to see from a "sliding argument" similar to the one above (but easier) that in fact $\theta < (id; \mathcal{L}((d)^s); K_1, \dots, K_t)$. We are then in a position to repeat the above argument to get $\theta < (g; \mathcal{L}^{(2)}((d)^s); K_1, \dots, K_t)$ for some g. Continuing in this manner, we establish the main lemma.

Finally, if $(d)^{(s)} = ()^{(s)}$, then the proof of the auxiliary lemma shows that for almost all $f\colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$, there is a $C_0 \subseteq \omega_1$ such that $\forall^* \vec{h}$, $F(f; \vec{h}) < (id; f; \tilde{d}; \vec{h}; N_{C_0})$, where $\tilde{d}$ is the maximal description. The remainder of the argument is as in the case above.

This establishes the main lemma. Λ

The Main Theorem

In order to state our main theorem, we need to introduce an ordering on a certain countable set.

Definition 3.8. For a fixed index (which we suppress writing), we define an ordering $<$ on tuples of the form $((d)^{(s)}; K_1, \dots, K_t)$, where $(d)^{(s)} \in \bar{\mathcal{D}}(K_1, \dots, K_t)$, and $K_1 \dots, K_t \in W_1 \cup S_1$. We let $<$ be the transitive relation generated by the relations:

(i) $(\bar{\mathcal{L}}((d)^{(s)}); K_1, \dots, K_t, K_{t+1}) < ((d)^{(s)}; K_1, \dots, K_t)$ for all $K_{t+1} \in W_1 \cup S_1$. Here $\mathcal{L}((d)^{(s)}) = \mathcal{L}((d)^{(s)}; K_1, \dots, K_t)$ refers to $\mathcal{L}((d)^{(s)})$ computed relative to the sequence $K_1, \dots, K_t$.

(ii) $((d_2)^{(s)}; K_1, \dots, K_t) < ((d_1)^{(s)}; K_1, \dots, K_t)$ if $d_2 < d_1$ or $d_2 = d_1$ and s appears in the first expression but not in the second.

It is not difficult to check that this relation is well-founded. Indeed, if this were not the case, then since $\mathcal{D}(K_1, \dots, K_t)$ is finite for fixed $K_1, \dots, K_t$, we would have a sequence

$$((d_1)^{(s)}; K_1, \dots, K_{t_1}) > ((d_2)^{(s)}; K_1, \dots, K_{t_2}) > \dots$$

where $t_1 < t_2 < \dots$, $d_i \in \mathcal{D}(K_1, \dots, K_{t_i})$, and $d_i > d_{i+1}$ for all i. Hence, for each i we have $\forall^* h_1, \dots, h_{t_{i+1}}\ h(d_i, h_1, \dots, h_{t_{i+1}}) > h(d_{i+1}; h_1, \dots, h_{t_{i+1}})$. By countable additivity we could then fix a sequence $h_1, h_2, \dots$, such that $h(d_1; h_1, \dots, h_{t_1}) > h(d_2; h_1, \dots, h_{t_2}) > \dots$, a contradiction.

We let $|((d)^{(s)}; K_1, \dots, K_t)|$ denote the rank of $((d)^{(s)}; K_1, \dots, K_t)$ in this ordering. Also, let $f((d)^{(s)}; K_1, \dots, K_t)$ denote the rank of $((d)^{(s)}; K_1, \dots, K_t)$ in this ordering, but computed in a somewhat non-standard fashion, that is, $f(t) = (\sup_{u<t} f(u)) + 1$ for tuples $t, u \in$ field$(<)$. Notice that $f((d)^{(s)}; K_1, \dots, K_t)$ is always a successor ordinal, and $f((d)^{(s)}; K_1, \dots, K_t) = |((d)^{(s)}; K_1, \dots, K_t)| + 1$.

We are now in a position to state our main theorem.

THEOREM 3.3 (Main Theorem). *Let* $(d)^{(s)} \in \bar{\mathcal{D}}(K_1, \ldots, K_t)$ *satisfy D. Then* $(id; (d)^{(s)}; K_1, \ldots, K_t) \leq \aleph_{\omega+f((d)^{(s)}; K_1, \ldots, K_t)}$.

PROOF. Fix $(d)^{(s)}$, $K_1, \ldots, K_t$ as above, and an ordinal $\alpha < (id; (d)^{(s)}; K_1, \ldots, K_t)$. If $(d)^{(s)}$ is minimal with respect to $\bar{\mathcal{L}}$, then from the main lemma 3.1 $\alpha < \boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$, and $f((d)^{(s)}; K_1, \ldots, K_t) = 1$, and we are done. So, assume $(d)^{(s)}$ is not minimal with respect to $\bar{\mathcal{L}}$. From the main lemma, there is a $g\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ such that $\alpha < (g; \bar{\mathcal{L}}((d)^{(s)}); K_1, \ldots, K_t)$. From Martin's theorem 2.1 and the local embedding theorem 1.1 it follows that there is a well-founded tree T on $\boldsymbol{\delta}^1_3$ and a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for $\beta \in C$, $g(\beta) \leq |T \restriction (\sup_j j(\beta))|$, where the supremum ranges over embeddings corresponding to measures in the canonical families W_1 and S_1. We fix such T, C.

We define now a well-ordering $<_0$ on the ordinals less than $(\sup_j j; d'; K_1, \ldots, K_t)$, where $d' = \bar{\mathcal{L}}((d)^{(s)})$. We define $\theta_1 <_0 \theta_2$ to hold iff

$$\forall^* f, h_1, \ldots, h_t \ |T \restriction (\sup_j j((id; f; d'; \vec{h})))(\theta_1(f; \vec{h}))| < |T \restriction \sup_j j((id; f; d'; \vec{h})))(\theta_2(f; \vec{h}))|,$$

where we recall $T \restriction \gamma(\delta)$ denotes the subtree of $T \restriction \gamma$ consisting of the elements of $T \restriction \gamma$ below δ in the tree ordering. Now, if β is any ordinal less than $(g; d'; K_1, \ldots, K_t)$, it follows from the definitions of T, C (since almost all $f\colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ have range in C) that there is a $\theta < (\sup_j j; d', K_1, \ldots, K_t)$ such that

$$\forall^* f, h_1, \ldots, h_t \ \beta(f; \vec{h}) = |T \restriction \sup_j j((id; f; d'; \vec{h}))(\theta(f; \vec{h}))|.$$

From this it follows that $| <_0 | \geq (g; d'; K_1, \ldots, K_t)$. Hence,

$$(g; d'; K_1, \ldots, K_t) < (\sup_j j; d'; K_1, \ldots, K_t)^+.$$

Now, if j is the embedding corresponding to the measure $K_{t+1} \in W_1 \cup S_1$, it follows immediately from the definition of these ordinals that

$$(j; d'; K_1, \ldots, K_t) = (id; d'; K_1, \ldots, K_t, K_{t+1}).$$

Also, by countably additivity

$$(\sup_j j; d'; K_1, \ldots, K_t) = \sup_j (j; d'; K_1, \ldots, K_t).$$

Hence,

$$(g; d'; K_1, \ldots, K_t) < [\sup_{K_{t+1}} (id; d'; K_1, \ldots, K_t, K_{t+1})]^+,$$

the supremum ranging over measures $K_{t+1} \in W_1 \cup S_1$. Since α was arbitrary, we have that

$$(id; (d)^{(s)}; K_1, \ldots, K_t) \leq [\sup_{K_{t+1}} (id; \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t, K_{t+1})]^+.$$

By induction, we may assume that

$$(id; \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t, K_{t+1}) \leq \aleph_{\omega+f(\mathcal{L}((d)^{(s)}); K_1, \ldots, K_{t+1})}.$$

Hence, from the definition of $<$ and f we have:

$$(id;(d)^{(s)};K_1,\dots,K_t)\le\left[\sup_{K_{t+1}}\aleph_{\omega+f(\mathcal{L}((d)^{(s)});K_1,\dots,K_{t+1})}\right]^+=$$
$$\aleph_{\omega+[\sup\limits_{K_{t+1}} f(\mathcal{L}((d)^{(s)});K_1,\dots,K_{t+1})]+1}\le\aleph_{\omega+f((d)^{(s)};K_1,\dots,K_t)}.$$

□

A Rank Computation

Our final goal in this chapter is to compute a suitable upper bound for $f((d)^{(s)};K_1,\dots,K_t)$. Together with theorem 3.3, this will yield an upper bound for $\boldsymbol{\delta}^1_5$.

We extend the ordering $<$ to tuples of the form $(d;K_1,\dots,K_t)$ where d satisfies C with respect to $K_1,\dots,K_t$ (and we are considering a fixed index) by letting $<$ be the transitive relation generated by the relations:

(i) $(\mathcal{L}(d);K_1,\dots,K_t,K_{t+1})<(d;K_1,\dots,K_t)$ for all $K_{t+1}\in W_1\cup S_1$. Again, $\mathcal{L}(d)$ here is computed relative to $K_1,\dots,K_t$.

(ii) $(d_2;K_1,\dots,K_t)<(d_1;K_1,\dots,K_t)$ if $d_2<d_1$.

If is easy to check, as before, that this relation is well-founded. We also let $|(d;K_1,\dots,K_t)|$, $f(d;K_1,\dots,K_t)$ denote the ranks of $(d;K_1,\dots,K_t)$ in this ordering, computed in the usual and non-standard manners as before (so $f(d;K_1,\dots,K_t)=|(d;K_1,\dots,K_t)|+1$).

In order to compute the upper bound for $f(d;K_1,\dots,K_t)$ (from which the bound for $f((d)^{(s)};K_1,\dots,K_t)$ will follow immediately) we need to introduce some auxiliary functions.

Definition 3.9. We let $<^k$, for $k\in\omega$ or $k=\infty$ be the relation with field the set of tuples of the form $(d;K_1,\dots,K_t)$ where d satisfies C relative to $K_1,\dots,K_t$ and $k(d)\ge k$, and defined to be the transitive relation generated by the relations:

(i) $(\mathcal{L}^k(d);K_1,\dots,K_t,K_{t+1})<^k(d;K_1,\dots,K_t)$ for all $K_{t+1}\in W_1\cup S_1$.

(ii) $(d_2;K_1,\dots,K_t)<^k(d_1;K_1,\dots,K_t)$ for $d_2<d_1$.

We let $|(d;K_1,\dots,K_t)|_k$, $f^k(d;K_1,\dots,K_t)$ be the ranks of $(d;K_1,\dots,K_t)$ in $<^k$ as before. Note that $|(d;K_1,\dots,K_t)|=|(d;K_1,\dots,K_t)|_1$, $f(d;K_1,\dots,K_t)=f^1(d;K_1,\dots,K_t)$.

We next introduce the auxiliary functions $\bar f^k$:

Definition 3.10. For d satisfying C relative to $K_1,\dots,K_t$, $1\le k\le t$ or $k=\infty$, and $k(d)\ge k$, we define $\bar f^k(d;K_1,\dots,K_t)$ by reverse induction on k (for fixed $K_1,\dots,K_t$) through the following cases. In the formulas below, we let $\bar f^k(d)$, $\bar f^{k+1}(d)$ abbreviate $\bar f^k(d;K_1,\dots,K_t)$, $\bar f^{k+1}(d;K_1,\dots,K_t)$. Also, we interpret "$\bar f^{k+1}$" as $\bar f^\infty$ if $k=t$.

I: $k=\infty$. So, $d=d^{(f_m)}=(i)$ for $1\le i\le m$. We set $\bar f^\infty(d)=\omega^{\omega^i}$.

II: $1\le k\le t$.

(1) $k=k(d)$.

(a) d basic type type -1, so $d=(k;p)$. We set $\bar f^k(d)=\omega+p$.

(b) $d=(k;d_r,d_1,\dots,d_l)$ where $l=r-1$ and $K_k=S^r_1$. We set $\bar f^k(d)=\bar f^k(d')+1$ where $d'=(k;d_r,d_1,\dots,d_l)^s$ and $\bar f^k(d')$ is given in the following case.

(c) $d = (k; d_r, d_1, \dots, d_l)^s$ where $l = r - 1$.
We set

$$\bar f^k(d) = \sum_{\beta < \bar f^{k+1}(d_r)} (2\cdot\beta^{r-1} + r) + (2\cdot(\bar f^{k+1}(d_r))^{r-2} + r - 1)\cdot \bar f^{k+1}(d_1) + \cdots$$
$$+ (2\cdot \bar f^{k+1}(d_r) + 2)\cdot \bar f^{k+1}(d_{l-1}) + 2\cdot \bar f^{k+1}(d_l).$$

(d) $d = (k; d_r, d_1, \dots, d_l)$ where $l < r - 1$.
We set

$$\bar f^k(d) = \sum_{\beta < \bar f^{k+1}(d_r)} (2\cdot\beta^{r-1} + r) + (2\cdot(\bar f^{k+1}(d_r))^{r-2} + r - 1)\cdot \bar f^{k+1}(d_1) + \cdots$$
$$+ (2\cdot(\bar f^{k+1}(d_r))^{r-l-1} + r - l)\cdot(\bar f^{k+1}(d_l) + 1) - 1.$$

If $l = 0$, so $d = (k; d_r)$, set $\bar f^k(d) = \sum_{\beta < \bar f^{k+1}(d_r)+1} (2\cdot\beta^{r-1} + r) - 1.$

(e) $d = (k; d_r, d_1, \dots, d_l)^s$ where $l < r - 1$.
We set

$$\bar f^k(d) = \sum_{\beta < \bar f^{k+1}(d_r)} (2\cdot\beta^{r-1} + r) + (2\cdot(\bar f^{k+1}(d_r))^{r-2} + r - 1)\cdot \bar f^{k+1}(d_1) + \cdots$$
$$+ (2\cdot(\bar f^{k+1}(d_r))^{r-l-1} + r - l)\cdot(\bar f^{k+1}(d_l)).$$

(2) $k < k(d)$, $K_k = W_1^r$.
We set $\bar f^k(d) = \omega + r + \bar f^{k+1}(d) + 1$.
(3) $k < k(d)$, $K_k = S_1^r$. We set $\bar f^k(d) = \sum_{\beta < \bar f^{k+1}(d_r)} (2\cdot\beta^{r-1} + r)$.

This completes the definition of $\bar f^k$. We let $\bar f(d; K_1, \dots, K_t) \doteq \bar f^1(d; K_1, \dots, K_t)$. Finally, we extend $\bar f$ to tuples of the form $((d)^{(s)}; K_1, \dots, K_t)$ by setting $\bar f((d); K_1, \dots, K_t) = 2\cdot \bar f(d; K_1, \dots, K_t) + 1$, and $\bar f((d)^s; K_1, \dots, K_t) = 2\cdot \bar f(d; K_1, \dots, K_t)$.

We introduce now two sets of assertions about the functions $\bar f^k$:

A^k: Let d satisfy C relative to $K_1, \dots, K_t$, with $k(d) \geq k$. Then for all $K_{t+1} \in W_1 \cup S_1$, $\bar f^k(d; K_1, \dots, K_t) = \bar f^k(d; K_1, \dots, K_t, K_{t+1})$.

R^k: For d_1, d_2 as above with $d_1 < d_2$, we have $\bar f^k(d_1; K_1, \dots, K_t) < \bar f^k(d_2; K_1, \dots, K_t)$.

First consider A^k. Fix $K_1, \dots, K_t$, and we prove A^k for $1 \leq k \leq t$ or $k = \infty$ by reverse induction on k. Let d be defined and satisfy C with respect to $K_1, \dots, K_t$ with $k(d) \geq k$, and fix $K_{t+1} \in W_1 \cup S_1$. If d is basic, it follows immediately from (**I**), (1a) that $\bar f^k(d; K_1, \dots, K_t) = \bar f^k(d; K_1, \dots, K_t, K_{t+1})$. If $k(d) < t$, the result follows easily by induction. If d is non-basic and $k(d) = t$, inspection of the above formulas shows the result follows provided we know that for d basic with $k(d) = \infty$, we have $\bar f^\infty(d) = \bar f^{t+1}(d; K_1, \dots, K_{t+1})$. This, however, follows from (**I**), (3), and the fact that $\sum_{\beta < \omega^{\omega^i}} \beta^r = \omega^{\omega^i}$ for all $i, r \in \omega$.

We now consider R^k. Fix $K_1, \dots, K_t$, and we prove R^k by reverse induction on k. Let d_1, d_2 be defined and satisfy C with respect to $K_1, \dots, K_t$. We consider cases. In considering these cases, the reader should keep in mind fact .

The case where $k(d_1) = k(d_2) = \infty$ is immediate. Also, if $k < \min(k(d_1), k(d_2))$, R^k follows easily by induction and (2) or (3). Hence, we may assume $k = \min(k(d_1), k(d_2)) < \infty$.

Note first that $\bar{f}^k(k; d_r, d_1, \dots, d_l) \geq \bar{f}^k(k; d_r, d_1, \dots, d_l, d_{l+1})$, for any d_{l+1} (this includes the case $l = 0$, where we interpret the left-hand side as $\bar{f}^k(k; d_r)$). If $l + 1 < r - 1$, this follows from (1d) and the fact that $(2 \cdot \alpha^{r-l-1} + r - l) > (2 \cdot \alpha^{r-l-2} + r - l - 1) \cdot (\beta + 1)$ for all infinite $\alpha, \beta \in ON$ with $\beta < \alpha$. We apply this fact with $\alpha = \bar{f}^{k+1}(d_r)$, and $\beta = \bar{f}^{k+1}(d_{l+1})$, noting that $\beta < \alpha$ by induction. If $l = r - 2$, comparing (1d) and (1b), the inequality follows from $2 \cdot \bar{f}^{k+1}(d_{l+1}) + 1 < 2 \cdot \bar{f}^{k+1}(d_r) + 2 - 1$.

An inspection of (1c), (1e) gives immediately that $\bar{f}^k(k; d_r, d_1, \dots, d_l)^s \leq \bar{f}^k(k; d_r, d_1, \dots, d_l, d_{l+1})^s$, for any d_{l+1}, where $1 \leq l \leq r - 2$. Trivially, from (1b) and (1c) it follows that $\bar{f}^k(k; d_r, d_1, \dots, d_l) > \bar{f}^k(k; d_r, d_1, \dots, d_l)^s$ when $l = r - 1$, and from (1d), (1e) we see that this holds for all $1 \leq l \leq r - 1$ as well.

Suppose now that $d_1 < d_2$, and $k(d_1) = k(d_2) = k$. Say, $d_1 = (k; \bar{d}_r, \bar{d}_1, \dots, \bar{d}_{l_1})^{(s)}$ and $d_2 = (k; d'_r, d'_1, \dots, d'_{l_2})^{(s)}$. If $\bar{d}_r = d'_r$, and there does not exist an $i \leq \min(l_1, l_2)$ such that $\bar{d}_i \neq d'_i$, then $\bar{f}^k(d_1) < \bar{f}^k(d_2)$ follows from the cases already considered. If $\bar{d}_r = d'_r$, and there is a least $i \leq \min(l_1, l_2)$ such that $\bar{d}_i \neq d'_i$, then without loss of generality we may assume $d_1 = (k; d_r, d_1, \dots, \bar{d}_i)$ and $d_2 = (k; d_r, d_1, \dots, d'_i)^s$. If $i = r - 1$, then $\bar{f}^k(d_1) < \bar{f}^k(d_2)$ follows from (1b), (1c), and the fact that $2 \cdot \bar{f}^{k+1}(d'_i) > 2 \cdot \bar{f}^{k+1}(\bar{d}_i) + 1$. If $i < r - 1$, inspecting (1d), (1e), the result follows from $\bar{f}^{k+1}(\bar{d}_i) + 1 \leq \bar{f}^{k+1}(d'_i)$, and the -1 term in (1d). Finally, suppose $\bar{d}_r \neq d'_r$. Thus, we have $\bar{d}_r < d'_r$. Without loss of generality we may assume $d_1 = (k; \bar{d}_r)$. By induction, $\bar{f}^{k+1}(\bar{d}_r) + 1 \leq \bar{f}^{k+1}(d'_r)$. Comparing (1d) with (1b), (1c), (1d), (1e), the result is immediate as $\bar{f}^k(d_1)$ is of the form $\sum -1$, and $\bar{f}^k(d_2)$ is of the form $\sum' + \theta$, where $\sum' \geq \sum$, and $\theta \in ON$.

Suppose next that $k = k(d_1) < k(d_2)$. Say $d_1 = (k; \bar{d}_r, \bar{d}_1, \dots, \bar{d}_{l_1})^{(s)}$. We must have $\bar{d}_r < d_2$, so by induction $\bar{f}^{k+1}(\bar{d}_r) < \bar{f}^{k+1}(d_2)$. We may assume from the cases already considered that $d_1 = (k; \bar{d}_r)$. Since $\bar{f}^{k+1}(\bar{d}_r) + 1 \leq \bar{f}^{k+1}(d_2)$, comparing (1d) and (3) shows $\bar{f}^k(d_1) < \bar{f}^k(d_2)$ (the sum term in (1d) is less than or equal to the sum term in (3), and (1d) has the extra -1 term).

Finally, suppose that $k(d_1) > k(d_2) = k$. Again, let $d_2 = (k; d'_r, d'_1, \dots, d'_{l_2})^{(s)}$. We have in this case $d_1 \leq d'_r$, and so $\bar{f}^{k+1}(d_1) \leq \bar{f}^{k+1}(d'_r)$. We may assume $d_2 = (k; d'_r, d'_1)^s$, and comparing (3) with (1e), (1c) shows that $\bar{f}^k(d_1)$ is of the form $\sum$, and $\bar{f}^k(d_2)$ is of the form $\sum' + \theta$, where $\sum' \geq \sum$, and $\theta > 0$ (in fact θ is infinite). Thus, $\bar{f}^k(d_1) < \bar{f}^k(d_2)$.

This establishes R^k in all cases.

We let now $\tilde{A}^k$ and $\tilde{R}^k$ denote the versions of A^k, R^k stated for tuples of the form $((d)^{(s)}; K_1, \dots, K_t)$. From A^k, R^k, it follows immediately that $\tilde{A}^k$, $\tilde{R}^k$ hold. From $\tilde{A}^1$, $\tilde{R}^1$ and the definition of $<$ it now follows easily that if $((d_1)^{(s)}; K_1, \dots, K_{t_1}) < ((d_2)^{(s)}; K_1, \dots, K_{t_2})$, then $\bar{f}((d_1)^{(s)}; K_1, \dots, K_{t_1}) < \bar{f}((d_2)^{(s)}; K_1, \dots, K_{t_2})$. From this the following theorem is now immediate:

THEOREM 3.4. *If $(d)^{(s)}$ satisfies D relative to the sequence $K_1, \dots, K_t$, then $|((d)^{(s)}; K_1, \dots, K_t)| \leq \bar{f}((d)^{(s)}; K_1, \dots, K_t)$.*

Now, an inspection of the formulas for $\bar{f}^k$ and an easy induction gives the following: if $d = d^{(f_m)}$, then $\bar{f}((d)^{(s)}; K_1, \ldots, K_t) < \omega^{\omega^{(m+1)}}$ (we use here the simple fact that if $\alpha < \omega^{\omega^p}$ then $\sum_{\beta<\alpha} \beta^r < \omega^{\omega^p}$ for all $p, r \in \omega$). From this and the above theorem we now have:

THEOREM 3.5. *If $(d)^{(s)}$ satisfies D relative to $K_1, \ldots, K_t$, where $d = d^{(f_m)}$, then $f((d)^{(s)}; K_1, \ldots, K_t) < \omega^{\omega^{(m+1)}}$.*

We now have the desired upper bound for $\boldsymbol{\delta}^1_5$:

THEOREM 3.6. $\boldsymbol{\delta}^1_5 \leq (\aleph_{\omega^{\omega^\omega}})^+$.

PROOF. We collect our previous results. First, from corollary 1.1 we have $\boldsymbol{\delta}^1_5 \leq (\sup_m j_{W^m_3}(\boldsymbol{\delta}^1_3))^+$. Fix $m \in \omega$. Let $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ represent $[F]_{W^m_3}$. Martin's theorem 2.1 and a simple case of the argument of theorem 3.3 then give that $[F]_{W^m_3} < \left[\sup_{K_1}(id; ()^{(f_m)}; K_1)\right]^+$. From theorems 3.3, 3.5, $(id; ()^{(f_m)}; K_1) < \aleph_{\omega+\omega^{\omega^{(m+1)}}}$, and the theorem follows. $\square$

CHAPTER 4

Results on Uniform Cofinalities

Our goal in this section is to analyze the possible "uniform cofinalities" of functions $f\colon \omega_{n+1} \to \boldsymbol{\delta}^1_3$ (a definition to be made precise). Before doing this, we digress to analyze ordinals $\alpha < \omega_{n+1}$. That is, we analyze the possible uniform cofinalities of functions $l\colon (\omega_1)^n \to \omega_1$ representing such α. Recall our notational convention that $\mathrm{dom}(W_1^n)$ consists of increasing n-tuples, and such tuples may be written in any order. Thus, if $\pi = (j_1, \dots, j_n)$ is a permutation on n, and $f\colon \mathrm{dom}(W_1^n) \to ON$, we write $f(\alpha_1, \dots, \alpha_n)$ and $f(\alpha_{j_1}, \dots, \alpha_{j_n})$ interchangeably.

Recall that to each standard permutation $\pi = (n, j_2, \dots, j_n)$ of n we have a corresponding ordering $<_\pi$ of ω_1^n. We say $f\colon \mathrm{dom}(<_\pi) \to \omega_1$ (or $f\colon (\omega_1)^n \to \omega_1$) is of type π if f is of the correct type with respect to the ordering $<_\pi$ (i.e., order-preserving, discontinuous, and of uniform cofinality ω). We say f is of type (π, s), if f is order-preserving and $f(\vec{\alpha}) = \sup\{f(\vec{\beta})\colon \vec{\beta} <_\pi \vec{\alpha}\}$ if this set has limit order-type, and otherwise $f(\vec{\alpha})$ has uniform cofinality ω. If $\pi' = (n+1, j_2', \dots, j_{n+1}')$ is a standard permutation extending π (in the sense of §1), then we say an order-preserving function $f\colon \mathrm{dom}(<_\pi) \to \omega_1$ is of type (π, π') if there is an $f'\colon \mathrm{dom}(<_{\pi'}) \to \omega_1$ of the correct type such that for all $\vec{\alpha} = (\alpha_n, \alpha_{j_2}, \dots, \alpha_{j_n})$ of type π, $f(\vec{\alpha}) = \sup\{f'(\vec{\alpha}, \beta)\colon (\vec{\alpha}, \beta) \text{ has type } \pi'\}$, if this set has limit order-type, and otherwise $f(\vec{\alpha})$ has uniform cofinality ω.

Slightly more generally, we call $\pi = (n, j_2, \dots, j_m)$ a *standard partial permutation* if $m \leq n$, and π can be extended to a permutation of n. We define the partial permutation $\pi' = (n+1, j_2', \dots, j_{m+1}')$ to extend $\pi = (n, j_2, \dots, j_m)$ exactly as for permutations, that is, $(n+1, j_2', \dots, j_m')$ is order isomorphic to $(n, j_2, \dots, j_m)$. Corresponding to the partial permutation π we have also the prewellordering $<_\pi$ on n-tuples of ordinals $< \omega_1$ defined by $(\alpha_1, \dots, \alpha_n) <_\pi (\beta_1, \dots, \beta_n)$ iff $(\alpha_n, \alpha_{j_2}, \dots, \alpha_{j_m}) <^{lex} (\beta_n, \beta_{j_2}, \dots, \beta_{j_m})$. Note that this only depends on m of the ordinals of the tuple. Thus, if $f\colon (\omega_1)^n \to \omega_1$ is order-preserving with respect to $<_\pi$, we may write $f(\alpha_n, \alpha_{j_2}, \dots, \alpha_{j_m})$ for $f(\alpha_1, \dots, \alpha_n)$. We say $f\colon (\omega_1)^n \to \omega_1$ is of type π if f is of the correct type with respect to the ordering $<_\pi$. Similarly, f is of type (π, s) if f is order-preserving, and $f(\alpha_n, \alpha_{j_2}, \dots, \alpha_{j_m}) = \sup\{f(\vec{\beta})\colon \vec{\beta} <_\pi \vec{\alpha}\}$ whenever this set has limit order type (i.e., α_{j_m} is a limit ordinal), and otherwise $f(\vec{\alpha})$ has uniform cofinality ω. Likewise, if $\pi' = (n+1, j_2', \dots, j_{m+1}')$ is a partial permutation extending π, and $f\colon \mathrm{dom}(<_\pi) \to \omega_1$ is order-preserving, we say f is of type (π, π') if there is an $f'\colon \mathrm{dom}(<_{\pi'}) \to \omega_1$ of the correct type such that for all $\vec{\alpha} = (\alpha_n, \alpha_{j_2}, \dots, \alpha_{j_m})$ order-isomorphic to π, $f(\vec{\alpha}) = \sup\{f'(\vec{\alpha}, \beta)\colon (\vec{\alpha}, \beta) \text{ is order-isomorphic to } \pi'\}$ if the set of such $(\vec{\alpha}, \beta)$ has limit order type, and otherwise $f(\vec{\alpha})$ has uniform cofinality ω. Note that this only depends on the order type of π'.

For $\alpha < \omega_{n+1}$, we say α is of type π, (π, s), or (π, π'), if there is a function f of type π, (π, s), or (π, π') respectively representing α. For non-trivial α (e.g., $\alpha > \omega_n \cdot \omega$) this is equivalent to saying (considering the (π, π') case) that for some (or any) f representing α, there is a c.u.b. $C \subseteq \omega_1$ and an f' with f, f' order preserving from $<_\pi\restriction C$, $<_{\pi'}\restriction C$ to ω_1 with f' inducing f as above. We say f is almost everywhere of type (π, π'), etc., if the ordinal f represents is of that type.

If $\pi = (n, j_2, \ldots, j_m)$, and $m = n$, then it is not difficult to show that if α has type π, (π, s), or (π, π'), then the type of α is unique. For $m < n$, let $\tilde{\pi}' = (m{+}1, \ldots)$ be the permutation of length $m + 1$ order isomorphic to π'. In this case, if α has type (π, π'), then the type of α is determined by $(\pi, \tilde{\pi}')$. That is, if π'' and π' are order isomorphic, then α has type (π, π') iff α has type (π, π''). With a slight abuse, we also refer to α as having type $(\pi, \tilde{\pi}')$. Written this way, the type of α is again seen to be unique.

Definition 4.1. If $\omega_n \leq \alpha < \omega_{n+1}$ is of type π, (π, s), or (π, π') for some standard partial permutation $\pi = (n, i_2, \ldots, i_r)$, the we say α has *depth* r.

In other words, a representing function $f \colon (\omega_1)^n \to \omega_1$ for α depends, almost everywhere, on exactly r of the n arguments.

Lemma 4.1. *Let $\omega_n \leq \alpha < \omega_{n+1}$ be a limit ordinal. Then α has type either π, (π, s), or (π, π') for some standard partial permutations π, π' of n, $n{+}1$ respectively.*

Proof. Fix $\omega_n \leq \alpha < \omega_{n+1}$ and $f \colon (\omega_1)^n \to \omega_1$ representing α. For each $k = 1, 2, \ldots, n$ we consider the partition $\mathcal{P}_k$: we partition pairs of tuples $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_n)$ of ordinals $< \omega_1$ ordered by $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots \alpha_{k-1} < \beta_{k-1} < \beta_k < \alpha_k < \alpha_{k+1} < \beta_{k+1} \ldots < \alpha_n < \beta_n$ according to whether or not $f(\alpha_1, \ldots, \alpha_n) > f(\beta_1, \ldots, \beta_n)$. An easy argument shows that there is a unique k such that on the homogeneous side of the partition the property stated in $\mathcal{P}_k$ holds (otherwise we get a c.u.b. $C \subseteq \omega_1$ such that $f(\beta_1, \ldots, \beta_n) \leq f(\alpha_1, \ldots, \alpha_n)$ for $\alpha_1 < \beta_1 < \ldots < \alpha_n < \beta_n$ in C, which makes f constant almost everywhere, a contradiction). It also follows from $\alpha \geq \omega_n$ that this unique k is equal to n. Hence, on a measure one set $f(\alpha_1, \ldots, \alpha_n)$ depends first on α_n. Next we consider the partitions $\mathcal{P}_k$ for $k = 1, \ldots, n-1$ where we partition $(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n)$, $(\beta_1, \ldots, \beta_{n-1}, \alpha_n)$ with $\alpha_1 < \beta_1 < \ldots < \alpha_{k-1} < \beta_{k-1} < \beta_k < \alpha_k < \ldots < \alpha_{n-1} < \beta_{n-1} < \alpha_n$ according to whether or not $f(\alpha_1, \ldots, \alpha_n) > f(\beta_1, \ldots, \alpha_n)$. Again it follows easily that either there is a unique $k = j_2$ such that the property stated in $\mathcal{P}_{j_2}$ holds on the homogeneous side of the partition, or $f(\alpha_1, \ldots, \alpha_n) = f(\alpha_n)$ on a measure one set. Continuing, we define a partial permutation $\pi = (n, j_2, \ldots, j_m)$ such that $f \colon \mathrm{dom}(<_\pi) \to \omega_1$ is order preserving on a measure one set. It remains to analyze the "uniform cofinality" of f.

Let $\prec$ be a well ordering of ω_1 such that

$$\forall^* \alpha < \omega_1 \ |\prec\restriction \alpha| > f(1)(\alpha) \doteq \sup_{\alpha_1, \ldots, \alpha_{n-1}} f(\alpha_1, \ldots, \alpha_{n-1}, \alpha).$$

For $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (\omega_1)^n$, let $S'(\vec{\alpha}) = \{\beta < \alpha_n \colon |\beta|_{\prec\restriction\alpha_n} < f(\alpha_1, \ldots, \alpha_n)\}$. Thus, $f(\alpha_1, \ldots, \alpha_n) = \sup_{\beta \in S'(\vec{\alpha})} |\beta|_{\prec\restriction\alpha_n}$. Let now $S \colon (\omega_1)^n \to \omega_1$ represent the least equivalence class such that

$$\exists l \colon \{(\vec{\alpha}, \beta) \colon \beta \in S(\vec{\alpha})\} \to \omega_1 \ \forall^* \vec{\alpha} \ f(\vec{\alpha}) = \sup_{\beta \in S(\vec{\alpha})} l(\vec{\alpha}, \beta).$$

Thus $S(\alpha_1, \ldots, \alpha_n) \leq \alpha_n$ almost everywhere. Now, if $S(\alpha_1, \ldots, \alpha_n) < \alpha_k$ almost everywhere, then there is a well-ordering $\prec$ of ω_1 such that $S(\alpha_1, \ldots, \alpha_n) < |\prec\restriction \alpha_{k-1}|$ almost everywhere. From this it follows easily that for the least such S we either have (i) $S(\alpha_1,\dots,\alpha_n)$ is constant almost everywhere or (ii) $S(\alpha_1,\dots,\alpha_n)=\alpha_k$ almost everywhere, for some $k\in\{n,j_2,\dots,j_m\}$. The first case corresponds to $f\colon \mathrm{dom}(<_\pi)\to\omega_1$ of the correct type, thus f of type π. In the second case we either have $f(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_m})=\sup_{\beta<\alpha_{j_m}} f(\alpha_n,\,\alpha_{j_2},\dots,\alpha_{j_{m-1}},\beta)$ almost everywhere, in which case f is of type (π,s), or else $f(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_m})>\sup_{\beta<\alpha_{j_m}} f(\alpha_n,\,\alpha_{j_2},\dots,\alpha_{j_{m-1}},\beta)$ almost everywhere in which case f is of type (π,π') where $\pi'=(n+1,j_2',\dots,j_{m+1}')$ where $j_a'=j_a+1$ if $j_a\ge k$, $j_{m+1}'=k$, and $j_a'=j_a$ otherwise. □

Our next goal is to analyze the possible uniform cofinalities of functions $f\colon\omega_{n+1}\to\boldsymbol{\delta}^1_3$ with respect to a measure $\mathcal{V}$ on ω_{n+1}. We first make this precise.

Definition 4.2. Let $\theta\in ON$. A *uniform cofinality* for θ is a function $u\colon\theta\to ON$. We say a function $f\colon\theta\to ON$ has uniform cofinality u if there is a function $l\colon S\doteq\{(\alpha,\beta)\colon\beta<u(\alpha)\}\to ON$ such that $\forall\alpha<\theta\ f(\alpha)=\sup\{l(\alpha,\beta)\colon(\alpha,\beta)\in S\}$. If $\mathcal{V}$ is a measure on θ, a uniform cofinality with respect to $\mathcal{V}$ is an equivalence class $[u]_{\mathcal{V}}$ of a function $u\colon\theta\to ON$. We say $f\colon\theta\to ON$ has uniform cofinality $[u]_{\mathcal{V}}$ with respect to $\mathcal{V}$ if there is a $\mathcal{V}$ measure one set $A\subseteq\theta$ and an $l\colon S\doteq\{(\alpha,\beta)\colon\alpha\in A\wedge\beta<u(\alpha)\}\to ON$ such that $\forall\alpha\in A\ f(\alpha)=\sup\{l(\alpha,\beta)\colon(\alpha,\beta)\in S\}$.

We frequently abbreviate the above by writing "$f(\alpha)$ has uniform cofinality $u(\alpha)$," or "$f(\alpha)$ has uniform cofinality $u(\alpha)$ almost everywhere with respect to $\mathcal{V}$," and omitting the reference to $\mathcal{V}$ if it is understood. Note that this only depends on the equivalence class of u with respect to $\mathcal{V}$. We say two uniform cofinalities with respect to $\mathcal{V}$ are distinct if it is not possible for a function to have both uniform cofinalities simultaneously.

The next theorem (actually a slight generalization of it, c.f. theorem 4.2) plays an important role in the following analysis. Recall that if $\pi=(n,j_2,\dots,j_m)$ is a partial permutation, $h\colon\mathrm{dom}(<_\pi)\to\omega_1$ is order-preserving, and $l\le m$, then the l^{th} invariant is defined by (for $(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_l})$ of type $(n,j_2,\dots,j_l)$):

$$h(l)(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_l})=\sup\{h(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_l},\beta_{j_{l+1}},\dots,\beta_{j_m})\colon (\alpha_n\dots,\alpha_{j_l},\vec{\beta})\text{ has type }\pi\}.$$

Theorem 4.1. *Let $\mathcal{V}$ be the measure on ω_{n+1} corresponding to the partial permutation $\pi=(n,j_2,\dots,j_m)$ and functions $h\colon(\omega_1)^n\to\omega_1$ of type π, (π,s), or (π,π') for some π' extending π. Let $f\colon\omega_{n+1}\to\boldsymbol{\delta}^1_3$ have range in the limit ordinals. Then one of the following occurs:*

1. *$f(\alpha)$ has uniform cofinality ω almost everywhere, or uniform cofinality ω_1 almost everywhere, or uniform cofinality ω_2 almost everywhere.*
2. *There is an $l\le m$ such that if $\tilde{\pi}=(l,i_2,\dots,i_l)$ is the permutation of l order isomorphic to $(n,j_2,\dots,j_l)$, then for almost all $\alpha<\omega_{n+1}$ represented by $h\colon\mathrm{dom}(<_\pi)\to\omega_1$ of the appropriate type (depending on $\mathcal{V}$), $f(\alpha)$ has uniform cofinality represented with respect to W^l_1 by $h'\colon\mathrm{dom}(<_{\tilde{\pi}})\to\omega_1$ defined by $h'(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_m})=h(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_m})$ if $l=m$, and if $l<m$ by $h'(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_l})=h(l)(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_l})$.*
3. *There is an $l<m$ such that if $\tilde{\pi}$ as in (2) and $\tilde{\pi}'$ is the permutation extending $\tilde{\pi}$ order isomorphic to $(n,j_2,\dots,j_{l+1})$, then for almost all $\alpha<\omega_{n+1}$ represented by $h\colon\mathrm{dom}(<_\pi)\to\omega_1$ of the appropriate type, $f(\alpha)$*

has uniform cofinality the supremum of the ordinals $\beta < \omega_{l+2}$ which are represented by functions $h'\colon \mathrm{dom}(<_{\tilde{\pi}'}) \to \omega_1$ of the correct type such that $h'(\alpha_n, \alpha_{j_2}, \ldots, \alpha_{j_{l+1}}) < h(l)(\alpha_n, \alpha_{j_2}, \ldots, \alpha_{j_l})$.

(4) *$\mathcal{V}$ is of type (π, π') and for almost all $\alpha < \omega_{n+1}$ represented by $h\colon \mathrm{dom}(<_\pi) \to \omega_1$ of type (π, π'), $f(\alpha)$ has uniform cofinality the supremum of the ordinals $\beta < \omega_{m+2}$ which are represented by functions $h'\colon \mathrm{dom}(<_{\tilde{\pi}'}) \to \omega_1$ of the correct type such that $h'(\alpha_n, \alpha_{j_2}, \ldots, \alpha_{j_{m+1}}) < h(\alpha_n, \alpha_{j_2}, \ldots, \alpha_{j_m})$ almost everywhere, where $\tilde{\pi}'$ is the permutation order-isomorphic to π'.*

Note that cases (3) and (4) are in essence the same case, separated for notational convenience.

Proof. Fix $f\colon \omega_{n+1} \to \boldsymbol{\delta}^1_3$ having range in the limit ordinals, and $\mathcal{V}$ a measure on ω_{n+1} of type π, (π, s), or (π, π') where $\pi = (n, j_2, \ldots, j_m)$ and π' extends π in the last case. Let $\prec$ be a well-ordering of ω_ω of length $> \sup_{\alpha<\omega_{n+1}} f(\alpha)$. For each $\alpha < \omega_{n+1}$ let $S_\alpha \subseteq \omega_\omega$ be defined by $S_\alpha = \{\beta < \omega_\omega\colon\ |\beta|_\prec < f(\alpha)\}$. So, $f(\alpha) = \sup_{\beta\in S_\alpha} |\beta|_\prec$. Let now $S\colon \omega_{n+1} \to \omega_\omega$ represent the least equivalence class (with respect to $\mathcal{V}$) such that there exists a function $l\colon \{(\alpha, \beta) : \alpha < \omega_{n+1} \wedge \beta \in S(\alpha)\} \to \boldsymbol{\delta}^1_3$ such that for $\mathcal{V}$ almost all $\alpha < \omega_{n+1}$, $f(\alpha) = \sup_{\beta\in S(\alpha)} l(\alpha, \beta)$. Thus, $S(\alpha) \leq \omega_\omega$ almost everywhere. If $S(\alpha) = \omega_\omega$ almost everywhere then we clearly have case (1), (with uniform cofinality ω) since we may take $f'(n, \alpha) = \sup_{\beta\in S(\alpha), \beta<\omega_n} l(\beta)$.

By countable additivity, let k be least such that $S(\alpha) \leq \omega_{k+1}$ almost everywhere. If $S(\alpha) = \omega_{k+1}$ almost everywhere, then we are in case (1) with uniform cofinality ω_1 if $k = 0$, and otherwise uniform cofinality ω_2. So assume $\omega_k < S(\alpha) < \omega_{k+1}$. For $\alpha < \omega_{n+1}$ represented by $h\colon \mathrm{dom}(<_\pi) \to \omega_1$ of the appropriate type, let $s_\alpha\colon (\omega_1)^k \to \omega_1$ represent $S(\alpha)$ with respect to W^k_1. Although we can not choose the functions s_α uniformly in α, the map $\alpha \to [s_\alpha]$ is well-defined.

First consider $s_\alpha(1)\colon \omega_1 \to \omega_1$. If for almost all $\alpha < \omega_{n+1}$ $[s_\alpha(1)] < [h(1)]$, then $[s_\alpha(1)]$ is constant on a measure one set with respect to $\mathcal{V}$. If this constant value $[s]$ of $[s_\alpha(1)]$ is $< \omega_1$, then $k = 0$ and case (1) applies. Otherwise, taking a well-ordering of ω_1 of length $> [s(1)]$ easily contradicts the minimality of S unless $s(1)(\beta) = \beta$ almost everywhere. But then $[s] \leq \omega_k$, a contradiction.

If $\forall^*\alpha < \omega_{n+1}$ $[s_\alpha(1)] > [h(1)]$, then we consider the partition $\mathcal{P}$: we partition $h\colon \mathrm{dom}(<_\pi) \to \omega_1$ of the appropriate type, and $g\colon \omega_1 \to \omega_1$ with $h(1)(\beta) < g(\beta) < N_h(h(1)(\beta))$ according to whether $[g] > [s_\alpha(1)]$, where $\alpha = [h]$. On the homogeneous side of the partition the property holds, and we let $C \subseteq \omega_1$ be homogeneous for $\mathcal{P}$. Letting $\prec$ be a well-ordering of ω_1 such that $|\prec\restriction \beta| > N_C(\beta)$ for almost all β then easily contradicts the minimality of S.

Hence, we may assume $\forall^*\alpha$ $[h(1)] = [s_\alpha(1)]$. Let $1 \leq p \leq \min(k, m)$ be maximal such that $\forall^*\alpha$ $s_\alpha(p)\colon \omega_1^p \to \omega_1$ is defined and $[h(p)] = [s_\alpha(p)]$.

If $p = k$, then $[s_\alpha] = [h(p)]$, and case (2) holds.

Assume now that $p < k$ and consider the following cases:

(i) $p < m$ and $\forall^*\alpha$ $S(\alpha) = [s_\alpha]$ has depth p (see definition 4.1). We claim that case (3) holds. Since $[s_\alpha]$ has depth p and $[s_\alpha(p)] = [h(p)]$, there is a partial permutation $\pi_1 = (k, i_2, \ldots, i_p)$ order isomorphic to $(n, j_2, \ldots, j_p)$ such that $\forall^*\alpha$ $\forall^*\alpha_1, \ldots, \alpha_k$ $s_\alpha(\alpha_1, \ldots, \alpha_k) = h(p)(\alpha_k, \alpha_{i_2}, \ldots, \alpha_{i_p})$. Let $q \in \{i_2, \ldots, i_p\} \cup \{k\}$ be such that $\forall^*\alpha_1, \ldots, \alpha_k$ $s(\alpha_1, \ldots, \alpha_k) = \sup_{\beta<\alpha_q} h(p+1)(\alpha_k, \alpha_{i_2}, \ldots, \alpha_{i_p}, \beta)$. Suppose now $s'\colon (\omega_1)^k \to \omega_1$ and $[s'] < [s_\alpha]$. If $q = 1$ or $q-1 \in \{i_2,\dots,i_p\}$, it follows that there is a $\theta\colon (\omega_1)^p \to \omega_1$ such that $[\theta] < [h(p)]$ and $\forall^*\alpha_1,\dots,\alpha_k\ s'(\alpha_1,\dots,\alpha_k) \le \theta(\alpha_k,\alpha_{i_2},\dots,\alpha_{i_p})$. This, however, allows us to define S', l' with $S'(\alpha) < \omega_{p+1} \le \omega_k$, violating the minimality of S. Namely, for $\alpha = [h]$, $h\colon \mathrm{dom}(<_\pi) \to \omega_1$ of the appropriate type, let $S'(\alpha) = [h(p)]$, and $l'(\alpha,[\theta]) = \sup\{l(\alpha,[s'])\colon [s'] < [s_\alpha] \wedge \forall^*\alpha_1,\dots,\alpha_k\ s'(\alpha_1,\dots,\alpha_k) < \theta(\alpha_k,\alpha_{i_2},\dots,\alpha_{i_p})\}$. So, $q > 1$ and $q-1 \notin \{i_2,\dots,i_p\}$. For $\alpha = [h]$ as above, and $[s'] < [s_\alpha]$, an easy sliding argument (e.g., corollary 3.1) shows there is a $\theta\colon (\omega_1)^{p+1} \to \omega_1$ of type π_2 order isomorphic to $(n, j_2,\dots,j_{p+1})$, with $[\theta(p)] = [h(p)]$ and such that $\forall^*\alpha_1,\dots,\alpha_k\ s'(\alpha_1,\dots,\alpha_k) \le \theta(\alpha_k,\alpha_{i_2},\dots,\alpha_{i_p},\alpha_q)$. The conclusion of case (3) follows (and also, $k = p+1$).

(ii) $p = m$ and $\forall^*\alpha\ [s_\alpha]$ has depth p. If $\mathcal{V}$ is of type π, $h\colon \mathrm{dom}(<_\pi) \to \omega_1$ is of type π, and $[s'] < [s_\alpha]$ where $\alpha = [h]$, then since h has uniform cofinality ω it follows that $\exists\theta\colon (\omega_1)^m \to \omega_1$ with $[\theta] < [h(m)]$ such that $\forall^*\alpha_1,\dots,\alpha_k\ s'(\alpha_1,\dots,\alpha_k) < \theta(\alpha_k,\alpha_{i_2},\dots,\alpha_{i_m})$. This allows us to define S', l' with $S'(\alpha) < \omega_{m+1} \le \omega_k$, violating the minimality of S. Suppose next that $\mathcal{V}$ is of type (π, s). Let $s'\colon (\omega_1)^k \to \omega_1$ with $[s'] < [s_\alpha]$. Let $\pi_1 = (k, i_2,\dots,i_m)$ again be the partial permutation order isomorphic to π such that $s\colon \mathrm{dom}(<_{\pi_1}) \to \omega_1$ is order-preserving almost everywhere. Thus, $\forall^*\alpha_1,\dots,\alpha_k\ \exists\beta < \alpha_{i_m}\ s'(\alpha_1,\dots,\alpha_k) < h(m)(\alpha_k,\alpha_{i_2},\dots,\alpha_{i_{m-1}},\beta)$. Thus, β is bounded by a function of α_{i_m-1} only. If $i_m = 1$ or $i_m - 1 \in \{i_2,\dots,i_{m-1}\}$, then $[s'(m-1)] < [s_\alpha(m-1)]$, and we violate the minimality of S. Otherwise, if $[s'] < [s_\alpha]$ then there is a $\theta\colon (\omega_1)^m \to \omega_1$ with $[\theta(m-1)] = [h(m-1)]$ and such that $\forall^*\alpha_1,\dots,\alpha_k\ s'(\alpha_1,\dots,\alpha_k) < \theta(\alpha_k,\alpha_{i_2},\dots,\alpha_{i_{m-1}},\alpha_{i_m-1})$. This again violates the minimality of S. Assume finally that $\mathcal{V}$ is of type (π,π'). Letting π_1 be as above, we use an argument similar to case (i), considering h' of type π' inducing h. If there is a $q \notin \{i_2,\dots,i_m\} \cup \{k\}$ such that $(k, i_2,\dots,i_m,q)$ is order isomorphic to π', then we are easily seen to be in case (4). Otherwise, we violate the minimality of S.

(iii) $p < m$ and $\forall^*\alpha\ [s_\alpha]$ has depth $> p$. By maximality of p, $[s_\alpha(p+1)] \ne [h(p+1)]$ for almost all $\alpha < \omega_{n+1}$. First suppose $[s_\alpha(p+1)] > [h(p+1)]$ for almost all α. We consider the partition $\mathcal{P}$: we partition $h\colon \mathrm{dom}(<_\pi) \to \omega_1$ of the appropriate type and $g\colon \mathrm{dom}(<_{\tilde\pi'}) \to \omega_1$ (where $\tilde\pi'$ is the permutation order isomorphic to $(n, j_2,\dots,j_{p+1})$) of the correct type with $h(p+1)(\alpha_n, \alpha_{j_2},\dots,\alpha_{j_{p+1}}) < g(\alpha_n, \alpha_{j_2},\dots,\alpha_{j_{p+1}}) < N_h(h(p+1)(\alpha_n, \alpha_{j_2},\dots, \alpha_{j_{p+1}}))$ according to whether $[g] > [s_\alpha(p+1)]$ where $\alpha = [h]$. From lemma 3.2 it follows that on the homogeneous side of the partition the property holds. Fixing a c.u.b. set C homogeneous for $\mathcal{P}$, and a well-ordering $\prec$ of ω_1 such that $|\prec\restriction\beta| > N_C(\beta)$ for almost all β then allows us to replace S, l by a representation S', l' such that for almost all α, if $s'_\alpha\colon \omega_1^k \to \omega_1$ represents $S'(\alpha)$, then $[s'_\alpha(p+1)] \le [h(p+1)]$. This contradicts the minimality of S.

If $[s_\alpha(p+1)] < [h(p+1)]$ almost everywhere, we argue as above except using the partition $\mathcal{P}$ where we require $\sup_{\beta<\alpha_{j_{p+1}}} h(p+1)(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_p},\beta) < g(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_{p+1}}) < N_h(\sup_{\beta<\alpha_{j_{p+1}}} h(p+1)(\alpha_n,\alpha_{j_2},\dots, \alpha_{j_p},\beta))$. We again violate the minimality of S unless $s_\alpha(p+1)((\alpha_n,\alpha_{j_2},\dots, \alpha_{j_{p+1}}) = \sup_{\beta<\alpha_{j_{p+1}}} h(p+1)(\alpha_n,\alpha_{j_2},\dots,\alpha_{j_p},\beta)$ almost everywhere. Thus, if $[s'] < [s_\alpha]$, then $\forall^*\alpha_1,\dots,\alpha_k\ \exists\beta < \alpha_{i_{p+1}}\ s'(\alpha_1,\dots,\alpha_k) < h(p+1)(\alpha_k,\alpha_{i_2},\dots,\alpha_{i_p},\beta)$, where $s_\alpha\colon \mathrm{dom}(<_{\pi_1}) \to \omega_1$ is order preserving, and $\pi_1 = (k, i_2,\dots,i_{p+1})$. If $i_{p+1} = 1$ or $i_{p+1} - 1 \in \{i_2,\dots,i_p\}$, then

$[s'(p)] < [s_\alpha(p)]$ which leads to a violation of the minimality of S. Otherwise, there is a $\theta\colon (\omega_1)^{p+1} \to \omega_1$ with $[\theta(p)] = [h(p)]$ such that $\forall^* \alpha_1, \dots, \alpha_k \; s'(\alpha_1, \dots, \alpha_k) < \theta(\alpha_k, \alpha_{i_2}, \dots, \alpha_{i_p}, \alpha_{i_{p+1}-1})$. This establishes case (3), with the $\tilde{\pi}'$ of case (3) order isomorphic to $(k, i_2, \dots, i_p, i_{p+1} - 1)$.

(iv) $p = m$ and $\forall^* \alpha \; [s_\alpha]$ has depth $> p$. The argument proceeds as in (iii) except we now consider $h'\colon \mathrm{dom}(<_{\pi'}) \to \omega_1$ of type π' inducing $h\colon \mathrm{dom}(<_\pi) \to \omega_1$ of type (π, π'). Since $[s_\alpha]$ depends only on h, an easy partition argument shows that $[s_\alpha(m+1)] < [h'(m+1)]$ for almost all h'. Using the minimality of S then gives that $\forall^* h' \; \forall^* \alpha_1, \dots, \alpha_k \; s_\alpha(\alpha_1, \dots, \alpha_k) = \sup_{\beta < \alpha_{i_{m+1}}} h'(\alpha_k, \alpha_{i_2}, \dots, \alpha_{i_m}, \beta)$, where again $\pi_1 = (k, i_2, \dots, i_{m+1})$ is such that $s_\alpha\colon \mathrm{dom}(<_{\pi_1}) \to \omega_1$ is order-preserving. Proceeding as in the second half of (iii) shows that either case (4) holds, or we violate the minimality of S. $\square$

We next introduce the notion of a type-1 tree of uniform cofinalities. This generalizes somewhat the concept of a type-1 tree of measures introduced in §1 (definition 1.6). The essential difference is that now the most general uniform cofinalities are used, rather than just uniform cofinality ω. Roughly speaking, the tree assigns to each node of the tree a uniform cofinality with respect to the measure corresponding to the immediately previous node. In §5 we will introduce type-2 trees of uniform cofinalities, and in §8 we introduce type-3 trees of uniform cofinalities. These are important, as they are the essential ingredients in analyzing arbitrary measures on $(\boldsymbol{\delta}^1_3)^-$, $\boldsymbol{\delta}^1_3$, and $(\boldsymbol{\delta}^1_5)^-$ respectively (we will carry out the last two of these analyses in §5, 8).

Definition 4.3. A type-1 *tree of uniform cofinalities* (of depth n) is a function $\mathcal{R}$ satisfying the following:

(1) $\langle p_1, i_1 \rangle \in \mathrm{dom}(\mathcal{R})$ for $0 \le i_1 \le a$ for some integer a, and $p_1 =$ the unique permutation of length 1, namely $p_1 = (1)$. For $i_1 = 0$, $\mathcal{R}(\langle p_1, i_1 \rangle) =$ the formal symbol (s); and for $i_1 > 0$, $\mathcal{R}(\langle p_1, i_1 \rangle)$ is either the formal symbol (ω) (which stands for "uniform cofinality ω"), or a standard permutation p_2 of length 2 (hence $p_2 = (2,1)$).

(2) In general, $\mathrm{dom}(\mathcal{R})$ will consist of tuples $\vec{s} = \langle p_1, i_1, \dots, i_{m-1}, p_m, i_m \rangle$, $m \le n$, such that $\vec{s} \restriction m-1 = \langle p_1, i_1, \dots, p_{m-1}, i_{m-1} \rangle \in \mathrm{dom}(\mathcal{R})$, $p_m = \mathcal{R}(\langle p_1, i_1, \dots, p_{m-1}, i_{m-1} \rangle)$ is a standard permutation $(m, j_2, \dots, j_m)$ of m extending p_{m-1}, and $0 \le i_m \le a$ for some integer a (which depends on $\langle p_1, i_1, \dots, p_{m-1}, i_{m-1} \rangle$). $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m \rangle)$ will be either (s), (ω), or a permutation p_{m+1} extending p_m. Also, $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m \rangle) = (s)$ iff $i_m = 0$. Furthermore, $\langle p_1, i_1, \dots, i_{m-1}, p_m, i_m \rangle$ is maximal in $\mathrm{dom}(\mathcal{R})$ iff $i_m = 0$ or $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m \rangle) = (\omega)$.

Remark 4.1. We will show below that the last requirement that $\langle p_1, i_1, \dots, p_m, i_m \rangle$ be maximal in $\mathrm{dom}(\mathcal{R})$ iff $i_m = 0$ or $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m \rangle) = (\omega)$ causes no loss of generality (because we can add new nodes ending in 0), and is for notational convenience.

Corresponding to a type-1 tree of uniform cofinalities $\mathcal{R}$ of depth n we have an ordering $<^{\mathcal{R}}$ defined on sequences $\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m \rangle$, $m \le n$, satisfying:

(1) $\alpha_1, \dots, \alpha_m < \omega_1$

(2) $(\alpha_1, \dots, \alpha_m)$ is order-isomorphic to the permutation p_m, where $(p_1, \dots, p_m)$ is the unique sequence such that $\langle p_1, i_1, \dots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R})$.

The ordering $<^{\mathcal{R}}$ is the lexicographic ordering on $\mathrm{dom}(\mathcal{R})$. We abbreviate the statement of (2) by saying $(\alpha_1, \ldots, \alpha_m)$ (or $\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle$) is of type $\langle p_1, i_1, \ldots, p_m, i_m \rangle$.

Definition 4.4. Given a type-1 tree of uniform cofinalities $\mathcal{R}$ with corresponding ordering $<^{\mathcal{R}}$, we say a function $f \colon \mathrm{dom}(\mathcal{R}) \to \omega_1$ is of *type* $\mathcal{R}$ if f satisfies:

(1) f is order-preserving from $<^{\mathcal{R}}$ to ω_1.
(2) If $\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle \in \mathrm{dom}(<^{\mathcal{R}})$ has limit rank with respect to $<^{\mathcal{R}}$, then $f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle) = \sup\{f(\vec{t}) \colon \vec{t} <^{\mathcal{R}} \langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle\}$. Otherwise, $f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle)$ has uniform cofinality ω.

Note the following simple facts about a function f of type $\mathcal{R}$. First, if $\vec{s} = \langle p_1, i_1, \ldots, p_m, i_m \rangle$ is not maximal in $\mathrm{dom}(\mathcal{R})$ (so $i_m > 0$), and $\mathcal{R}(\vec{s}) = p_{m+1}$, then for $\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle$ of type $\langle p_1, i_1, \ldots, p_m, i_m \rangle$ and limit $\alpha_1, \ldots, \alpha_m$, we have: $f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle) = \sup\{f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m, \beta, 0 \rangle) \colon (\alpha_1, \ldots, \alpha_m, \beta)$ is of type $p_{m+1}\}$. Next, if $\langle p_1, i_1, \ldots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R})$ and $i_m = 0$, then for all sequences $\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle$ of type $\langle p_1, i_1, \ldots, p_m, i_m \rangle$ we have: $f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle) = \sup_{\beta < \alpha_m} f(\langle \alpha_1, i_1, \ldots, i_{m-1}, \beta, 0 \rangle)$ if α_m is a limit ordinal and otherwise $f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle)$ has uniform cofinality ω. Also, if $\langle p_1, i_1, \ldots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R})$ and $\mathcal{R}(\langle p_1, i_1, \ldots, p_m, i_m \rangle) = (\omega)$, then for all $\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle$ of type $\langle p_1, i_1, \ldots, p_m, i_m \rangle$ we have $f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle)$ has uniform cofinality ω.

We say the tuple of ordinals $\vec{\beta} = (\ldots, \beta^{\vec{s}}, \ldots)_{\vec{s} \in \mathrm{dom}(\mathcal{R})}$ is of type $\mathcal{R}$ if there is a function $f \colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ representing $\vec{\beta}$ in the following sense: for all $\vec{s} = \langle p_1, i_1, \ldots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R})$, $\beta^{\vec{s}} = [f^{\vec{s}}]_{W_1^m}$, where $f^{\vec{s}}(\alpha_1, \ldots, \alpha_m) \doteq f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle)$. If $f \colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ is of type $\mathcal{R}$, we write $[f]$ for the tuple of ordinals $(\ldots, \beta^{\vec{s}}, \ldots)_{\vec{s} \in \mathrm{dom}(\mathcal{R})}$ represented by f.

We now associate to each type-1 tree of measures $\mathcal{R}$ a measure $M^{\mathcal{R}}$ on $(\omega_\omega)^{<\omega}$.

Definition 4.5. Let $\mathcal{R}$ be a type-1 tree of measures. Then $M^{\mathcal{R}}$ is the measure on $(\omega_\omega)^{<\omega}$ defined as follows. A has $M^{\mathcal{R}}$ measure one iff $\exists$ c.u.b. $C \subseteq \omega_1$ $\forall f \colon \mathrm{dom}(<^{\mathcal{R}}) \to C$ of type $\mathcal{R}$, $[f] \in A$.

We refer to the measures $M^{\mathcal{R}}$ as the type-1 *general homogeneous measures* (they are the measures arising in the homogeneous tree construction on a $\mathbf{\Pi}^1_2$ set, using the most general "type" of function).

The next lemma shows that the requirement that $\vec{s} = \langle p_1, i_1, \ldots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R})$ be non-maximal in $\mathrm{dom}(\mathcal{R})$ whenever $\mathcal{R}(\vec{s}) = p_{m+1}$ causes no loss of generality, as we can add $\langle p_1, i_1, \ldots, p_m, i_m, p_{m+1}, 0 \rangle$ to $\mathrm{dom}(\mathcal{R})$. Although the proof is essentially trivial, we require some terminology.

By a *pruned* type-1 tree of uniform cofinalities we mean a function $\mathcal{R}'$ satisfying definition 4.3 except we remove the requirement that $\vec{s} = \langle p_1, i_1, \ldots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R}')$ and $\mathcal{R}'(\vec{s}) = p_{m+1}$ implies that $\vec{s}$ is non-maximal in $\mathrm{dom}(\mathcal{R}')$. Thus, a pruned tree is obtained from a type-1 tree by removing some (possibly all) sequences $\vec{s}$ ending in 0 from its domain. If $\mathcal{R}$ is a type-1 tree, and $\mathcal{R}'$ is obtained by deleting all sequences $\vec{s}$ ending in 0 from $\mathrm{dom}(\mathcal{R})$, then we say $\mathcal{R}'$ is the pruned subtree of $\mathcal{R}$.

If $\mathcal{R}'$ is a pruned type-1 tree, we define $<^{\mathcal{R}'}$ exactly as for type-1 trees. We say $f \colon \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ is of type $\mathcal{R}'$ if f is order-preserving, satisfies (2) of definition 4.4 for $\vec{s} = \langle p_1, i_1, \ldots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R}')$ with $\vec{s}$ non-maximal in $\mathrm{dom}(\mathcal{R})$

or $\vec{s}$ maximal in $\mathrm{dom}(\mathcal{R})$ and $\mathcal{R}(\vec{s}) = (s)$ or (ω). If $\vec{s}$ is maximal in $\mathrm{dom}(\mathcal{R})$ and $\mathcal{R}(\vec{s}) = p_{m+1}$, then we require that for all $(\alpha_1, \dots, \alpha_m)$ of type p_{m+1} that $f(\langle \alpha_1, i_1, \dots, \alpha_m, i_m \rangle)$ has uniform cofinality $\{\beta \colon (\alpha_1, \dots, \alpha_m, \beta)$ has type $p_{m+1}\}$ if this set has limit order type, and otherwise uniform cofinality ω (and also require $f(\langle \alpha_1, i_1, \dots, \alpha_m, i_m \rangle) > \sup\{f(\vec{t}) \colon t <^{\mathcal{R}'} \langle \alpha_1, i_1, \dots, \alpha_m, i_m \rangle\}$).

Note that if $\mathcal{R}$ is a type-1 tree, and $\mathcal{R}'$ is the pruned subtree of $\mathcal{R}$, then any $f \colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ induces an $f' \colon \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ of type $\mathcal{R}'$ by restricting f to $\mathrm{dom}(<^{\mathcal{R}'})$. The next lemma says we can reverse the process.

LEMMA 4.2. *Suppose $\mathcal{R}$ is a type-1 tree of uniform cofinalities, and $\mathcal{R}'$ is the pruned subtree. Suppose $f' \colon \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ is of type $\mathcal{R}'$, and has range in the additively closed ordinals. Then there is an $f \colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ with f inducing f' by restriction. Moreover, f is unique in the following sense: if f, g both induce f' then for all $\langle p_1, i_1, \dots, p_m, 0 \rangle \in dom(\mathcal{R})$, $\forall^* \alpha_1, \dots, \alpha_m$ $f(\langle \alpha_1, i_1, \dots, \alpha_m, 0 \rangle) = g(\langle \alpha_1, i_1, \dots, \alpha_m, 0 \rangle)$.*

PROOF. Suppose $\vec{s} = \langle p_1, i_1, \dots, p_m, i_m \rangle$ is maximal in $\mathrm{dom}(\mathcal{R}')$ and $\mathcal{R}(\vec{s}) = p_{m+1}$ (so the only extension of $\vec{s}$ in $\mathrm{dom}(\mathcal{R})$ is $\langle p_1, i_1, \dots, p_m, i_m, p_{m+1}, 0 \rangle$). Let h be continuous in the last argument and witness the uniform cofinality of $f'^{\vec{s}}$. Thus, $\mathrm{dom}(h)$ consists of tuples $(\alpha_1, \dots, \alpha_m, \beta)$ of type p_{m+1} if $\{\beta \colon (\alpha_1, \dots, \alpha_m, \beta)$ has type $p_{m+1}\}$ has limit order type, and otherwise $\beta \in \omega$. Define then $f(\langle \alpha_1, i_1, \dots, \alpha_m, i_m, \alpha_{m+1}, 0 \rangle) =$ the $\omega \cdot h(\alpha_1, \dots, \alpha_{m+1})^{st}$ ordinal greater than $\sup\{f(\vec{t}) \colon \vec{t} <^{\mathcal{R}'} \langle \alpha_1, i_1, \dots, \alpha_m, i_m \rangle\}$ if $\{\beta \colon (\alpha_1, \dots, \alpha_m, \beta)$ has type $p_{m+1}\}$ has limit order type. Otherwise, define $f(\langle \alpha_1, i_1, \dots, \alpha_m, i_m, \alpha_{m+1}, 0 \rangle =$ the $\omega \cdot \alpha_{m+1}^{st}$ ordinal greater than this supremum. This easily works. Uniqueness is easily checked. $\square$

If $\mathcal{R}$ is a type-1 tree of uniform cofinalities, then by the preceding lemma we can identify $M^{\mathcal{R}}$ with a measure on tuples $(\dots, \beta^{\vec{s}}, \dots)$ where $\vec{s} \in \mathrm{dom}(\mathcal{R})$ and $\vec{s}$ does not end in 0. We will henceforth view $\mathrm{dom}(M^{\mathcal{R}})$ in this manner.

We define the tree of uniform cofinalities $\mathcal{R}_2$ to extend the tree $\mathcal{R}_1$ in the obvious manner, that is, $\mathcal{R}_1 = \mathcal{R}_2 \restriction \mathrm{dom}(\mathcal{R}_1)$. We say $\mathcal{R}_2$ is an *immediate extension* of $\mathcal{R}_1$ if there is only one sequence $\langle p_1, i_1, \dots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R}_2) - \mathrm{dom}(\mathcal{R}_1)$ with $i_m > 0$ (via lemma 4.2 we may think of $\mathcal{R}_2$ as having only this sequence extra in its domain). Notice that if $\mathcal{R}_2$ extends $\mathcal{R}_1$, then the measure $M^{\mathcal{R}_2}$ projects to the measure $M^{\mathcal{R}_1}$ in a natural way provided we have a way of identifying $\mathrm{dom}(\mathcal{R}_1)$ with a subtree of $\mathrm{dom}(\mathcal{R}_2)$ (not necessarily the identity map). For our purposes, this identification will always be of the following form. Suppose $\langle p_1, i_1, \dots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R}_2) - \mathrm{dom}(\mathcal{R}_1)$, and let a be the largest integer such that $\langle p_1, i_1, \dots, p_m, a \rangle \in \mathrm{dom}(\mathcal{R}_1)$. Then for some $1 \le k \le a$ we identify $\langle p_1, i_1, \dots, p_m, l \rangle \in \mathrm{dom}(\mathcal{R}_1)$ with $\langle p_1, i_1, \dots, p_m, l \rangle \in \mathrm{dom}(\mathcal{R}_2)$ for $l < k$, and with $\langle p_1, i_1, \dots, p_m, l+1 \rangle \in \mathrm{dom}(\mathcal{R}_2)$ for $l \ge k$. With a slight abuse of notation, we abbreviate this by saying $\mathcal{R}_2$ has the "extra sequence" $\vec{s} = \langle p_1, i_1, \dots, p_m, k \rangle$ in its domain, and write $\vec{s} \in \mathrm{dom}(\mathcal{R}_2) - \mathrm{dom}(\mathcal{R}_1)$. When we speak of $\mathcal{R}_2$ extending $\mathcal{R}_1$, we will always implicitly have such an identification.

We say the function $\mathcal{R}'$ is an (immediate) *partial extension* of $\mathcal{R}$ if $\mathcal{R} = \mathcal{R}' \restriction \mathrm{dom}(\mathcal{R})$, there is a unique $\langle p_1, i_1, \dots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R}') - \mathrm{dom}(\mathcal{R})$, and for this sequence $i_m > 0$, but $\mathcal{R}'(\langle p_1, i_1, \dots, p_m, i_m \rangle)$ is not defined (more precisely, we can say $\mathcal{R}'(\langle p_1, i_1, \dots, p_m, i_m \rangle)$ is assigned a symbol for "undefined"). The ordering $<^{\mathcal{R}'}$ is defined for such $\mathcal{R}'$ exactly as for $<^{\mathcal{R}}$ (where now $\langle p_1, i_1, \dots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R}')$).

Additionally, we say a function $f'\colon \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ is of *semi-type* $\mathcal{R}'$ if f' is order preserving and $f' \restriction \mathrm{dom}(<^{\mathcal{R}})$ is of type $\mathcal{R}$. Thus, the uniform cofinality of $f'(\langle \alpha_1, i_1, \dots, \alpha_m, i_m \rangle)$ is not specified. We say the function $\mathcal{R}'$ is a partial type-1 tree of uniform cofinalities if it is a partial extension of some type-1 tree. Thus, there is a unique sequence $\vec{s} \in \mathrm{dom}(\mathcal{R}')$ for which $\mathcal{R}'(\vec{s})$ is not defined, and $\vec{s}$ does not end in 0.

Definition 4.6. A type ≤ 1 tree of uniform cofinalities is a pair $\mathcal{B} = \langle n, \mathcal{R} \rangle$ where $n \in \omega$ and $\mathcal{R}$ is a type-1 tree of uniform cofinalities.

To the type ≤ 1 tree $\mathcal{B} = \langle n, \mathcal{R} \rangle$ we associate the ordering $<^{\mathcal{B}}$ on $\mathrm{dom}(<^{\mathcal{B}}) = \{1, \dots, n\} \cup \mathrm{dom}(<^{\mathcal{R}})$. The integers $1, \dots, n$ are ordered naturally, and precede all elements of $\mathrm{dom}(<^{\mathcal{R}})$. We then define the measure $M^{\mathcal{B}}$ in the usual way, so $M^{\mathcal{B}} = W_1^n \times M^{\mathcal{R}}$. We say $\mathcal{B}' = \langle n', \mathcal{R}' \rangle$ immediately extends $\mathcal{B}$ if either $n' = n+1$ and $\mathcal{R}' = \mathcal{R}$, or $n' = n$ and $\mathcal{R}'$ immediately extends $\mathcal{R}$. In the first case we will also implicitly have an embedding from n into $n+1$. Thus, the measure $M^{\mathcal{B}'}$ projects to the measure $M^{\mathcal{B}}$ naturally.

By a partial type ≤ 1 tree of uniform cofinalities we mean a pair $\mathcal{B}' = \langle n, \mathcal{R}' \rangle$ where $\mathcal{R}'$ is a partial type-1 tree. Also, we say $\mathcal{B}'$ is an immediate partial extension of the type ≤ 1 tree $\mathcal{B} = \langle n, \mathcal{R} \rangle$ if $\mathcal{B}' = \langle n+1, \mathcal{R} \rangle$ or $\mathcal{B}' = \langle n, \mathcal{R}' \rangle$ and $\mathcal{R}'$ is an immediate partial extension of $\mathcal{R}$. We define $f\colon \mathrm{dom}(<^{\mathcal{B}}) \to \omega_1$ being of type $\mathcal{B}$ and $f'\colon \mathrm{dom}(<^{\mathcal{B}'}) \to \omega_1$ being of semi-type $\mathcal{B}'$ as for $\mathcal{R}$ and $\mathcal{R}'$.

We now recast theorem 4.1 into the present framework. The proof is essentially identical to that of theorem 4.1 with only notational changes. It says that possible uniform cofinalities for functions $f\colon \mathrm{dom}(M^{\mathcal{B}}) \to \boldsymbol{\delta}^1_3$ with respect to $M^{\mathcal{B}}$ correspond to the partial extensions $\mathcal{B}'$ of $\mathcal{B}$. Recall that if $\mathcal{B} = \langle n, \mathcal{R} \rangle$, $h\colon \mathrm{dom}(<^{\mathcal{B}}) \to \omega_1$, and $\vec{s} = \langle q_1, j_1, \dots, q_l, j_l \rangle \in \mathrm{dom}(\mathcal{R})$, then $h^{\vec{s}}\colon (\omega_1)^l \to \omega_1$ is defined by: $h^{\vec{s}}(\alpha_1, \dots, \alpha_l) = h(\langle \alpha_1, j_1, \dots, \alpha_l, j_l \rangle)$. We extend this notation by allowing $\vec{s}$ to be an integer k in $\{1, \dots, n\}$, in which case $h^{\vec{s}}$ or $[h^{\vec{s}}]$ denotes $h(k)$.

Theorem 4.2. *Let $\mathcal{B} = \langle n, \mathcal{R} \rangle$ be a type ≤ 1 tree of uniform cofinalities with corresponding measure $M^{\mathcal{B}}$. Let $f\colon \mathrm{dom}(M^{\mathcal{B}}) \to \boldsymbol{\delta}^1_3$ have range in the limit ordinals. Then f has uniform cofinality with respect to $M^{\mathcal{B}}$ described by one of the following cases.*

1. *f has uniform cofinality ω.*
2. *There is an immediate partial extension $\mathcal{B}'$ of $\mathcal{B}$, with say $\vec{s} \in \mathrm{dom}(\mathcal{R}') - \mathrm{dom}(\mathcal{R})$, such that for almost all $\vec{\theta} \in \mathrm{dom}(M^{\mathcal{B}})$, $f(\vec{\theta})$ has uniform cofinality $\{\gamma \colon \exists h'\colon \mathrm{dom}(<^{\mathcal{B}'}) \to \omega_1$ of semi-type $\mathcal{B}'$ inducing $h\colon \mathrm{dom}(<^{\mathcal{B}}) \to \omega_1$ with $[h] = \vec{\theta}$ and $\gamma = [h'^{\vec{s}}]\}$.*

Proof. If f has uniform cofinality ω_1 almost everywhere we may take $\mathcal{B}' = \langle n+1, \mathcal{R} \rangle$ with $n+1$ as the extra element in $\mathrm{dom}(<^{\mathcal{B}'})$. If $f(\vec{\theta})$ has uniform cofinality $u(\vec{\theta}) < \omega_1$ almost everywhere, then an easy argument shows that for some $1 \leq k \leq n$, for almost all $\vec{\theta} = (\alpha_1, \dots, \alpha_n, \dots, \beta^{\vec{s}}, \dots)_{\vec{s} \in \mathrm{dom}(\mathcal{R})}$, $f(\vec{\theta})$ has uniform cofinality α_k. This is described by the partial extension $\mathcal{B} = \langle n+1, \mathcal{R} \rangle$, where we identify $j < k$ in $\mathrm{dom}(\mathcal{B})$ with $j \in \mathrm{dom}(\mathcal{B}')$, and $j \geq k$ in $\mathrm{dom}(\mathcal{B})$ with $j+1 \in \mathrm{dom}(\mathcal{B}')$.

So suppose $f(\vec{\theta})$ has uniform cofinality $> \omega_1$ almost everywhere. Following the proof of theorem 4.1 we arrive at the conclusion of theorem 4.1 with the function h' in the statement of 4.1 replaced by $h^{\langle q_1, j_1, \dots, j_l \rangle}\colon (\omega_1)^l \to \omega_1$

for some $\langle q_1, j_1, \ldots, q_l, j_l\rangle \in \mathrm{dom}(\mathcal{R})$ (where $h\colon \mathrm{dom}(<^{\mathcal{B}}) \to \omega_1$ represents $\vec{\theta}$). For example, the conclusion of case (2) of theorem 4.1 becomes: there is a $\langle q_1, j_1, \ldots, q_l, j_l\rangle \in \mathrm{dom}(\mathcal{R})$ such that for $M^{\mathcal{B}}$ almost all $\vec{\theta}$ represented by $h\colon \mathrm{dom}(<^{\mathcal{B}}) \to \omega_1$ of type $\mathcal{B}$, $f(\vec{\theta})$ has uniform cofinality represented with respect to W_1^l by $h^{\langle q_1, j_1, \ldots, j_l\rangle}(\alpha_1, \ldots, \alpha_l) = h(\langle \alpha_1, j_1, \ldots, \alpha_l, j_l\rangle)$. In this case, the uniform cofinality of f is described by the partial extension $\mathcal{B}' = \langle n, \mathcal{R}'\rangle$, where $\mathcal{R}'$ has the extra sequence $\langle q_1, j_1, \ldots, q_l, j_l\rangle$ in its domain, and we identify $\langle q_1, j_1, \ldots, q_l, j\rangle \in \mathrm{dom}(\mathcal{R})$ with $\langle q_1, j_1, \ldots, q_l, j\rangle \in \mathrm{dom}(\mathcal{R}')$ for $j < j_l$, and with $\langle q_1, j_1, \ldots, q_l, j{+}1\rangle \in \mathrm{dom}(\mathcal{R}')$ for $j \geq j_l$. The other cases can also easily be seen to correspond to partial extensions $\mathcal{B}'$ of $\mathcal{B}$. □

We refer to the $\mathcal{B}'$ of theorem 4.2 as the partial extension of $\mathcal{B}$ by f (or by its uniform cofinality u).

Finally, we state a sliding lemma for type-1 trees. It is a straightforward generalization of lemma 3.2.

Lemma 4.3. *Let $\mathcal{R}$ be a type-1 tree of uniform cofinalities, $f\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$, and $\langle p_1, i_1, \ldots, p_m, i_m\rangle \in \mathrm{dom}(\mathcal{R})$.*

(1) *Let $g\colon \omega_1^m \to \omega_1$ be given such that $\forall^*_{W_1^m} \alpha_1, \ldots, \alpha_m\ g(\alpha_1, \ldots, \alpha_m) < f(\langle \alpha_m, i_1, \alpha_{j_2}, i_2, \ldots, \alpha_{j_{m-1}}, i_{m-1}\rangle)$, where $p_m = (m, j_2, \ldots, j_m)$. Let $\mathcal{R}'$ be the partial extension of $\mathcal{R}$ with the extra sequence $\langle p_1, i_1, \ldots, p_m, k+1\rangle$ in its domain, where k is maximal such that $\langle p_1, i_1, \ldots, p_m, k\rangle \in \mathrm{dom}(\mathcal{R})$. Then there is an $f'\colon \mathrm{dom}(<^{\mathcal{R}'}) \to \mathrm{ran}(f)$ of semi-type $\mathcal{R}'$ inducing $f_2\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ such that $[f_2] = [f]$ and $\forall^*_{W_1^m} \alpha_1, \ldots, \alpha_m$ $g(\vec{\alpha}) < f'(\langle \alpha_m, i_1, \alpha_{j_2}, i_2, \ldots, \alpha_{j_m}, k+1\rangle)$.*

(2) *Let $g\colon \omega_1^m \to \omega_1$ be given such that $\forall^*_{W_1^m} \alpha_1, \ldots, \alpha_m\ g(\alpha_1, \ldots, \alpha_m) < f(\langle \alpha_m, i_1, \alpha_{j_2}, i_2, \ldots, \alpha_{j_{m-1}}, i_{m-1}, \alpha_{j_m}, i_m\rangle)$, where $i_m > 0$ and p_m as above. Let $\mathcal{R}'$ be as above, where we now identify $\langle p_1, i_1, \ldots, p_m, j\rangle \in \mathrm{dom}(\mathcal{R})$ with $\langle p_1, i_1, \ldots, p_m, j\rangle \in \mathrm{dom}(\mathcal{R}')$ for $j < i_m$, and with $\langle p_1, i_1, \ldots, p_m, j{+}1\rangle \in \mathrm{dom}(\mathcal{R}'')$ for $j \geq i_m$ (i.e., the "extra integer" is inserted between $i_m - 1$ and i_m). Then there is an $f'\colon \mathrm{dom}(<^{\mathcal{R}'}) \to \mathrm{ran}(f)$ of semi-type $\mathcal{R}'$ inducing $f_2\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ such that $[f_2] = [f]$ and $\forall^*_{W_1^m} \alpha_1, \ldots, \alpha_m\ g(\vec{\alpha}) < f'(\langle \alpha_m, i_1, \alpha_{j_2}, i_2, \ldots, \alpha_{j_m}, i_m\rangle)$.*

Proof. Consider first (1). Let $C \subseteq \omega_1$ be c.u.b. such that $\forall^* \alpha_1, \ldots, \alpha_m$ $g(\alpha_1, \ldots, \alpha_m) < f(\langle \alpha_m, i_1, \alpha_{j_2}, i_2, \ldots, i_{m-1}, N_C(\alpha_{j_m}), 0\rangle)$. Let $l(\alpha) =$ the $2 \cdot \alpha^{th}$ element of C. In particular, $l(\alpha) = \alpha$ almost everywhere. Define $\tilde{f}\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ by: $\tilde{f}(\langle \beta_1, l_1, \beta_2, l_2, \ldots, \beta_n, l_n\rangle) = f(\langle l(\beta_1), l_1, \ldots, l(\beta_n), l_n\rangle)$. Clearly $\tilde{f}$ is order-preserving from $\mathrm{dom}(<^{\mathcal{R}})$ to ω_1 and $[\tilde{f}] = [f]$. Extend $\tilde{f}$ to $\tilde{f}'\colon \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ by defining $\tilde{f}'(\langle \alpha_m, i_1, \alpha_{j_2}, i_2, \ldots, \alpha_{j_m}, k+1\rangle) = f(\langle \alpha_m, i_1, \alpha_{j_2}, i_2, \ldots, i_{m-1}, \beta, 0\rangle)$, where β is the $l(\alpha_{j_m}) + 1^{st}$ element of C. From the definition of C it follows that $\forall^*_{W_1^m} \alpha_1, \ldots, \alpha_m\ g(\vec{\alpha}) < \tilde{f}'(\langle \alpha_m, i_1, \alpha_{j_2}, i_2, \ldots, \alpha_{j_m}, k+1\rangle)$. Finally, $\tilde{f}'$ is of semi-type $\mathcal{R}'$ except for the requirement that for $\langle q_1, l_1, \ldots, q_n, l_n\rangle \in \mathrm{dom}(\mathcal{R})$ and $\beta_1, \ldots, \beta_l$ in a certain W_1^1 measure zero set, we must have $f(\langle \beta_1, l_1, \ldots, \beta_n, l_n\rangle)$ having uniform cofinality ω. However, it is now trivial to redefine $\tilde{f}'$ on these measure zero sets to obtain f' satisfying also this last condition. Namely, define $f'(\langle \beta_1, l_1, \ldots, \beta_n, l_n\rangle)$ to be the ω^{th} ordinal greater than $\sup\{\tilde{f}'(\vec{t})\colon \vec{t} <^{\mathcal{R}'} \langle \beta_1, l_1, \ldots, \beta_n, l_n\rangle\}$.

For (2), we start with a c.u.b. $C \subseteq \omega_1$ such that $\forall^* \alpha_1, \ldots, \alpha_m \; g(\alpha_1, \ldots, \alpha_m) < f(\langle \alpha_m, i_1, \alpha_{j_2}, i_2, \ldots, \alpha_{j_m}, i_m, N_C(\alpha_q), 0 \rangle)$, where if $p_{m+1} \doteq \mathcal{R}(\langle p_1, i_1, \ldots, p_m, i_m \rangle) = (m+1, \ldots, r)$, then $q = r - 1$. The argument then follows exactly as above. $\square$

CHAPTER 5

Analysis of Measures on $\boldsymbol{\delta}^1_3$

Our next goal is to introduce a family of measures on $\boldsymbol{\delta}^1_3$ which we prove captures all measures on $\boldsymbol{\delta}^1_3$. These measures are similar to those in the family $\mathcal{W}_3$ with two main exceptions. First, the most general uniform cofinalities must be used, which leads to the notion of a type-2 tree of uniform cofinalities, and secondly, descriptions and certain functions from $\boldsymbol{\delta}^1_3$ to $\boldsymbol{\delta}^1_3$ are used in their definition. First we define the notion of a type-2 tree of uniform cofinalities.

If $\mathcal{B}' = \langle n, \mathcal{R}' \rangle$ is a partial type ≤ 1 tree of uniform cofinalities, we say the type-1 tree $\mathcal{B}$ *completes* $\mathcal{B}'$ if $\mathrm{dom}(\mathcal{B}')$ and $\mathrm{dom}(\mathcal{B})$ agree on sequences not ending in 0. In other words, if $\vec{s} = \langle q_1, j_1, \dots, q_n, j_n \rangle$ is the unique sequence in $\mathrm{dom}(\mathcal{B}')$ for which $\mathcal{B}'(\vec{s})$ is undefined, then $\mathcal{B}$ completes $\mathcal{B}'$ by giving a value $\mathcal{B}(\vec{s}) = q_{n+1}$ (technically, this also requires $\langle q_1, j_1, \dots, q_n, j_n, q_{n+1}, 0 \rangle$ to be in $\mathrm{dom}(\mathcal{B})$, though sequences ending in zero play little role via lemma 4.2).

DEFINITION 5.1. A *type-2 tree of uniform cofinalities* is a function $\mathcal{U}$ satisfying the following:

(1) $\langle i_1 \rangle \in \mathrm{dom}(\mathcal{U})$ for $1 \leq i_1 \leq a$ for some integer a. $\mathcal{U}(\langle i_1 \rangle)$ is either the formal symbol (ω), or a partial type ≤ 1 tree of uniform cofinalities which is an immediate partial extension of the empty tree.

(2) In general, $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m \rangle \in \mathrm{dom}(\mathcal{U})$ iff $\langle i_1, p_1, \dots, i_{m-1} \rangle \in \mathrm{dom}(\mathcal{U})$, p_{m-1} is a type ≤ 1 tree of uniform cofinalities which completes the partial type ≤ 1 tree $\mathcal{U}(\langle i_1, p_1, \dots, i_{m-1} \rangle)$, and $0 \leq i_m \leq a$ for some integer a (which depends on $\langle i_1, p_1, \dots, i_{m-1} \rangle$). $\mathcal{U}(\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m \rangle)$ is the formal symbol (s) iff $i_m = 0$. For $i_m > 0$, $\mathcal{U}(\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m \rangle)$ is either (ω) or a partial type ≤ 1 tree of uniform cofinalities which is an immediate partial extension of p_{m-1}.

The only immediate partial extensions of the empty type-1 tree are $\mathcal{B}' = \langle 1, \emptyset \rangle$ and $\mathcal{B}' = \langle 0, \mathcal{R}' \rangle$, where $\mathcal{R}'$ is the partial type-1 tree having $\langle q_1, 1 \rangle$ as the only sequence not ending in 0 in its domain, where $q_1 = (1)$ is the unique permutation of length 1. The first of these describes cofinality ω_1, and the second, cofinality ω_2 (c.f. lemma 4.2). Thus, we can think of $\mathcal{U}(\langle i_1 \rangle)$ as being one of the three possible cofinalities of an ordinal below $\boldsymbol{\delta}^1_3$. In general, $\mathcal{U}(\langle i_1, p_1, \dots, p_{m-1}, i_m \rangle)$ is a possible uniform cofinality with respect to the measure $M^{p_{m-1}}$. p_m will then range over the possible types for an ordinal in this uniform cofinality.

Notice that $\vec{s} = \langle i_1, p_1, \dots, p_{m-1}, i_m \rangle$ is maximal in $\mathrm{dom}(\mathcal{U})$ iff $\mathcal{U}(\vec{s}) = (s)$ or (ω), the first case occurring iff $i_m = 0$.

If $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m \rangle \in \mathrm{dom}(\mathcal{U})$, then each type-$\leq 1$ tree p_l is an immediate extension of p_{l-1} for all $l \leq m-1$. Recall $M^{p_{m-1}}$ is a measure on tuples $(\dots, \alpha^{\vec{s}}, \dots)$ where $\vec{s} \in \mathrm{dom}(\mathcal{R})$ and does not end in 0. There are exactly $m-1$ such

$\vec{s} \in \operatorname{dom}(\mathcal{R})$. Moreover, the sequence $p_1, \ldots, p_{m-1}$ determines an order of enumeration of the $\vec{s}$, and hence of the components of the tuple $(\ldots, \alpha^{\vec{s}}, \ldots)$. Thus, given the sequence $\langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, i_m \rangle$, we may identify $M^{p_{m-1}}$ with a measure on sequences $(\alpha_1, \ldots, \alpha_{m-1})$. That is, $\alpha_i = \alpha^{s_i}$, where s_i is the unique sequence not ending in 0 in $\operatorname{dom}(p_i) - \operatorname{dom}(p_{i-1})$. We say the sequence $(\alpha_1, \ldots, \alpha_{m-1})$ is of type $\langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, i_m \rangle$ if there is a tuple $(\ldots, \alpha^{\vec{s}}, \ldots)_{\vec{s} \in \operatorname{dom}(\mathcal{R})}$ of type $\mathcal{R}$ which when enumerated in this manner produces $(\alpha_1, \ldots, \alpha_{m-1})$. If the sequence $\langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, i_m \rangle$ is understood, we simply say $(\alpha_1, \ldots, \alpha_{m-1})$ is of type p_{m-1}. We also refer to the sequence $\langle i_1, \alpha_1, \ldots, i_{m-1}, \alpha_{m-1}, i_m \rangle$ as having type $\langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, i_m \rangle$, or type p_{m-1}.

Corresponding to the type-2 tree of cofinalities $\mathcal{U}$ we define an ordering $<^{\mathcal{U}}$.

Definition 5.2. Let $\mathcal{U}$ be a type-2 tree of uniform cofinalities. Then $<^{\mathcal{U}}$ is lexicographic ordering on sequences $\langle i_1, \alpha_1, \ldots, i_{m-1}, \alpha_{m-1}, i_m \rangle$ such that for some $\langle i_1, p_1, \ldots, p_{m-1}, i_m \rangle \in \operatorname{dom}(\mathcal{U})$, $(\alpha_1, \ldots, \alpha_{m-1})$ has type $\langle i_1, p_1, \ldots, p_{m-1}, i_m \rangle$.

Definition 5.3. Let $\mathcal{U}$ be a type-2 tree of uniform cofinalities. We say a function $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}_3^1$ is of *type* $\mathcal{U}$ if f satisfies:

(1) f is order preserving from $<^{\mathcal{U}}$ into $\boldsymbol{\delta}_3^1$.
(2) If $\vec{\alpha} = \langle i_1, \alpha_1, \ldots, i_{m-1}, \alpha_{m-1}, i_m \rangle \in \operatorname{dom}(<^{\mathcal{U}})$ is of limit rank with respect to $<^{\mathcal{U}}$, then $f(\vec{\alpha}) = \sup\{f(\vec{\alpha}') : \vec{\alpha}' <^{\mathcal{U}} \vec{\alpha}\}$. Otherwise, $f(\vec{\alpha})$ has uniform cofinality ω.

As with type-1 tree, we note the following simple facts. If the sequence $\vec{s} = \langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, i_m \rangle$ is not maximal in $\operatorname{dom}(\mathcal{U})$, then for all $\vec{\alpha} = \langle i_1, \alpha_1, \ldots, i_{m-1}, \alpha_{m-1}, i_m \rangle$ of type $\vec{s}$ of limit rank,

$$f(\vec{\alpha}) = \sup_\beta f(\langle i_1, \alpha_1, \ldots, i_{m-1}, \alpha_{m-1}, i_m, \beta, 0 \rangle),$$

where the supremum ranges over β such that $\langle i_1, \alpha_1, \ldots, i_{m-1}, \alpha_{m-1}, i_m, \beta, 0 \rangle \in \operatorname{dom}(<^{\mathcal{U}})$. If $i_m = 0$, then for $\vec{\alpha}$ of type $\vec{s}$, $f(\vec{\alpha}) = \sup\limits_{\beta < \alpha_{m-1}} f(\langle i_1, \alpha_1, \ldots, i_{m-1}, \beta, 0 \rangle)$ for α_{m-1} limit, and otherwise has uniform cofinality ω. Finally, if $\mathcal{U}(\vec{s}) = \omega$, then $f(\vec{\alpha})$ has uniform cofinality ω.

We now define, in the usual way, the measure $M^{\mathcal{U}}$ associated to the type-2 tree $\mathcal{U}$, now using the weak partition relation on $\boldsymbol{\delta}_3^1$.

Definition 5.4. Let $\mathcal{U}$ be a type-2 tree of uniform cofinalities. Then $M^{\mathcal{U}}$ is the measure on $(\boldsymbol{\delta}_3^1)^{<\omega}$ defined as follows. A has $M^{\mathcal{U}}$ measure one iff $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}_3^1$ $\forall f\colon \operatorname{dom}(<^{\mathcal{U}}) \to C$ of type $\mathcal{U}$, $(\ldots, \beta^{\vec{s}}, \ldots)_{\vec{s} \in \operatorname{dom} \mathcal{U}} \in A$. Here, for $\vec{s} = \langle i_1, p_1, \ldots, p_{m-1}, i_m \rangle \in \operatorname{dom}(\mathcal{U})$, $\beta^{\vec{s}}$ is represented with respect to $M^{p_{m-1}}$ by $f^{\vec{s}}$, where for $(\alpha_1, \ldots, \alpha_{m-1})$ of type p_{m-1}, $f^{\vec{s}}(\alpha_1, \ldots, \alpha_{m-1}) = f(\langle i_1, \alpha_1, \ldots, \alpha_{m-1}, i_m \rangle)$.

We define the notion of the type-2 tree $\mathcal{U}_2$ extending or immediately extending the type-2 tree $\mathcal{U}$ exactly as with type-1 trees (thus an immediate extension has exactly one extra sequence not ending in 0 in its domain).

We define a function $\mathcal{U}'$ being a *partial* type-2 tree of uniform cofinalities exactly as in the type-1 case (i.e., there is a unique sequence $\vec{s} \in \operatorname{dom}(\mathcal{U})$ not ending in 0 for which $\mathcal{U}(\vec{s})$ is not defined). Likewise, we define $\mathcal{U}'$ being an immediate partial extension of the type-2 tree $\mathcal{U}$, and $f\colon \operatorname{dom}(\mathcal{U}') \to \boldsymbol{\delta}_3^1$ being of *semi-type* $\mathcal{U}'$ exactly as in the type-1 case. Similarly, we define the notion of the type-2 tree $\mathcal{U}$ *completing* the partial type-2 tree $\mathcal{U}'$.

As with type-1 trees, we define a *pruned* subtree $\mathcal{U}'$ of the type-2 tree $\mathcal{U}$ to be the function obtained by deleting some (possibly all) maximal sequences ending in 0 from the domain of $\mathcal{U}$. We define $f\colon \operatorname{dom}(\mathcal{U}') \to \boldsymbol{\delta}^1_3$ being of type $\mathcal{U}'$ as in the type-1 case, thus for $\vec{s} = \langle i_1, p_1, \ldots, p_{m-1}, i_m\rangle$ maximal in $\operatorname{dom}(\mathcal{U}')$ and $\mathcal{U}'(\vec{s}) = p'_m$ (a partial extension of p_{m-1}), and $\vec{\theta} = \langle i_1, \alpha_1, \ldots, \alpha_{m-1}, i_m\rangle$ of type $\vec{s}$, $f(\vec{\theta})$ has uniform cofinality $\{\beta\colon (\alpha_1, \ldots, \alpha_{m-1}, \beta)$ is of semi-type $p'_m\}$ if this set has limit order-type, and otherwise $f(\vec{\theta})$ has uniform cofinality ω. To say $(\alpha_1, \ldots, \alpha_{m-1}, \beta)$ has semi-type p'_{m-1} means there is a $h\colon \operatorname{dom}(<^{p'_m}) \to \omega_1$ of semi-type p'_m representing this tuple.

As in the type-1 case, the requirement that $\vec{s} = \langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, i_m\rangle$ be non-maximal in $\operatorname{dom}(\mathcal{U})$ for $i_m > 0$ and $\mathcal{U}(\vec{s}) \neq (\omega)$ causes no loss of generality via the following lemma.

LEMMA 5.1. *Let $\mathcal{U}$ be a type-2 tree of uniform cofinalities, and let $\mathcal{U}'$ be a pruned subtree of $\mathcal{U}$. Suppose $f'\colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ is of type $\mathcal{U}'$, and has range in the additively closed ordinals. Then there is an $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ which induces f' by restriction. Furthermore, f is unique in the sense that if g also induces f' then for all $\langle i_1, p_1, \ldots, p_{m-1}, 0\rangle \in \operatorname{dom}(\mathcal{U})$, $f(\vec{\theta}) = g(\vec{\theta})$ for almost all $\vec{\theta} = \langle i_1, \alpha_1, \ldots, \alpha_{m-1}, 0\rangle$ with respect to $M^{p_{m-1}}$.*

PROOF. The existence of f is immediate since f' has type $\mathcal{U}'$ (c.f. the proof of lemma 4.2). Suppose g is another such function and let $\langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, 0\rangle \in \operatorname{dom}(\mathcal{U})$. A partition argument on ω_1 shows that if $l\colon \operatorname{dom}(M^{p_{m-1}}) \to \boldsymbol{\delta}^1_3$ is such that $\forall^*\alpha_1, \ldots, \alpha_{m-1}\ l(\alpha_1, \ldots, \alpha_{m-1}) < f(\langle i_1, \alpha_1, \ldots, i_{m-1}, \alpha_{m-1}, 0\rangle)$ then there is a function $\tilde{f}$ such that $\forall^*\alpha_1, \ldots, \alpha_{m-2}$ with respect to $M^{p_{m-2}}$, $\tilde{f}(\alpha_1, \ldots, \alpha_{m-2}) < f'^{\langle i_1, p_1, \ldots, i_{m-1}\rangle}(\alpha_1, \ldots, \alpha_{m-2}) \doteq f'(\langle i_1, \alpha_1, \ldots, \alpha_{m-2}, i_{m-1}\rangle)$ and such that $\forall^*\alpha_1, \ldots, \alpha_{m-1}\ l(\alpha_1, \ldots, \alpha_{m-1}) < \tilde{f}(\alpha_1, \ldots, \alpha_{m-2})$. It follows that $[f^{\langle i_1, p_1, \ldots, p_{m-1}, 0\rangle}]$, and also $[g^{\langle i_1, p_1, \ldots, p_{m-1}, 0\rangle}]$, are determined by f', and the lemma follows. □

The next lemma is another "sliding" lemma. It is the level 2 analog of lemma 4.3.

LEMMA 5.2. *Let $\mathcal{U}$ be a type-2 tree of uniform cofinalities, and consider $\langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, i_m\rangle \in \operatorname{dom}(\mathcal{U})$. Let $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ be of type $\mathcal{U}$.*

(1) *Let $g\colon \operatorname{dom}(M^{p_{m-1}}) \to \boldsymbol{\delta}^1_3$ be such that*

$$\forall^*_{M^{p_{m-1}}}\alpha_1, \ldots, \alpha_{m-1}\ g(\alpha_1, \ldots, \alpha_{m-1}) < f(\langle i_1, \alpha_1, \ldots, \alpha_{m-2}, i_{m-1}\rangle).$$

Let $\mathcal{U}'$ be the partial extension of $\mathcal{U}$ having $\langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, a+1\rangle$ as the extra element in its domain, where a is maximal such that $\langle i_1, p_1, \ldots, i_{m-1}, p_{m-1}, a\rangle \in \operatorname{dom}(\mathcal{U})$. Then there is a $f'\colon \operatorname{dom}(<^{\mathcal{U}'}) \to \operatorname{ran}(f)$ of semi-type $\mathcal{U}'$ inducing $f_2\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ such that $[f_2] = [f]$ and

$$\forall^*_{M^{p_{m-1}}}\alpha_1, \ldots, \alpha_{m-1}\ g(\alpha_1, \ldots, \alpha_{m-1}) < f'(\langle i_1, \alpha_1, \ldots, i_{m-1}, \alpha_{m-1}, a+1\rangle).$$

(2) *Let g be such that*

$$\forall^*_{M^{p_{m-1}}}\alpha_1, \ldots, \alpha_{m-1}\ g(\alpha_1, \ldots, \alpha_{m-1}) < f(\langle i_1, \alpha_1, \ldots, i_{m-1}, \alpha_{m-1}, i_m\rangle).$$

Let $\mathcal{U}'$ be as above, except we now identify $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, j\rangle \in \mathrm{dom}(\mathcal{U})$ with $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, j\rangle \in \mathrm{dom}(\mathcal{U}')$ for $j < i_m$, and with $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, j+1\rangle \in \mathrm{dom}(\mathcal{U}')$ for $j \geq i_m$ (i.e., the "extra integer" is inserted between $i_m - 1$ and i_m). Then there is an $f'\colon \mathrm{dom}(<^{\mathcal{U}'}) \to \mathrm{ran}(f)$ of semi-type $\mathcal{U}'$ inducing f_2 of type $\mathcal{U}$ with $[f_2] = [f]$ and

$$\forall^*_{M^{p_{m-1}}} \alpha_1, \dots, \alpha_{m-1}\ g(\alpha_1, \dots, \alpha_{m-1}) < f'(\langle i_1, \alpha_1, \dots, i_{m-1}, \alpha_{m-1}, i_m\rangle).$$

PROOF. We consider (1), the proof of (2) being similar. Fix $\mathcal{U}$, f, g as above. Let $\langle q_1, j_1, \dots, q_n, j_n\rangle$ be the unique sequence not ending in 0 in $\mathrm{dom}(p_{m-1}) - \mathrm{dom}(p_{m-2})$ (the case where $p_{m-2} = \langle n, \mathcal{R}\rangle$, $p_{m-1} = \langle n+1, \mathcal{R}\rangle$ is entirely similar. Also, if $m = 2$, then we consider $\langle q_1, j_1\rangle$, the unique sequence not ending in 0 in $\mathrm{dom}(p_1)$).

We consider the partition $\mathcal{P}$: we partition functions $h\colon \mathrm{dom}(<^{p_{m-1}}) \to \omega_1$ of type p_{m-1}, with the extra value $h'(\langle \beta_1, j_1, \dots, \beta_n, j_n\rangle)$ of uniform cofinality ω inserted between $h(\langle \beta_1, j_1, \dots, \beta_n, j_n\rangle)$ and $N_h(h(\langle \beta_1, j_1, \dots, \beta_n, j_n\rangle))$ according to whether $g(\alpha_1, \dots, \alpha_{m-1}) < f(\langle i_1, \alpha_1, \dots, \alpha_{m-2}, i_{m-1}, \gamma, 0\rangle)$, where $\alpha_1, \dots, \alpha_{m-1}$ are the ordinals represented by h, and $\gamma = [{h'}^{\langle q_1, j_1, \dots, q_n, j_n\rangle}]$. From lemma 4.3 it follows easily that on the homogeneous side of the partition the stated property holds. Fix a c.u.b. $C \subseteq \omega_1$ homogeneous for $\mathcal{P}$. Define $\tilde{f}$ as follows: for $\langle l_1, r_1, \dots, r_{k-1}, l_k\rangle \in \mathrm{dom}(\mathcal{U})$, and $\langle l_1, \delta_1, \dots, \delta_{k-1}, l_k\rangle$ of type $\langle l_1, r_1, \dots, r_{k-1}, l_k\rangle$, set $\tilde{f}(\langle l_1, \delta_1, \dots, \delta_{k-1}, l_k\rangle) = f(\langle l_1, \delta'_1, \dots, \delta'_{k-1}, l_k\rangle)$. Here, if $h\colon \mathrm{dom}(<^{r_{k-1}}) \to \omega_1$ of type r_{k-1} represents $(\delta_1, \dots, \delta_{k-1})$, then $(\delta'_1, \dots, \delta'_{k-1})$ is represented by $l \circ h$, where $l(\gamma) =$ the $\omega \cdot \gamma^{th}$ element of C. Note that $l \circ h$ is also of type r_{k-1} so this is well-defined. Easily, $[\tilde{f}] = [f]$ and $\tilde{f}$ is order preserving with respect to $<^{\mathcal{U}}$. Extend $\tilde{f}$ to f' of semi-type $\mathcal{U}'$ by defining $f'(\langle i_1, \alpha_1, \dots, \alpha_{m-1}, a+1\rangle) = f(\langle i_1, \alpha'_1, \dots, \alpha'_{m-1}, 0\rangle)$, where if $h\colon \mathrm{dom}(<^{p_{m-1}}) \to \omega_1$ represents $(\alpha_1, \dots, \alpha_{m-1})$, then $(\alpha'_1, \dots, \alpha'_{m-2})$ are represented by the corresponding components of $l \circ h$, and α'_{m-1} is represented by $N_C \circ l \circ h^{\langle q_1, j_1, \dots, q_n, j_n\rangle}$. Note that $(\alpha'_1, \dots, \alpha'_{m-1})$ is still of type p_{m-1}, so this is well-defined. Easily $f'\colon \mathrm{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ is order-preserving and from the definition of C it follows that $\forall^*_{M^{p_{m-1}}} \alpha_1, \dots, \alpha_{m-1}$ $g(\alpha_1, \dots, \alpha_{m-1}) < f'(\langle i_1, \alpha_1, \dots, i_{m-1}, \alpha_{m-1}, a+1\rangle)$. f' is also of semi-type $\mathcal{U}'$ except for the requirement that $f'(\langle l_1, \delta_1, \dots, \delta_{k-1}, l_k\rangle)$ have uniform cofinality ω for certain $(\delta_1, \dots, \delta_{k-1})$ in measure zero sets with respect to the $M^{r_{k-1}}$. As in lemma 4.3, it now trivial to redefine f' on these measure zero sets to have also this requirement. $\square$

We now extend to this more general setting the notion of description.

DEFINITION 5.5. Fix a type-2 tree of uniform cofinalities $\mathcal{U}$ and a sequence of measures $K_1, \dots, K_t \in W_1 \cup S_1$. A *type-2 generalized description* d is an object satisfying: d is of the form $d = \langle i_1, d_1, i_2, d_2, \dots, i_{m-1}, d_{m-1}, i_m\rangle$, where for some $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle \in \mathrm{dom}(\mathcal{U})$ we have:

(1) Each $d_i \in \mathcal{D}$ and is defined and satisfies condition C relative to $K_1, \dots, K_t$.
(2) $\forall^* h_1, \dots, h_t$ if $\alpha_1 = h(d_1; h_1, \dots, h_t), \dots, \alpha_{m-1} = h(d_{m-1}; h_1, \dots, h_t)$, then $\langle i_1, \alpha_1, \dots, i_{m-1}, \alpha_{m-1}, i_m\rangle$ has type $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle$. In this case we say d has type $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle$.

Note that if d is defined relative to $\mathcal{U}$, $K_1, \dots, K_t$, then d is defined relative to $\mathcal{U}$, $K_1, \dots, K_u$ for any sequence $K_1, \dots, K_u$ extending $K_1, \dots, K_t$.

Given a generalized description $d = \langle i_1, d_1, i_2, d_2, \dots, i_{m-1}, d_{m-1}, i_m\rangle$ defined relative to $\mathcal{U}$, $K_1, \dots, K_t$, and given $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}_3^1$ of type $\mathcal{U}$, we define $h(f; d; K_1, \dots, K_t)$ by integrating with respect to the measures $K_1, \dots, K_t$ as before, where we now define

$$h(f; d; h_1, \dots, h_t) = f(\langle i_1, \alpha_1, \dots, i_{m-1}, \alpha_{m-1}, i_m\rangle),$$

where $\alpha_k = h(d_k; h_1, \dots, h_t)$. Similarly, given also $g\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ we define $h(g; f; d; K_1, \dots, K_t)$, using

$$h(g; f; d; h_1, \dots, h_t) = g(f(\langle i_1, \alpha_1, \dots, i_{m-1}, \alpha_{m-1}, i_m\rangle)).$$

Note that these definitions only depend on $[f]$, the equivalence class of f with respect to the measures $M^{p_{m-1}}$ for $\langle i_1, p_1, \dots, p_{m-1}, i_m\rangle \in \operatorname{dom}(\mathcal{U})$.

If d_1, d_2 are generalized descriptions defined relative to $\mathcal{U}$, $K_1, \dots, K_t$, we set $d_1 < d_2$ provided $\forall^* f\ h(f; d_1; K_1, \dots, K_t) < h(f; d_2; K_1, \dots, K_t)$. Here, and later, $\forall^* f$ refers to the measure $M^{\mathcal{U}}$ on $[f]$ for $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}_3^1$ of type $\mathcal{U}$.

Theorem 4.2 says that if $\mathcal{U}$ is a type-2 tree of uniform cofinalities, $\vec{s} = \langle i_1, p_1, \dots, p_{m-1}, i_m\rangle \in \operatorname{dom}(\mathcal{U})$, and $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}_3^1$ is of type $\mathcal{U}$, then there is a partial extension $\mathcal{B}'$ of p_{m-1} describing the uniform cofinality of $f^{\vec{s}}$ with respect to $M^{p_{m-1}}$. The next lemma says, in a certain sense, that descriptions also suffice to capture the uniform cofinality.

LEMMA 5.3. *Let $\mathcal{U}$ be a type-2 tree of uniform cofinalities, $\langle i_1, p_1, \dots, p_{m-1}, i_m\rangle \in \operatorname{dom}(\mathcal{U})$, and $d = \langle i_1, d_1, \dots, i_{m-1}, d_{m-1}, i_m\rangle$ a generalized type-2 description of type $\langle i_1, p_1, \dots, p_{m-1}, i_m\rangle$, defined relative to $K_1, \dots, K_t$. Let $g\colon \operatorname{dom}(M^{p_{m-1}}) \to \boldsymbol{\delta}_3^1$ have range in the limit ordinals. Then either g has uniform cofinality ω almost everywhere or there is a $d^\star = d^{\star(f_k)}$ defined and satisfying C relative to $K_1, \dots, K_t$ such that $\forall^* h_1, \dots, h_t\ g(h(d_1; \vec{h}), \dots, h(d_{m-1}; \vec{h}))$ has uniform cofinality $h(d^\star; \vec{h})$.*

PROOF. This may be either shown directly, or (more easily) by appealing to theorem 3.2. Let g have uniform cofinality $u\colon \operatorname{dom}(M^{p_{m-1}}) \to \omega_{k+1}$ with respect to $M^{p_{m-1}}$, where k is chosen least. Let $d^\star = d^{\star(f_k)}$ be minimal such that $\forall^* h_1, \dots, h_t$ $u(h(d_1; \vec{h}), \dots, h(d_{m-1}; \vec{h})) \le h(d^\star; \vec{h})$. If inequality holds almost everywhere, then for some $l\colon \omega_1 \to \omega_1$ we have $\forall^* h_1, \dots, h_t\ u(\dots) < h(\mathcal{L}(d^\star); h_1, \dots, h_t; N_l)$. However, $\forall^* h_1, \dots, h_t\ u(\dots)$ is represented by a function $h\colon (\omega_1)^k \to \omega_1$ having range in points closed under l. Thus, $\forall^* h_1, \dots, h_t\ u(\dots) \le h(\mathcal{L}(d^\star); \vec{h})$, a contradiction (if $d^\star$ is minimal with respect to $\mathcal{L}$, then g has uniform cofinality ω almost everywhere). $\square$

We also extend the $\mathcal{L}$ operation to generalized descriptions. Suppose $d = \langle i_1, d_1, \dots, i_{m-1}, d_{m-1}, i_m\rangle$ is defined relative to $\mathcal{U}$, $K_1, \dots, K_t$, and has type $\vec{s} = \langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle$. We define $\mathcal{L}(d)$ through the following cases:

- **(i):** If $i_m = 0$, $\mathcal{L}(d_{m-1})$ is defined, and $\langle i_1, d_1, \dots, i_{m-1}, \mathcal{L}(d_{m-1}), 0\rangle$ has type $\langle i_1, p_1, \dots, i_{m-1}, q, 0\rangle$ for some $\langle i_1, p_1, \dots, i_{m-1}, q, 0\rangle \in \operatorname{dom}(\mathcal{U})$, then $\mathcal{L}(d) = \langle i_1, p_1, \dots, i_{m-1}, \mathcal{L}(d_{m-1}), k\rangle$ where k is maximal such that $\langle i_1, p_1, \dots, i_{m-1}, q, k\rangle \in \operatorname{dom}(\mathcal{U})$.
- **(ii):** If $i_m = 0$ but $\mathcal{L}(d_{m-1})$ is not defined or $\langle i_1, d_1, \dots, i_{m-1}, \mathcal{L}(d_{m-1}), 0\rangle$ does not have type $\langle i_1, p_1, \dots, i_{m-1}, q, 0\rangle$ for some $\langle i_1, p_1, \dots, i_{m-1}, q, 0\rangle \in \operatorname{dom}(\mathcal{U})$, then $\mathcal{L}(d) = \langle i_1, p_1, \dots, i_{m-1} - 1\rangle$. Note that $i_{m-1} \ge 1$, as otherwise $\langle i_1, p_1, \dots, i_{m-1}\rangle$ is maximal in $\operatorname{dom}(\mathcal{U})$. If $m - 1 = 1$, and $i_{m-1} = 1$, we leave $\mathcal{L}(d)$ undefined.

(iii): Suppose $i_m > 0$ and $\mathcal{U}(\vec{s}) = \omega$. Then we set $\mathcal{L}(d) = \langle i_1, d_1, \dots, i_{m-1}, d_{m-1}, i_m - 1\rangle$. If $m = 1$ and $i_m = 1$, we leave $\mathcal{L}(d)$ undefined.

(iv): Suppose $i_m > 0$, and $\mathcal{U}(\vec{s}) = \mathcal{B}' \neq \omega$. By lemma 5.3, Let $d^\star$ represent the uniform cofinality given by $\mathcal{B}'$ with respect to $M^{p_{m-1}}$. Suppose also in this case that $\mathcal{L}(d^\star)$ is defined and $\langle i_1, d_1, \dots, i_{m-1}, d_{m-1}, i_m, \mathcal{L}(d^\star), 0\rangle$ has type $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m, q, 0\rangle$ for some sequence in $\mathrm{dom}(\mathcal{U})$. Then we set $\mathcal{L}(d) = \langle i_1, d_1, \dots, i_{m-1}, d_{m-1}, i_m, \mathcal{L}(d^\star), k\rangle$ where k is maximal so that $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m, q, k\rangle \in \mathrm{dom}(\mathcal{U})$.

(v): As in (iv), except $\mathcal{L}(d^\star)$ is not defined or $\langle i_1, d_1, \dots, i_{m-1}, d_{m-1}, i_m, \mathcal{L}(d^\star), 0\rangle$ does not have type $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m, q, 0\rangle$ for some sequence in $\mathrm{dom}(\mathcal{U})$. We set $\mathcal{L}(d) = \langle i_1, p_1, \dots, i_{m-1}, d_{m-1}, i_m - 1\rangle$. If $m = 1$ and $i_m = 1$, we leave $\mathcal{L}(d)$ undefined.

The following theorem generalizes theorem 3.1 to the current context.

THEOREM 5.1. *Let d be a generalized type-2 description defined relative to the type-2 tree of uniform cofinalities $\mathcal{U}$, and the sequence of measures $K_1, \dots, K_t$.*

1. *Suppose $f\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}_3^1$ is of type $\mathcal{U}$, $\theta \in ON$, and $\forall^* h_1, \dots, h_t$ $\theta(h_1, \dots, h_t) < h(f; d; \vec{h})$. If d is not minimal with respect to $\mathcal{L}$ (i.e., $\mathcal{L}(d)$ is defined), then there is a $f'\colon \mathrm{dom}(<^{\mathcal{U}}) \to \mathrm{ran}(f)$ such that $[f'] = [f]$ and $\forall^* h_1, \dots, h_t$ $\theta(\vec{h}) < h(N_{f'}; f'; \mathcal{L}(d); \vec{h})$. If d is minimal with respect to $\mathcal{L}$, then there is a f' as above such that $\forall^* h_1, \dots, h_t$ $\theta(\vec{h}) < \inf(f')$.*
2. *Suppose $\theta \in ON$ and $\forall^* f\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}_3^1$ $\forall^* h_1, \dots, h_t$ $\theta(f, \vec{h}) < h(f; d; \vec{h})$. If d is not minimal with respect to $\mathcal{L}$, then there exists a $g\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ such that $\forall^* f$ $\forall^* h_1, \dots, h_t$ $\theta(f, \vec{h}) < h(g; f; \mathcal{L}(d); \vec{h})$. If d is minimal with respect to $\mathcal{L}$ we have $\theta < \boldsymbol{\delta}_3^1$.*

PROOF. Consider first (1). Fix f of type $\mathcal{U}$, and θ as in the statement of the lemma. Suppose $d = \langle i_1, d_1, \dots, i_{m-1}, d_{m-1}, i_m\rangle$ is defined relative to $\mathcal{U}$, $K_1, \dots, K_t$, and has type $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle$. We consider the case $i_m > 0$, the case $i_m = 0$ being similar. If $\mathcal{U}(\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle) = \omega$, the result follows easily, so assume $\mathcal{U}(\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle) = \mathcal{B}'$ is a partial extension of p_{m-1}. To be specific, suppose that $\langle q_1, j_1, \dots, j_{n-1}, q_n, l+1\rangle$ is the extra sequence not ending in 0 in $\mathrm{dom}(\mathcal{B}')$, where l is maximal such that $\langle q_1, j_1, \dots, j_{n-1}, q_n, l\rangle \in \mathrm{dom}(p_{m-1})$ (the other cases are entirely similar). Let u be the uniform cofinality with respect to $M^{p_{m-1}}$ described by $\mathcal{B}'$. Thus, since f has type $\mathcal{U}$, $\forall^*_{M^{p_{m-1}}} \alpha_1, \dots, \alpha_{m-1}$ $f(\langle i_1, \alpha_1, \dots, i_{m-1}, \alpha_{m-1}, i_m\rangle)$ has uniform cofinality $u(\alpha_1, \dots, \alpha_{m-1})$. Let $d^\star$ be as in lemma 5.3, thus $\forall^* h_1, \dots, h_t$ $h(d^\star; \vec{h}) = u(h(d_1; \vec{h}), \dots, h(d_{m-1}; \vec{h})$.

Now, $\forall^* h_1, \dots, h_t$ $\alpha_1 = h(d_1; \vec{h})$, …, $\alpha_{m-1} = h(d_{m-1}; \vec{h})$ are represented by $h\colon \mathrm{dom}(<^{p_{m-1}}) \to \omega_1$ of type p_{m-1}. In fact, $M^{p_{m-1}} = \nu_d$, where $\nu_d(A) = 1$ iff $\forall^* h_1, \dots, h_t$ $(\alpha_1, \dots, \alpha_{m-1}) \in A$, with $\alpha_1, \dots, \alpha_{m-1}$ as above. This follows since for any c.u.b. $C \subseteq \omega_1$, $\forall^* h_1, \dots, h_t$ $(\alpha_1, \dots, \alpha_{m-1})$ are represented by an $h\colon \mathrm{dom}(<^{p_{m-1}}) \to C$ of type p_{m-1}. We consider now cases corresponding to the definition of $\mathcal{L}(d)$.

Assume first that $\mathcal{L}(d^\star)$ is defined and $\langle i_1, d_1, \dots, d_{m-1}, i_m, \mathcal{L}(d^\star), 0\rangle$ has type $\langle i_1, p_1, \dots, p_{m-1}, i_m, q, 0\rangle \in \mathrm{dom}(\mathcal{U})$ for some q. From theorem 3.2 we have that there is a c.u.b. $C \subseteq \omega_1$ such that

$$\forall^* h_1, \dots, h_t\ \theta(\vec{h}) < f(\langle i_1, \alpha_1, \dots, \alpha_{m-1}, i_m, \beta', 0\rangle),$$

where $\alpha_i = h(d_i, \vec{h})$ and $\beta' = h(\mathcal{L}(d^\star); \vec{h}; N_C)$. From lemma 5.2, there is a f' of type $\mathcal{U}$ with $[f'] = [f]$ and such that

$$\forall^*_{M^q} \alpha_1, \dots, \alpha_{m-1}, \beta \; N_{f'}(f'(\langle i_1, \alpha_1, \dots, \alpha_{m-1}, i_m, \beta, k\rangle)) > \\ f(\langle i_1, \alpha_1, \dots, \alpha_{m-1}, i_m, \beta', 0\rangle),$$

where k is maximal such that $\langle i_1, p_1, \dots, p_{m-1}, i_m, q, k\rangle \in \mathrm{dom}(\mathcal{U})$, and if $\beta = [h]$, then $\beta' = [N_C \circ h]$. Thus,

$$\forall^* h_1, \dots, h_t \; \theta(\vec{h}) < f(\langle i_1, \alpha_1, \dots, \alpha_{m-1}, i_m, \beta', 0\rangle) \leq \\ N_{f'}(f'(\langle i_1, \alpha_1, \dots, \alpha_{m-1}, i_m, \beta, k\rangle)) = h(N_{f'}; f'; \mathcal{L}(d); \vec{h}).$$

Assume next that $\mathcal{L}(d^\star)$ is defined but $\langle i_1, d_1, \dots, d_{m-1}, i_m, \mathcal{L}(d^\star), 0\rangle$ does not have type $\langle i_1, p_1, \dots, p_{m-1}, i_m, q, 0\rangle \in \mathrm{dom}(\mathcal{U})$ for some q. Theorem 3.2 again produces a c.u.b. $C \subseteq \omega_1$ such that

$$\forall^* h_1, \dots, h_t \; \theta(\vec{h}) < f(\langle i_1, \alpha_1, \dots, \alpha_{m-1}, i_m, \beta', 0\rangle),$$

with β' as above. Now, for almost all $h_1, \dots, h_t$ we must have

$$h(\mathcal{L}(d^\star); \vec{h}) \leq [h^{\langle q_1, j_1, \dots, q_n, l\rangle}],$$

where $h\colon \mathrm{dom}(M^{p_{m-1}}) \to \omega_1$ of type p_{m-1} represents $\alpha_1 = h(d_1; \vec{h}), \dots, \alpha_{m-1} = h(d_{m-1}; \vec{h})$, as otherwise by lemma 4.3 $\langle i_1, d_1, \dots, d_{m-1}, i_m, \mathcal{L}(d^\star), 0\rangle$ has type $\langle i_1, p_1, \dots, i_m, q, 0\rangle$ for some $\langle i_1, p_1, \dots, i_m, q, 0\rangle \in \mathrm{dom}(\mathcal{U})$. By lemma 5.2, it now follows that there is an f' of type $\mathcal{U}$ with $[f'] = [f]$ and such that

$$\forall^* h_1, \dots, h_t \; \theta(\vec{h}) < N_{f'}(f'(\langle i_1, \alpha_1, \dots, \alpha_{m-1}, i_m - 1\rangle)) = h(N_{f'}; f'; \mathcal{L}(d); \vec{h}).$$

Finally, if $\mathcal{L}(d^\star)$ is not defined, then

$$\exists \delta < \omega_1 \forall^* h_1, \dots, h_t \; \theta(\vec{h}) < f(\langle i_1, \alpha_1, \dots, \alpha_{m-1}, i_m, \delta, 0\rangle).$$

A simple case of lemma 5.2 produces the desired f'.

We note lastly that (1) of the theorem easily implies (2) by a partition argument: fix $[f] \to \theta(f)$ representing θ with respect to $M^{\mathcal{U}}$ and partition $f\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ according to whether $\forall^* h_1, \dots, h_t \; \theta(f, \vec{h}) < h(N_f; \mathcal{L}(d); \vec{h})$. Part (1) easily shows that on the homogeneous side of the partition the property holds. Fixing a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ homogeneous for the partition and letting $g = N_C$ then establishes (2).

□

Let $T \subseteq \omega \times \boldsymbol{\delta}^1_3$ be the Martin tree constructed in §2. Thus, for any $f\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ there is a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ and an $x \in \omega^\omega$ such that T_x is well-founded and such that $\forall \alpha \in C \; f(\alpha) < |T_x \restriction \sup_j j(\alpha)|$, where the supremum ranges over embeddings j corresponding to measures in $W_1 \cup S_1$.

Definition 5.6. A *type-2 complex* is a sequence of the form

$$\mathcal{C} = \langle \mathcal{U}, (x_0, \dots, x_{t-1}), (d_0, \dots, d_t), (K_1, \dots, K_t)\rangle$$

where $\mathcal{U}$ is a type-2 tree of uniform cofinalities, T_{x_i} is well-founded for $i = 0, \dots, t-1$, $K_1, \dots, K_t \in W_1 \cup S_1$, and $d_0, \dots, d_t$ are generalized descriptions with d_i defined relative to $\mathcal{U}$, $K_1, \dots, K_i$, and $d_0 > d_1 > \dots > d_t$. Also, $d_0 = \langle i_1\rangle$ where i_1 is maximal such that $\langle i_1\rangle \in \mathrm{dom}(\mathcal{U})$.

For each complex $\mathcal{C}$ as above, and $f\colon \operatorname{dom}(\mathcal{U}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$, we associate an ordinal $h(f;\mathcal{C})$. We set $h(f;\mathcal{C}) = |T_{x_0} \restriction \sup_j j(h(f;d_0;))(\alpha_1)|$ where α_1 is represented with respect to K_1 by the function which assigns to $[h_1]$ the value $\alpha_1(h_1) = |T_{x_1} \restriction \sup_j j(h(f;d_1;h_1))(\alpha_2(h_1))|$, where $\alpha_2(h_1)$ is represented with respect to K_2 by the function which assigns to $[h_2]$ the value $\alpha_2(h_1,h_2), \ldots$, and $\alpha_{t-1}(h_1,\ldots,h_{t-1}) = |T_{x_{t-1}} \restriction \sup_j j(h(f;d_{t-1};h_1,\ldots,h_{t-1}))(\alpha_t(h_1,\ldots,h_{t-1}))|$, where $\alpha_t(h_1,\ldots,h_{t-1})$ is represented with respect to K_t by the function which assigns to $[h_t]$ the value $\alpha_t(h_1,\ldots,h_t) = h(f;d_t;h_1,\ldots,h_t)$.

If $\mathcal{C}$, f are as above, and $\gamma \in ON$, we also define the ordinal $h(f;\mathcal{C};\gamma)$. This is defined exactly as $h(f;\mathcal{C})$ above, except we use $\alpha_t(h_1,\ldots,h_t) = \gamma$. We assume γ is sufficiently small relative to f so that this is well-defined. Note that $h(f;\mathcal{C};\gamma)$ does not depend on d_t.

To each complex $\mathcal{C}$ and measure μ on $\theta < \boldsymbol{\delta}^1_3$, we associate a measure $\mathcal{V}^{\mathcal{C},\mu}$ on $\boldsymbol{\delta}^1_3$ defined as follows. $A \subseteq \boldsymbol{\delta}^1_3$ has measure one if for almost all $\gamma < \theta$ with respect to μ there is a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to C$ of type $\mathcal{U}$, $h(f;\mathcal{C};\gamma) \in A$.

Note that the definition of this measure does not depend on d_t. It is easy to check that this definition is well-defined, and defines a measure on $\boldsymbol{\delta}^1_3$.

We now state the main theorem of this section.

THEOREM 5.2. *Let $\mathcal{V}$ be a measure on $\boldsymbol{\delta}^1_3$. Then there is a complex $\mathcal{C}$ and a measure μ on some ordinal $\theta < \boldsymbol{\delta}^1_3$ such that $\mathcal{V} = \mathcal{V}^{\mathcal{C},\mu}$.*

The remainder of this section is devoted to a proof of theorem 5.2.

We fix a countably additive measure $\mathcal{V}$ on $\boldsymbol{\delta}^1_3$ for the remainder of the proof. We assume that $\mathcal{V}$ does not concentrate on any $\theta < \boldsymbol{\delta}^1_3$. We give the first step of the argument separately to illustrate more simply the ideas involved. One main idea involves a pressing down argument from the measure space $\mathcal{V}$. This idea was also used in [**3**], where a new proof of the strong partition relation on ω_1 was given.

Let $g_0\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ satisfy:

(a) There is a $\mathcal{V}$ measure one set A such that $g_0 \restriction A$ is monotonically increasing.
(b) g_0 restricted to any measure one set is not bounded below $\boldsymbol{\delta}^1_3$.
(c) Any g which is $\mathcal{V}$ almost everywhere less than g_0 does not satisfy both (a) and (b).

We easily see that g_0 exists by considering the identity function. Let $g_0(\mathcal{V})$ be the measure on $\boldsymbol{\delta}^1_3$ defined by $g_0(\mathcal{V})(A) = 1$ iff $\mathcal{V}(g_0^{-1}(A)) = 1$. Then $g_0(\mathcal{V})$ is semi-normal, that is, gives every c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ measure one (otherwise we violate (c)). Since there are only finitely many regular cardinals (i.e., 3) below $\boldsymbol{\delta}^1_3$, $g_0(\mathcal{V})$ is equal to one the normal measures N on $\boldsymbol{\delta}^1_3$. Fix now a measure one set $A \subseteq \boldsymbol{\delta}^1_3$ on which (a) is satisfied. Define $h_0\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ by $h_0(\beta) = \sup\{\alpha \in A\colon g_0(\alpha) \leq \beta\}$. Let $C \subseteq \boldsymbol{\delta}^1_3$ be a c.u.b. set closed under h_0. Thus, for any c.u.b. $D \subseteq C$ almost all α with respect to $\mathcal{V}$ lie in an interval of the form $[\beta, h_0(\beta))$ for some $\beta \in D$, and any two such intervals are disjoint. Let now $x_0 \in \omega^\omega$ be such that T_{x_0} is well-founded, and $|T_{x_0} \restriction \sup_j j(\alpha)| > h_0(\alpha)$ for α in some c.u.b. set. Without loss of generality, we may assume this inequality holds everywhere on C. Define r_0 on the measure one set of α with $\alpha \in A$ and $g_0(\alpha) \in C$ by $r_0(\alpha) =$ the unique $\beta < \sup_j j(g_0(\alpha))$ such that $|T_{x_0} \restriction \sup_j j(g_0(\alpha))(\beta)| = \alpha$ (If N is the cofinality ω normal measure, then $r_0(\alpha) < \alpha$). Assume for the sake of illustration that N is the cofinality ω_2 normal measure.

Since $\mathcal{V}$ is countably additive, we may fix a measure $K_1 \in S_1$ such that $\forall^*\alpha$ $r_0(\alpha) \leq j_{K_1}(g_0(\alpha))$. Let $\mathcal{U}_0$ be the type-2 tree of uniform cofinalities with only $\langle 1 \rangle \in \mathrm{dom}(\mathcal{U}_0)$ (other than sequences ending with 0), where $\mathcal{U}_0(\langle 1 \rangle)$ corresponds to uniform cofinality ω_2. Let d_0 be the generalized description (defined with respect to the empty sequence of measures) $d_0 = \langle 1 \rangle$. Let also $d_1 = d_0$, which is defined with respect to K_1. Note that an $f\colon\ <^{\mathcal{U}_0} \to \boldsymbol{\delta}_3^1$ can be identified with simply an ordinal $\beta = f(\langle 1 \rangle) < \boldsymbol{\delta}_3^1$ of cofinality ω_2. Hence $M^{\mathcal{U}} = N$. Note also that $h(f; d_0;\ ) = f(\langle 1 \rangle)$ and $h(f; d_1; K_1) = j_{K_1}(f(\langle 1 \rangle))$.

Let $\mathcal{C}_0 = \langle \mathcal{U}_0, (x_0), (d_0, d_1), (K_1) \rangle$. Thus we have defined a complex $\mathcal{C}_0$ and functions $g_0, r_0\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ satisfying:

(1) $g_0(\mathcal{V}) = M^{\mathcal{U}_0}$ ($= N$, the cofinality ω_2 normal measure on $\boldsymbol{\delta}_3^1$).
(2) $\forall^*_{\mathcal{V}}\alpha < \boldsymbol{\delta}_3^1\ r_0(\alpha) < h(f_{g_0(\alpha)}; d_1; K_1)$. Here $f_{g_0(\alpha)}\colon\ <^{\mathcal{U}_0} \to \boldsymbol{\delta}_3^1$ represents $g_0(\alpha)$, which here simply means $f_{g_0(\alpha)} = g_0(\alpha)$.
(3) $\forall^*_{\mathcal{V}}\alpha < \boldsymbol{\delta}_3^1\ \alpha = |T_{x_0} \restriction \sup_j j(h(f_{g_0(\alpha)}; d_0;\ ))(r_0(\alpha))| = h(f; \mathcal{C}_0; r_0(\alpha))$.

This completes the first step in the analysis of $\mathcal{V}$. We now proceed to the general case. With (1)-(3) above as motivation, we make the following definition:

DEFINITION 5.7. A *situation* for $\mathcal{V}$ is a triple $\mathcal{S} = (\mathcal{C}, g, r)$ satisfying:

(1) $\mathcal{C} = \langle \mathcal{U}, (x_0, \dots, x_{t-1}), (d_0, \dots, d_t), (K_1, \dots, K_t) \rangle$ is a complex and g, r: $\boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$. Let $<^{\mathcal{U}}$, $M^{\mathcal{U}}$ be the ordering and measure associated to $\mathcal{C}$. Then $g(\mathcal{V}) = M^{\mathcal{U}}$.
(2) $\forall^*_{\mathcal{V}}\alpha < \boldsymbol{\delta}_3^1\ r(\alpha) < h(f_{g(\alpha)}; d_t; K_1, \dots, K_t)$. Here $f_{g(\alpha)}\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}_3^1$ represents $g(\alpha)$ with respect to the measures $M^{p_{m-1}}$ for $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m \rangle \in \mathrm{dom}(\mathcal{U})$.
(3) $\forall^*_{\mathcal{V}}\alpha < \boldsymbol{\delta}_3^1\ \alpha = h(f_{g(\alpha)}; \mathcal{C}; r_0(\alpha))$. This is easily seen to be well-defined.

Among all situations for $\mathcal{V}$, we now select one for which the "bounding function" $\alpha \to h(f_{g(\alpha)}; d_t; K_1, \dots, K_t)$ is minimal. We denote this situation by $\mathcal{S} = (\mathcal{C}, g, r)$. Thus, we have also fixed the type-2 tree of uniform cofinalities $\mathcal{U}$, and associated ordering $<^{\mathcal{U}}$ and measure $M^{\mathcal{U}}$. We will show that for some measure μ on an ordinal $\theta < \boldsymbol{\delta}_3^1$ that $\mathcal{V} = \mathcal{V}^{\mathcal{C},\mu}$.

We first note that it will be sufficient to show that for some $\theta < \boldsymbol{\delta}_3^1$ that $\forall^*_{\mathcal{V}}\alpha < \boldsymbol{\delta}_3^1\ r(\alpha) < \theta$. For if this is the case, let $\mu = r(\mathcal{V})$. We claim then that $\mathcal{V} = \mathcal{V}^{\mathcal{C},\mu}$. If not, let $A \subseteq \boldsymbol{\delta}_3^1$ have $\mathcal{V}$ measure one, but $\forall^*_{\mu}\gamma < \theta\ \exists$ c.u.b. $C \subseteq \boldsymbol{\delta}_3^1$ $\forall f\colon \mathrm{dom}(<^{\mathcal{U}}) \to C$ of type $\mathcal{U}$, $h(f; \mathcal{C}; \gamma) \notin A$. Fix $B \subseteq \gamma$ with $\mu(B) = 1$ such that $\forall \gamma \in B\ \exists$ c.u.b. $C \subseteq \boldsymbol{\delta}_3^1\ \forall f\colon \mathrm{dom}(<^{\mathcal{U}}) \to C$ of type $\mathcal{U}$, $h(f; \mathcal{C}; \gamma) \notin A$. A simple argument (using the coding lemma for prewellorderings of length θ) shows that $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}_3^1\ \forall \gamma \in B\ \forall f\colon \mathrm{dom}(<^{\mathcal{U}}) \to C$ of type $\mathcal{U}$, $h(f; \mathcal{C}; \gamma) \notin A$. Fix such a C. Since $\mathcal{V}(A) = 1$ and $\mu(B) = 1$, $\exists \alpha \in A$ ($f_{g(\alpha)}\colon \mathrm{dom}(<^{\mathcal{U}}) \to C$ and $r(\alpha) \in B$). Since $\alpha = h(f_{g(\alpha)}, \mathcal{C}; r(\alpha))$, we have a contradiction.

The remainder of the argument involves considering cases. In each case we either show that r is bounded almost everywhere or else violate the minimality of the situation. To fix notation, suppose $d_t = \langle i_1, d^t_1, \dots, i_{m-1}, d^t_{m-1}, i_m\rangle$ is of type $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle$.

Case I. $\mathcal{L}(d_t)$ is defined.

We define a partial extension $\mathcal{U}'$ of $\mathcal{U}$ as follows. Let $d^\star$ be as in lemma 5.3 applied to d_t. We refer the reader to the definition of $\mathcal{L}(d_t)$ after lemma 5.3. If

$$i_m = 0 \text{ and } \mathcal{L}(d_t) = \langle i_1, d^t_1, \dots, i_{m-1}, \mathcal{L}(d^t_{m-1}), k\rangle,$$

then $\mathcal{U}'$ has the extra sequence $\langle i_1, p_1, \dots, i_{m-1}, q, k+1\rangle$ in its domain, where $\langle i_1, d^t_1, \dots, i_{m-1}, \mathcal{L}(d^t_{m-1}), k\rangle$ has type $\langle i_1, p_1, \dots, i_{m-1}, q, k\rangle$ and k is maximal such that $\langle i_1, p_1, \dots, i_{m-1}, q, k\rangle \in \operatorname{dom}(\mathcal{U})$. If

$$i_m = 0 \text{ and } \mathcal{L}(d_t) = \langle i_1, p_1, \dots, i_{m-1} - 1\rangle,$$

then $\mathcal{U}'$ has the extra sequence $\langle i_1, p_1, \dots, p_{m-2}, i_{m-1}\rangle$ inserted between

$$\langle i_1, p_1, \dots, p_{m-2}, i_{m-1} - 1\rangle \in \operatorname{dom}(\mathcal{U}) \text{ and } \langle i_1, p_1, \dots, p_{m-2}, i_{m-1}\rangle \in \operatorname{dom}(\mathcal{U}).$$

Suppose now $i_m > 0$. If $\mathcal{U}(\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle)) = \omega$, or $\mathcal{L}(d^\star)$ is not defined, or $\langle i_1, d^t_1, \dots, i_m, \mathcal{L}(d^\star), 0\rangle$ does not have type $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m, q, 0\rangle \in \operatorname{dom}(\mathcal{U})$, then $\mathcal{U}'$ has the extra sequence $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle$ inserted between

$$\langle i_1, \dots, i_{m-1}, p_{m-1}, i_m - 1\rangle \in \operatorname{dom}(\mathcal{U}) \text{ and } \langle i_1, \dots, i_{m-1}, p_{m-1}, i_m\rangle \in \operatorname{dom}(\mathcal{U}).$$

Finally, if

$$\langle i_1, d^t_1, \dots, i_m, \mathcal{L}(d^\star), 0\rangle \text{ has type } \langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m, q, 0\rangle \in \operatorname{dom}(\mathcal{U}),$$

then $\mathcal{U}'$ has the extra sequence $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m, q, k+1\rangle$ in its domain, where k is maximal such that $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m, q, k\rangle \in \operatorname{dom}(\mathcal{U})$.

To unify notation, let $\vec{s} = \langle i'_1, p'_1, \dots, i'_{l-1}, p'_{l-1}, i'_l\rangle$ be the unique sequence not ending in 0 in $\operatorname{dom}(\mathcal{U}') - \operatorname{dom}(\mathcal{U})$. For example, if

$$\mathcal{L}(d_t) = \langle i_1, d^t_1, \dots, i_{m-1}, d^t_{m-1}, i_m, \mathcal{L}(d^\star), k\rangle,$$

then

$$\vec{s} = \langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m, q, k+1\rangle,$$

where $\mathcal{L}(d_t)$ has type $\langle i_1, p_1, \dots, i_{m-1}, p_{m-1}, i_m, q, k\rangle$.

By theorem 5.1, $\forall^*_{\mathcal{V}}\alpha\ \exists f'\colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{U}'$ inducing $f_{g(\alpha)}$ of type $\mathcal{U}$ representing $g(\alpha)$ and such that $r(\alpha) < h(N_{f'}; f'; \mathcal{L}(d_t); K_1, \dots, K_t)$.

Subcase (1): There is a measure one set $A \subseteq \boldsymbol{\delta}^1_3$ with respect to $\mathcal{V}$ such that for $\alpha \in A$, if $f_{g(\alpha)}\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ represents $g(\alpha)$, then there is an $f'\colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{U}'$ inducing $f_{g(\alpha)}$, and such that for all $\alpha' \in A$ with $g(\alpha') = g(\alpha)$ we have that $r(\alpha') < h(N_{f'}; f'; \mathcal{L}(d_t); K_1, \dots, K_t)$.

Fix such a measure one set A. We then consider the partition $\mathcal{P}$: we partition $f'\colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{U}'$ and f' of uniform cofinality ω on sequences in $\operatorname{dom}(<^{\mathcal{U}'}) - \operatorname{dom}(<^{\mathcal{U}})$, with f' inducing $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$, according to whether or not $\exists\alpha \in A$ such that $g(\alpha) = [f]$ and $\forall\alpha' \in A$ with $g(\alpha') = g(\alpha)$, $r(\alpha') < h(N_{f'}; f'; \mathcal{L}(d_t); K_1, \dots, K_t)$. From the hypothesis of subcase (1) it follows that on the homogeneous side of the partition the above property holds. Fix now a homogeneous $C \subseteq \boldsymbol{\delta}^1_3$. We then have $\forall^*_{\mathcal{V}}\alpha < \boldsymbol{\delta}^1_3$ $r(\alpha) < h(N_C; f_{g(\alpha)}; \mathcal{L}(d_t); K_1, \dots, K_t)$.

Let $x_t \in \omega^\omega$ be such that T_{x_t} is well-founded and $|T_{x_t} \restriction \sup_j j(\gamma)| > N_C(\gamma)$ on a c.u.b. set. For a $\mathcal{V}$ measure one set of α, Let $r'(\alpha)$ be the least ordinal such that $\forall^* h_1, \dots, h_t\ r(\alpha)(\vec{h}) = |T_{x_t} \restriction \sup_j j(h(f_{g(\alpha)}; \mathcal{L}(d_t); \vec{h}))(r'(\alpha)(\vec{h}))|$. It follows that for some $K_{t+1} \in W_1 \cup S_1$ that $r'(\alpha) < h(f_{g(\alpha)}; \mathcal{L}(d_t); K_1, \dots, K_t, K_{t+1})$, for almost all α. If we let $\mathcal{S}'$ be the situation $(\mathcal{C}'; g; r')$ where

$$\mathcal{C}' = \langle \mathcal{U}, (x_0, \dots, x_{t-1}, x_t), (d_0, \dots, d_{t-1}, \mathcal{L}(d_t), \mathcal{L}(d_t)), (K_1, \dots, K_t, K_{t+1}) \rangle,$$

it follows that we have produced a situation for the measure $\mathcal{V}$ with $r'(\alpha) < h(f_{g(\alpha)}; \mathcal{L}(d_t); K_1, \dots, K_t, K_{t+1})$. This violates the minimality assumption on our original situation.

Subcase (2): For every measure one set $A \subseteq \boldsymbol{\delta}^1_3$ with respect to $\mathcal{V}$, for almost all $\alpha \in A$, if $f_{g(\alpha)} \colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ represents $g(\alpha)$ then for all $f' \colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{U}'$ inducing f, $\exists \alpha' \in A$ such that $g(\alpha') = g(\alpha)$ and $r(\alpha') \geq h(N_{f'}; f'; \mathcal{L}(d_t); K_1, \dots, K_t)$.

Let $g' \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ represent the least equivalence class with respect to $\mathcal{V}$ which has a representative satisfying:

(a) $\forall^*_{\mathcal{V}} \alpha < \boldsymbol{\delta}^1_3\ g'(\alpha)$ is represented by a function $f' \colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{U}'$ inducing $f \colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ representing $g(\alpha)$.
(b) There is a $\mathcal{V}$ measure one set $A \subseteq \boldsymbol{\delta}^1_3$ such that for α_1, $\alpha_2 \in A$ with $g(\alpha_1) = g(\alpha_2)$, if $r(\alpha_1) \leq r(\alpha_2)$ then $g'(\alpha_1) \leq g'(\alpha_2)$.
(c) There does not exist a $\mathcal{V}$ measure one set $A \subseteq \boldsymbol{\delta}^1_3$ such that for almost all $\alpha \in A$ $\exists f' \colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{U}'$ inducing $f \colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ representing $g(\alpha)$ and such that for all $\alpha' \in A$ with $g(\alpha') = g(\alpha)$, $g'(\alpha) \leq [f']$.

Note that there is a function $\tilde{g}$ satisfying (a)-(c) above. To see this, consider $\tilde{g}(\alpha) =$ the equivalence class of the least $f' \colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{U}'$ such that f' induces $f \colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ representing $g(\alpha)$ and $r(\alpha) < h(N_{f'}; f'; \mathcal{L}(d_t); K_1, \dots, K_t)$. Then $\tilde{g}$ satisfies (a)-(c), so g' is well defined.

Consider the measure $g'(\mathcal{V})$. By countable additivity of $\mathcal{V}$ and theorem 4.2 there is type-2 tree of uniform cofinalities, which we also denote by $\mathcal{U}'$, extending the partial tree $\mathcal{U}'$ (i.e., a uniform cofinality for sequences of type $\vec{s} \in \operatorname{dom}(\mathcal{U}') - \operatorname{dom}(\mathcal{U})$ is now specified) such that $\forall^*_{\mathcal{V}} \alpha < \boldsymbol{\delta}^1_3\ g'(\alpha)$ is represented by an $f' \colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}'$. Also, if $C \subseteq \boldsymbol{\delta}^1_3$ is c.u.b. then $\forall^*_{\mathcal{V}} \alpha < \boldsymbol{\delta}^1_3\ g'(\alpha)$ is represented by $f' \colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ having range in C. For if not, by composing $f'^{\vec{s}}$ with $l(\alpha) \doteq \sup\{\beta < \alpha : \beta \in C\}$ we would get a function g'' also satisfying (a)-(c) above and strictly smaller than g' almost everywhere. This implies that $g'(\mathcal{V})$ is the measure induced from the weak partition relation on $\boldsymbol{\delta}^1_3$ and functions $f' \colon \operatorname{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}'$.

Let $\tilde{d}_t$ be the generalized type-2 description defined as follows. If $\mathcal{L}(d_t) = \langle i_1, d^t_1, \dots, i_{m-1}, d^t_{m-1}, i_m, \mathcal{L}(d^\star), k \rangle$, then $\tilde{d}_t = \langle i_1, d^t_1, \dots, i_{m-1}, d^t_{m-1}, i_m, \mathcal{L}(d^\star), k+1 \rangle$. If $\mathcal{L}(d_t) = \langle i_1, d^t_1, \dots, i_{m-1}, d^t_{m-1}, i_m - 1 \rangle$, then $\tilde{d}_t = \langle i_1, d^t_1, \dots, i_{m-1}, d^t_{m-1}, i_m \rangle$. If $\mathcal{L}(d_t) = \langle i_1, d^t_1, \dots, i_{m-1}, \mathcal{L}(d^t_{m-1}), k \rangle$, then $\tilde{d}_t = \langle i_1, d^t_1, \dots, i_{m-1}, \mathcal{L}(d^t_{m-1}), k+1 \rangle$. Finally, if $\mathcal{L}(d_t) = \langle i_1, d^t_1, \dots, i_{m-1} - 1 \rangle$, then $\mathcal{L}(d_t) = \langle i_1, d^t_1, \dots, i_{m-1} \rangle$. Notice the definition of $\tilde{d}_t$ parallels the definition of $\mathcal{U}'$ in all cases.

Consider now the following partition $\mathcal{P}$: we partition $f' \colon \mathrm{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}'$ with the extra value $\tilde{f}(\langle i'_1, \alpha_1, \dots, i'_{l-1}, \alpha_{l-1}, i'_l\rangle)$ (for $\langle i'_1, \alpha_1, \dots, i'_{l-1}, \alpha_{l-1}, i'_l\rangle$ of type $\vec{s}$) inserted between $f'(\langle i'_1, \alpha_1, \dots, i'_{l-1}, \alpha_{l-1}, i'_l\rangle)$ and $N_{f'}(f'(\langle i'_1, \alpha_1, \dots, i'_{l-1}, \alpha_{l-1}, i'_l\rangle))$, with $\tilde{f}(\langle i'_1, \alpha_1, \dots, i'_{l-1}, \alpha_{l-1}, i_l\rangle)$ of uniform cofinality ω, according to whether

$$h(N_{\tilde{f}}; f'; \tilde{d}_t; K_1, \dots, K_t) \geq \sup\{r(\alpha) \colon \alpha \in A \wedge g'(\alpha) \leq [f']\},$$

where A is a $\mathcal{V}$ measure one set on which (b) holds for g'. It follows readily from properties (b) and (c) for g' that on the homogeneous side of $\mathcal{P}$ the stated property holds. Fixing a c.u.b. C homogeneous for the partition, we then have $\forall^*_{\mathcal{V}} \alpha < \boldsymbol{\delta}^1_3$ if $f'_{g'(\alpha)} \colon \mathrm{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}'$ represents $g'(\alpha)$, then $r(\alpha) < h(N_C; f'_{g'(\alpha)}; \tilde{d}_t; K_1, \dots, K_t)$. Let $x_t \in \omega^\omega$ be such that T_{x_t} is well-founded and $|T_{x_t} \restriction \sup_j j(\alpha)| > N_C(\alpha)$ on a c.u.b. set. Let r' be such that

$$\forall^*_{\mathcal{V}} \alpha \; r'(\alpha) < h(\sup_j j; f'_{g'(\alpha)}; \tilde{d}_t; K_1, \dots, K_t)$$

and

$$\forall^*_{\mathcal{V}} \alpha \; \forall^* h_1, \dots, h_t \; r(\alpha)(\vec{h}) = |T_{x_t} \restriction h(\sup_j j; f'_{g'(\alpha)}; \tilde{d}_t; \vec{h})(r'(\alpha)(\vec{h}))|.$$

By countable additivity, let $K_{t+1} \in W_1 \cup S_1$ be such that

$$\forall^*_{\mathcal{V}} \alpha \; r'(\alpha) < h(f'_{g'(\alpha)}; \tilde{d}_t; K_1, \dots, K_t, K_{t+1}).$$

Let $\mathcal{C}' = \langle \mathcal{U}'; (x_0, \dots, x_{t-1}, x_t); (d_0, \dots, d_{t-1}, \tilde{d}_t, \tilde{d}_t); (K_1, \dots, K_t, K_{t+1})\rangle$. Then $\mathcal{S}' = (\mathcal{C}'; g'; r')$ is a situation for $\mathcal{V}$ with a smaller bounding function $r'(\alpha) < h(f'_{g'(\alpha)}; \tilde{d}_t; K_1, \dots, K_t, K_{t+1})$. This violates the minimality of $\mathcal{S}$.

<u>Case (II)</u>: $\mathcal{L}(d_t)$ is not defined.

Note that d_t is either of the form $\langle 1, d^t_1, 0\rangle$ or $\langle 1\rangle$.

From theorem 5.1 we have $\forall^*_{\mathcal{V}} \alpha < \boldsymbol{\delta}^1_3 \; \exists f \colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ $\exists \gamma < f(\langle 1\rangle)$ $r(\alpha) < \gamma$. As in case (I), we consider cases.

First suppose there is a measure one set A with respect to $\mathcal{V}$ such that for $\alpha \in A$ and $f_{g(\alpha)} \colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ representing $g(\alpha)$, $\exists \gamma < f(\langle 1\rangle) \sup\{r(\alpha') \colon \alpha' \in A \wedge g(\alpha') = g(\alpha)\} < \gamma$. An easy partition argument then shows $\exists \gamma < \boldsymbol{\delta}^1_3 \; \forall^*_{\mathcal{V}} \alpha \; r(\alpha) < \gamma$.

Otherwise, for all A of $\mathcal{V}$ measure one, for almost all $\alpha \in A$, if $f_{g(\alpha)}$ of type $\mathcal{U}$ represents $g(\alpha)$ and $\gamma < f_{g(\alpha)}(\langle 1\rangle)$, then there is an $\alpha' \in A$ with $g(\alpha') = g(\alpha)$ and $r(\alpha') > \gamma$. A pressing down argument as in case (I), but easier, then gives a situation $(\mathcal{C}', g', r')$ for $\mathcal{V}$ which violates the minimality of $\mathcal{S}$. Here $\mathcal{U}'$ will have the extra sequence $\langle 1\rangle$ inserted below $\langle 1\rangle \in \mathrm{dom}(\mathcal{U})$ (that is, $\langle i_1\rangle \in \mathrm{dom}(\mathcal{U})$ is identified with $\langle i_1 + 1\rangle \in \mathrm{dom}(\mathcal{U}')$). Also, for almost all α and $f'_{g'(\alpha)}$ of type $\mathcal{U}'$ representing $g'(\alpha)$, we will have $\forall^* h_1, \dots, h_t, h_{t+1} \; r'(\alpha)(h_1, \dots, h_{t+1}) < f'_{g'(\alpha)}(\langle 1\rangle)$ for some $K_{t+1} \in W_1 \cup S_1$.

This completes the proof of theorem 5.2. Λ

CHAPTER 6

The Strong Partition Relation on $\boldsymbol{\delta}^1_3$

We use the analysis of measures (theorem 5.2) in the previous chapter to get a coding for the subsets of $\boldsymbol{\delta}^1_3$ sufficiently nice to establish the strong partition relation on $\boldsymbol{\delta}^1_3$. The argument for converting an analysis of the measures on κ to a coding of the subsets of κ is due to Kunen (see [**11**]). Throughout this chapter, $T \subseteq (\omega \times \boldsymbol{\delta}^1_3)^{<\omega}$ denotes the Martin tree of §2.

Recall the main result of the last section, theorem 5.2, showed that any measure $\mathcal{V}$ on $\boldsymbol{\delta}^1_3$ is of the form $\mathcal{V} = \mathcal{V}^{\mathcal{C},\mu}$ for some complex

$$\mathcal{C} = \langle \mathcal{U}, (x_0, \dots, x_{t-1}), (d_0, \dots, d_t), (K_1, \dots, K_t) \rangle,$$

and measure μ on some $\theta < \boldsymbol{\delta}^1_3$. Recall that $A \subseteq \boldsymbol{\delta}^1_3$ has measure one with respect to $\mathcal{V}^{\mathcal{C},\mu}$ if $\forall^*_\mu \gamma < \theta\ \exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to C$ of type $\mathcal{U}$, $h(f;\mathcal{C};\gamma) \in A$ (where $h(f;\mathcal{C};\gamma)$ is defined just before lemma 5.2). We observed in the proof of theorem 5.2 that this is equivalent to: $\exists D \subseteq \theta$ of μ measure one $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3\ \forall \gamma \in D\ \forall f\colon \operatorname{dom}(<^{\mathcal{U}}) \to C$ of type $\mathcal{U}$ $h(f;\mathcal{C};\gamma) \in A$. Also, it is clear that we may take as a base for the measure sets A such that $\exists D \subseteq \theta$ of μ measure one $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3\ \forall \alpha\ [\alpha \in A \Leftrightarrow \exists f\colon \operatorname{dom}(<^{\mathcal{U}}) \to C \cap \alpha$ of type $\mathcal{U}$ $h(f;\mathcal{C};\gamma) \in A]$. Note also that the computation of $h(f;\mathcal{C};\gamma)$ only requires that $T_{x_i} \restriction \eta(\nu)$ be well-founded for certain values of η, ν which are less than $\sup_j j(\sup f)$.

We first code (sufficiently many) c.u.b. subsets of $\boldsymbol{\delta}^1_3$. Every real $\sigma \in \omega^\omega$ can be viewed, in some standard manner, as a strategy for player II in a two person integer game. We let $\sigma(x)$ be the result of following this strategy against I's play of x. Let $K \subseteq (\omega \times \boldsymbol{\delta}^1_3)^{<\omega}$ be the Kunen tree constructed from the $\boldsymbol{\Pi}^1_3$ complete set P and $\boldsymbol{\Pi}^1_3$ scale $\{\phi_n\}$ on P (with ϕ_0 onto $\boldsymbol{\delta}^1_3$). Recall K may be constructed as follows. Let $U \subseteq (\omega \times \boldsymbol{\delta}^1_3)^{<\omega}$ be the tree of the scale $\{\phi_n\}$, and $V \subseteq (\omega \times \aleph_\omega)^{<\omega}$ a tree with $p[V]$ a complete $\boldsymbol{\Sigma}^1_3$ set. Define $\tilde{K} \subseteq (\omega \times \omega \times \boldsymbol{\delta}^1_3 \times \omega \times \aleph_\omega)^{<\omega}$ by:

$$(\sigma, x, \vec{\alpha}, y, \vec{\beta}) \in [\tilde{K}] \Leftrightarrow \forall n\ (x_n, \vec{\alpha}_n) \in [U] \wedge \sigma(x) = y \wedge (y, \vec{\beta}) \in [V].$$

Here $x_n(m) = x(\langle n, m\rangle)$, and $\vec{\alpha}_n(m) = \vec{\alpha}(\langle n, m\rangle)$, where $(n,m) \to \langle n, m\rangle$ is a bijection between ω^2 and ω. Weaving the last four coordinates of $\tilde{K}$ into a single coordinate produces the tree K. Say an ordinal $\alpha < \boldsymbol{\delta}^1_3$ is *good* if $\forall \beta < \alpha\ \exists x \in P$ $(\phi_0(x) = \beta \wedge \forall n\ \phi_n(x) < \alpha)$. Let $GO \subseteq \boldsymbol{\delta}^1_3$ denote the (c.u.b.) set of good ordinals, and GO' the set of limit points of GO. K then has the usual property that for all $f\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$, there is a σ with K_σ well-founded and such that for all good α of cofinality ω, $|K_\sigma \restriction \alpha| > f(\alpha)$.

DEFINITION 6.1. A real σ is *good* if K_σ is well-founded. Also σ is *good* below β if $\forall \gamma < \beta\ K_\sigma \restriction \gamma$ is well-founded. To each real σ we associate a closed set $C_\sigma \subseteq \boldsymbol{\delta}^1_3$ given by: $C_\sigma = \{\alpha \in GO' \colon \forall \beta < \alpha\ K_\sigma \restriction \beta$ is well-founded of rank $< \alpha\}$. We say a c.u.b. $C \subseteq GO'$ is *good* if there is a good real σ such that $C = C_\sigma$.

A key fact is the following:

LEMMA 6.1. *Every c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ contains good c.u.b. set.*

PROOF. Let $f(\alpha) =$ the least $\beta \in C$ greater than α. Let σ be a good real such that $|K_\sigma \restriction \alpha| > f(\alpha)$ for all good α of cofinality ω. Then $C_\sigma \subseteq C$. $\square$

DEFINITION 6.2. A set $A \subseteq \boldsymbol{\delta}^1_3$ is *simple* if there is a complex $\mathcal{C}$, a good c.u.b. $C = C_\sigma$, and a $D \subseteq \boldsymbol{\delta}^1_3$ with $\sup D < \boldsymbol{\delta}^1_3$ such that

$$A = \{\alpha \colon \exists \delta \in D \ \exists f \colon \operatorname{dom}(<^{\mathcal{U}}) \to C \cap \alpha \text{ of type } \mathcal{U} \ (\alpha = h(f; \mathcal{C}; \delta))\}.$$

Here $\mathcal{U}$ and the function $(\delta, f) \to h(f; \mathcal{C}; \delta)$ are as determined from $\mathcal{C}$.

Note that a simple set is described by a bounded set D, a complex $\mathcal{C}$ (which may be coded by a real), and a good real σ. We let $S_{D,\mathcal{C},\sigma}$ denote the corresponding simple set.

The Kunen argument now yields:

THEOREM 6.1. *Every $A \subseteq \boldsymbol{\delta}^1_3$ is a countable union of simple sets.*

PROOF. Following Kunen's argument, assume the theorem fails and fix a counterexample $A \subseteq \boldsymbol{\delta}^1_3$. Let $\mathcal{I} \subseteq \mathcal{P}(\boldsymbol{\delta}^1_3)$ be the countably additive ideal consisting of those $I \subseteq \boldsymbol{\delta}^1_3$ such that $I \cap A$ is contained in a countable union of simple sets each of which is contained in A, and $I \cap A^c$ is contained in a countable union of simple sets each of which is contained in A^c. By hypothesis, $\mathcal{I}$ is a proper ideal. By AD, the filter dual to $\mathcal{I}$ can be extended to a (countably additive) measure $\mathcal{V}$ [$\mathcal{V}$ is induced by the Martin measure on the degrees $\mathcal{D}$ and the map $f \colon \mathcal{D} \to \boldsymbol{\delta}^1_3$ given by $f(d) =$ least ordinal not in $\bigcup\{B_x \colon x \in d\}$, where the real $x \in \omega^\omega$ codes the element B_x of $\mathcal{I}$ in some manner]. Without loss of generality, suppose $\mathcal{V}(A) = 1$. From theorem 5.2, there is a measure μ on some $\theta < \boldsymbol{\delta}^1_3$, and a complex $\mathcal{C}$ such that $\mathcal{V} = \mathcal{V}^{\mathcal{C},\mu}$. Thus, there is a $D \subseteq \theta$, and a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that $\{\alpha < \boldsymbol{\delta}^1_3 \colon \exists \delta \in D \ \exists f \colon \operatorname{dom}(<^{\mathcal{U}}) \to C \cap \alpha \text{ of type } \mathcal{U} \ [\alpha = h(f; \mathcal{C}; \delta)]\} \subseteq A$. By lemma 6.1, let $C_\sigma \subseteq C$ be a good set. Thus, $S_{D,\mathcal{C},\sigma} \subseteq A$, and $\mathcal{V}(S_{D,\mathcal{C},\sigma}) = 1$. This contradicts $S_{D,\mathcal{C},\sigma} \in \mathcal{I}$. $\square$

Fix a $\boldsymbol{\Delta}^1_3$ coding $x \to R_x \subseteq (\aleph_\omega)^3$ for the subsets of $(\aleph_\omega)^3$, say by the coding of [**11**]. This gives a coding for the bounded subsets of $\boldsymbol{\delta}^1_3$ as well. To be specific, $x \in \omega^\omega$ codes the bounded set D_x as follows: $\delta \in D_x$ iff $\exists \alpha < \aleph_\omega$ ($(R_x)_\alpha$ is well-founded of rank δ). Here $R_\alpha = \{(\beta, \gamma) \colon (\alpha, \beta, \gamma) \in R\}$. Thus, every real x codes a bounded $D_x \subseteq \boldsymbol{\delta}^1_3$.

To each real z we now associate a set $A_z \subseteq \boldsymbol{\delta}^1_3$ as follows. First, decode z into an infinite sequence of triples $(x_n, y_n, \sigma_n) \in (\omega^\omega)^3$, in some standard manner. Let $D_n = D_{x_n}$ be the bounded subset of $\boldsymbol{\delta}^1_3$ coded by x_n as above. Let $C_n = C_{\sigma_n}$ be the closed subset of $\boldsymbol{\delta}^1_3$ coded by σ_n. If y_n does not have the correct "syntactical form" to code a complex for some n, set $S_n = \emptyset$. Otherwise, decode y_n into $\mathcal{C}_n = \langle \mathcal{U}, (w_0, \dots, w_{t-1}), (d_0, \dots, d_t), (K_1, \dots, K_t)\rangle$ which satisfies the requirements for being a complex except that we do not require that the T_{w_i} be well-founded. To D_n, C_n and $\mathcal{C}_n$ we associate the set $S_n = \{\alpha < \boldsymbol{\delta}^1_3 \colon \exists \delta \in D_n \ \exists f \colon \operatorname{dom}(<^{\mathcal{U}}) \to C_n \cap \alpha$ of type $\mathcal{U}$ [$\forall i \ T_{w_i} \restriction \eta(\nu)$ is well-founded for all η, ν such that $T_{w_i} \restriction \eta(\nu)$ is used in the definition of $h(f; \mathcal{C}; \delta) \wedge \alpha = h(f; \mathcal{C}; \delta)$]$\}$. More precisely, $\alpha \in S_n$ iff $\exists \delta \in D_n$ $\exists f \colon \operatorname{dom}(<^{\mathcal{U}}) \to C_n \cap \alpha$ of type $\mathcal{U}$ such that $h(f; \mathcal{C}; \delta)$ is well-defined and equal to α.

By this we mean α_1 is well-defined, in the well-founded part of $T_{x_0} \restriction \sup_j j(h(f;d_0;\ ))$ $= T_{x_0} \restriction \sup_j j(\sup f)$, and $\alpha = |T_{x_0} \restriction \sup_j j(h(f;d_0;\ ))(\alpha_1)|$. To say α_1 is well-defined, we in turn mean $\forall^*_{K_1} h_1\ \alpha_2(h_1)$ is well-defined, in the well-founded part of $T_{x_1} \restriction \sup_j j(h(f;d_1;h_1))$, and $\alpha_1(h_1) = |T_{x_1} \restriction \sup_j j(h(f;d_1;h_1))(\alpha_2(h_1))|$, etc. (c.f. the definition of $h(f;\mathcal{C};\gamma)$ before lemma 5.2). Note that if $\forall i\ T_{w_i} \restriction \sup_j j(\sup f))$ is well-founded then $h(f;\mathcal{C};\delta)$ is well defined. We then set $A_z = \bigcup_n S_n$. Thus every real z codes a set $A_z \subseteq \boldsymbol{\delta}^1_3$.

We next prove a result which shows that our coding is reasonable.

THEOREM 6.2. *The coding $z \to A_z \subseteq \boldsymbol{\delta}^1_3$ satisfies the following properties.*

(1) *For every $A \subseteq \boldsymbol{\delta}^1_3$ $\exists z \in \omega^\omega$ $(A = A_z)$. Thus, every set is coded.*

(2) *For any $\alpha < \boldsymbol{\delta}^1_3$, $\{z\colon \alpha \in A_z\} \in \boldsymbol{\Delta}^1_3$.*

PROOF. Part (1) follows immediately from the previous theorem. Fix $\alpha < \boldsymbol{\delta}^1_3$, and let $B_\alpha = \{z : \alpha \in A_z\}$. So, $z \in B_\alpha \Leftrightarrow \alpha \in A_z \Leftrightarrow \exists n\ \alpha \in S_{D_n(z),\mathcal{C}_n(z),\sigma_n(z)}$. Fixing n, it is enough to show that $E_\alpha = \{z\colon \alpha \in S_{D(z),\mathcal{C}(z),\sigma(z)}\} \in \boldsymbol{\Delta}^1_3$. Now, $z \in E_\alpha \Leftrightarrow \exists x$ [(x codes a function $f_x\colon \operatorname{dom}(<^{\mathcal{U}}) \to \alpha$ of type $\mathcal{U}$) $\wedge$ (f_x has range in $C_{\sigma(z)}$) $\wedge$ $(\exists \delta \in D(z)\ (\alpha = h(f;\mathcal{C}(z);\delta))]$. Note that $h(f;\mathcal{C}(z);\delta)$ depends only on $[f] = (\dots, \eta^{\langle i_1,p_1,\dots,i_{m-1},p_{m-1},i_m\rangle}, \dots)$, the ordinals represented by f with respect to the measures $M^{p_{m-1}}$. Thus we may write $h(\vec{\eta};\mathcal{C};\delta)$. Also, if $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ has range almost everywhere in some C, then there is an f' with range everywhere in C and with $[f'] = [f]$. Thus we may write: $z \in E_\alpha \Leftrightarrow \exists \vec{\eta} < \sup_j j(\alpha)$ [($\vec{\eta}$ is representable by a function $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ having range in $C_{\sigma(z)}$ almost everywhere)$\wedge(\exists \delta \in D(z)\ (\alpha = h(\vec{\eta};\mathcal{C}(z);\delta))]$. Since $\boldsymbol{\Delta}^1_3$ is closed under $< \boldsymbol{\delta}^1_3$ unions and intersections, it is enough to fix $\vec{\eta}$, and hence fix $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ representing $\vec{\eta}$, and show the expression inside the square brackets defines a $\boldsymbol{\Delta}^1_3$ set. We isolate a useful lemma.

LEMMA 6.2. *Let μ be a measure on $\aleph_\omega$, and suppose that $\boldsymbol{\Delta}^1_3$ sets $R_\beta(x) \subseteq \omega^\omega$ are given for $\beta < \aleph_\omega$. Then $S(x)$ defined by $S(x) \Leftrightarrow \forall^*_\mu \beta\ R_\beta(x)$ is $\boldsymbol{\Delta}^1_3$.*

PROOF. For notational simplicity, we consider the case where $\mu = S^1_1 =$ the ω cofinal normal measure on ω_2. The general case is entirely similar, since a general μ is equivalent to a measure $M^{\mathcal{B}}$ for some type ≤ 1 tree of uniform cofinalities, and thus is described by the strong partition relation on ω_1, c.u.b. sets $C \subseteq \omega_1$, and functions $f\colon \operatorname{dom}(\prec) \to C$ for some ordering $\prec$, and this is in effect all we will use about the measure S^1_1 (we will, in fact, only need this lemma for measures of the form $M^{\mathcal{B}}$).

We first prove the lemma when $\mu = W^1_1 =$ the normal measure on ω_1. Let $K_1 \subseteq (\omega \times \omega_1)^{<\omega}$ be a "Kunen tree" for ω_1 (i.e., for any $f\colon \omega_1 \to \omega_1$ $\exists \sigma$ (T_σ is well-founded $\wedge$ $|T_\sigma \restriction \alpha| > f(\alpha)$ on a c.u.b. set)). Any $y \in \omega^\omega$ with $(K_1)_y$ well-founded codes a c.u.b. set $C_y \subseteq \omega_1$, namely, $C_y = \{\alpha < \omega_1 : \forall \beta < \alpha\ |(K_1)_y \restriction \beta| < \alpha\}$. Thus, $S(x) \Leftrightarrow \forall^* \beta < \omega_1\ R_\beta(x) \Leftrightarrow \exists y$ [$(K_1)_y$ is well-founded $\wedge\ \forall \beta < \omega_1\ (\beta \in C_y \to R_\beta(x))$] $\Leftrightarrow \forall y$ [(T_y is well-founded) $\to \exists \beta < \omega_1\ (\beta \in C_y\ \wedge\ R_\beta(x))$]. This easily shows that $S(x) \in \boldsymbol{\Delta}^1_3$.

Suppose now that $\mu = S^1_1$. If $z = \langle \gamma, w\rangle$ where $\gamma < \omega_1$, $w \in \omega^\omega$, and $(K_1)_w$ is well-founded, let $f_z : \omega_1 \to \omega_1$ be given by $f_z(\alpha) = |(K_1)_z \restriction \alpha(\gamma)|$. Thus, for any

$f\colon \omega_1 \to \omega_1$ there is such a z with $f = f_z$ on a c.u.b. set. Now, $S(x) \Leftrightarrow \forall^*_\mu \beta < \omega_2$ $R_\beta(x) \Leftrightarrow \exists y\ [(K_1)_y$ is well-founded $\wedge\ \forall \omega_1 < \beta < \omega_2$ of cofinality ω ((any (some) $f\colon \omega_1 \to \omega_1$ representing β has range almost everywhere in $C_y) \to R_\beta(x))] \Leftrightarrow \forall y$ $[((K_1)_y$ is well-founded$) \to \exists \omega_1 < \beta < \omega_2$ of cofinality ω ((any (some) $f\colon \omega_1 \to \omega_1$ representing β has range almost everywhere in $C_y) \wedge R_\beta(x))]$. It is enough to fix $\omega_1 < \beta < \omega_2$, and fix $z = \langle \gamma, w\rangle$ such that f_z represents β, and show that $U(x,y) \Leftrightarrow$ (f_z has range almost everywhere in C_y) is $\boldsymbol{\Delta}^1_3$. However, $U(x,y) \Leftrightarrow$ $\forall^* \alpha < \omega_1\ (|(K_1)_z \restriction \alpha(\gamma)| \in C_y)$. This is easily $\boldsymbol{\Delta}^1_3$ from the first case. □

From the lemma it follows that $\{z\colon f$ has range almost everywhere in $C_{\sigma(z)}\}$ is $\boldsymbol{\Delta}^1_3$ for fixed f. Thus for fixed $\delta < \alpha < \boldsymbol{\delta}^1_3$ and fixed $f\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$, we must show that $F(z) \Leftrightarrow [\alpha = h(f;\mathcal{C}(z);\delta)]$ is $\boldsymbol{\Delta}^1_3$. We have:

$$F(z) \Leftrightarrow \exists \alpha_0, \alpha_1 < \alpha[(\alpha_0 = \sup f) \wedge |T_{y_0} \restriction \sup_j j(\alpha_0)(\alpha_1)| = \alpha \wedge (\alpha_1 = h_1(f;\mathcal{C}(z);\delta)].$$

Here $\mathcal{C}(z) = \langle \mathcal{U}, (y_0, \ldots, y_{t-1}), (d_0, \ldots, d_t), (K_1, \ldots, K_t)\rangle$ is the complex coded by z, and we define inductively: $h_1(f;\mathcal{C}(z);\delta)$ is represented with respect to K_1 by

$$[h_1] \to h_1(f;\mathcal{C}(z);\delta;h_1) = |T_{y_1} \restriction \sup_j j(h(f;d_1;h_1))(h_2(f;\mathcal{C}(z);\delta;h_1))|,$$

and $h_2(f;\mathcal{C}(z);\delta;h_1)$ is represented with respect to K_2 by

$$[h_2] \to h_2(f;\mathcal{C}(z);\delta;h_1,h_2) = |T_{y_2} \restriction \sup_j j(h(f;d_1;h_1,h_2))(h_3(f;\mathcal{C}(z);\delta;h_1,h_2))|,$$

etc., and where $h_t(f;\mathcal{C}(z);\delta;h_1,\ldots,h_t) = \delta$. Fix α_0, α_1, and a function $[h_1] \to \alpha_1([h_1])$ representing α_1 with respect to K_1. The first two conjuncts in the above expression for $F(z)$ easily define $\boldsymbol{\Delta}^1_3$ sets. Thus we consider

$$G(z) \Leftrightarrow (\alpha_1 = h_1(f;\mathcal{C}(z);\delta)) \Leftrightarrow \forall^*[h_1]\ (\alpha_1([h_1]) = h_1(f;\mathcal{C}(z);\delta;h_1).$$

Note that $h_1(f;\mathcal{C}(z);\delta;h_1)$ depends only on $[h_1]$. Thus, by lemma 6.2 we may fix $[h_1]$, and hence h_1 representing $[h_1]$, and show that $H(z)$ is $\boldsymbol{\Delta}^1_3$ where

$$\begin{aligned} H(z) \Leftrightarrow (\alpha_1([h_1]) = h_1(f;\mathcal{C}(z);\delta;h_1)) \Leftrightarrow \exists \beta_0,\beta_1 < \alpha[\beta_0 = h(f;d_1;h_1) \wedge \\ |T_{y_1} \restriction \sup_j j(\beta_0)(\beta_1)| = \alpha_1([h_1]) \wedge \beta_1 = h_2(f;\mathcal{C}(z);\delta;h_1)]. \end{aligned}$$

Continuing, we show that $F(z) \in \boldsymbol{\Delta}^1_3$. This completes the proof that $E_\alpha \in \boldsymbol{\Delta}^1_3$. □

To establish the strong partition relation on $\boldsymbol{\delta}^1_3$ we must code functions $f\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ in a sufficiently nice manner. Of course, any function may be viewed as a subset of $\boldsymbol{\delta}^1_3$, and we may then use our coding of subsets of $\boldsymbol{\delta}^1_3$. This, however, does not seem quite sufficient to get the desired coding of functions. Instead, we must make a minor modification to our coding of subsets.

DEFINITION 6.3. For each real $z \in \omega^\omega$ we define a relation $F_z \subseteq \boldsymbol{\delta}^1_3 \times \boldsymbol{\delta}^1_3$ as follows: First, decode z into a sequence of triples $(x_n, y_n, \sigma_n) \in (\omega^\omega)^3$. Let $D_n = D_{x_n}$ be the bounded subset of $\boldsymbol{\delta}^1_3 \times \boldsymbol{\delta}^1_3$ coded by x_n as before. Let $C_n = C_{\sigma_n}$ be the closed subset of $\boldsymbol{\delta}^1_3$ coded by σ_n. If y_n does not have the correct "syntactical form" to code a complex for some n, set $S_n = \emptyset$. Otherwise, decode y_n into $\mathcal{C}_n = \langle \mathcal{U}, (w_0, \ldots, w_{t-1}),(d_0, \ldots, d_t), (K_1, \ldots, K_t)\rangle$ which satisfies the requirements for being a complex except that we do not require that the T_{w_i} be well-founded. To D_n, C_n, and $\mathcal{C}_n$ we associate the set $S_n = \{(\alpha,\beta) \in \boldsymbol{\delta}^1_3 \times \boldsymbol{\delta}^1_3\colon \exists \delta_0, \delta_1 < \alpha$ $(\delta_0,\delta_1) \in D_n \wedge \exists f\colon \mathrm{dom}(<^{\mathcal{U}}) \to C_n \cap \alpha$ of type $\mathcal{U}$ $(\forall i\ T_{w_i} \restriction \eta(\nu)$ is well-founded for all η, ν such that $T_{w_i} \restriction \eta(\nu)$ is used in the definitions of $h(f;\mathcal{C}_n;\delta_0)$, $h(f;\mathcal{C}_n;\delta_1)$ and $(\alpha,\beta) = (h(f;\mathcal{C}_n;\delta_0),\, h(f;\mathcal{C}_n;\delta_1)))\}$. We then set $F_z = \bigcup_n S_n$.

The essential difference between the coding here and the previous coding of subsets (extended to a coding of pairs) is that we now require $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to C_n \cap \alpha$ rather than $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to C_n \cap \langle\alpha,\beta\rangle$, where $\langle\alpha,\beta\rangle =$ the ordinal coding (α,β).

We introduce two families of sets related to our coding. For $\alpha < \boldsymbol{\delta}_3^1$ let

$$\begin{aligned} V(\alpha) = \{z : \forall\gamma \le \alpha \exists \nu < \boldsymbol{\delta}_3^1\ [F_z(\gamma,\nu) \wedge \forall(\delta_0,\delta_1) < \alpha \forall f \forall n[(\delta_0,\delta_1) \in D_n(z) \\ \wedge (f\colon \operatorname{dom}(<^{\mathcal{U}_n(z)}) \to \alpha \text{ is of type } \mathcal{U}_n(z) \text{ with range in } C_{\sigma_n(z)}) \\ \wedge (\alpha = h(f;\mathcal{C}_n(z);\delta_0)] \Rightarrow (\nu = h(f;\mathcal{C}_n(z));\delta_1)]\}. \end{aligned}$$

For $\alpha,\beta < \boldsymbol{\delta}_3^1$, let

$$\begin{aligned} W(\alpha,\beta) = \{z : \forall\gamma \le \alpha \exists \nu \le \beta\ [F_z(\gamma,\nu) \wedge \forall(\delta_0,\delta_1) < \alpha \forall f \forall n[(\delta_0,\delta_1) \in D_n(z) \\ \wedge (f\colon \operatorname{dom}(<^{\mathcal{U}_n(z)}) \to \alpha \text{ is of type } \mathcal{U}_n(z) \text{ with range in } C_{\sigma_n(z)}) \\ \wedge (\alpha = h(f;\mathcal{C}_n(z);\delta_0)] \Rightarrow (\nu = h(f;\mathcal{C}_n(z);\delta_1))]. \end{aligned}$$

Note that $W(\alpha,\beta) \subseteq V(\alpha)$ for all α,β. Also, $z \in V(\alpha)$ requires that $F_z(\gamma)$ be defined for all $\gamma \le \alpha$, and the value $F_z(\gamma)$ be unique in a strong sense. Similarly for $W(\alpha,\beta)$.

We have:

$$\begin{aligned} W(\alpha,\beta)(z) \Leftrightarrow & \forall\gamma \le \alpha \exists\nu \le \beta[F_z(\gamma,\nu) \wedge \forall n \in \omega \forall(\delta_0,\delta_1) \in D_n \cap (\alpha \times \alpha) \\ & \forall \vec{\eta} < \sup_j j(\alpha) \text{ representable by a } f\colon \operatorname{dom}(<^{\mathcal{U}_n}) \to \boldsymbol{\delta}_3^1 \text{ of type } \mathcal{U}_n \\ & \text{having range almost everywhere in } C_{\sigma_n(z)} \\ & \{(h(\vec{\eta};\mathcal{C}_n(z);\delta_0) = \gamma) \Rightarrow (h(\vec{\eta};\mathcal{C}_n(z);\delta_1) = \nu)\}]. \end{aligned}$$

This shows that $W(\alpha,\beta) \in \boldsymbol{\Delta}_3^1$.

We have now our desired coding of functions $F : \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$.

THEOREM 6.3. *The coding $z \to F_z \subseteq \boldsymbol{\delta}_3^1 \times \boldsymbol{\delta}_3^1$ satisfies:*

(1) *For every function $F\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1\ \exists z(F = F_z)$, i.e., every function is coded.*

(2) *For all $\alpha,\beta < \boldsymbol{\delta}_3^1\ W(\alpha,\beta) \in \boldsymbol{\Delta}_3^1$.*

(3) *If $R \subseteq V(\alpha)$ is $\boldsymbol{\Sigma}_3^1$ for some $\alpha < \boldsymbol{\delta}_3^1$, then $\exists\beta < \boldsymbol{\delta}_3^1\ \forall z \in R\ F_z(\alpha) < \beta$.*

PROOF. (2) has already been established. The proof of (1) is a variation of the argument for coding subsets. To be specific, fix a function $F \subseteq \boldsymbol{\delta}_3^1 \times \boldsymbol{\delta}_3^1$. We call a subfunction $G \subseteq F$ *simple* if $G = S_{(x,y,\sigma)}$ for some reals x, y, σ, with S defined as in definition 6.3, with all the trees T_{w_i} well-founded. It is enough to show that F is the countable union of simple sets. If not, let $\mathcal{I} \subseteq \mathcal{P}(\boldsymbol{\delta}_3^1)$ be the ideal consisting of those $I \subseteq \boldsymbol{\delta}_3^1$ such that $F \restriction I$ is contained in a countable union of simple subfunctions of F. Then, $\mathcal{I}$ is a proper ideal, and we may extend the dual filter to a measure $\mathcal{V}$. We may identify $\mathcal{V}$ with a measure $\mathcal{V}'$ on $X = \{(\alpha,\beta)\colon \beta = F(\alpha)\}$. As in the first step of the proof of theorem 5.2, we let $g_0\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ be a representative for the least $\mathcal{V}$ equivalence class which has a representative satisfying:

(a) There is a measure one set $A \subseteq \boldsymbol{\delta}_3^1$ such that $g_0 \restriction A$ is monotonically increasing.

(b) There is no measure one set on which g_0 is bounded.

Clearly $g_0(\alpha) \leq \alpha$ almost everywhere. As before, $g_0(\mathcal{V})$ is a normal measure N on $\boldsymbol{\delta}^1_3$. Fix a measure one set $A \subseteq \boldsymbol{\delta}^1_3$ on which (a) holds, and $g_0(\alpha) \leq \alpha$ for $\alpha \in A$. Define

$$h_0(\gamma) = \sup\{\max(\alpha, \beta) \colon \alpha \in A \wedge g_0(\alpha) = \gamma \wedge F(\alpha) = \beta\}.$$

So, $h_0(\gamma) < \boldsymbol{\delta}^1_3$ for all $\gamma < \boldsymbol{\delta}^1_3$. Extend g_0 to g'_0 defined on X by $g'_0(\alpha, \beta) = g_0(\alpha)$. Thus $g'_0(\mathcal{V}') = N$ as well. Let x_0 be such that T_{x_0} is well-founded (recall T is the "Martin" tree of §2) and $|T_{x_0} \restriction \sup_j j(\gamma)| > h_0(\gamma)$ on a c.u.b. set. On a measure one set, let $r_0(\alpha, \beta)$, $s_0(\alpha, \beta) < \sup_j j(g_0(\alpha))$ be the ordinals such that

$$|T_{x_0} \restriction \sup_j j(g'_0(\alpha, \beta))(r_0(\alpha, \beta))| = \alpha, \text{ and } |T_{x_0} \restriction \sup_j j(g'_0(\alpha, \beta))(s_0(\alpha, \beta))| = \beta.$$

From here, the argument is essentially identical to that of theorem 5.2. Now a situation $\mathcal{S} = (\mathcal{C}, g', r, s)$ will involve two functions r, s, and part (2) of definition 5.7 becomes

$$(2') \colon \forall^*_{\mathcal{V}'}(\alpha, \beta) < \boldsymbol{\delta}^1_3 \ \max(r(\alpha, \beta), s(\alpha, \beta)) < h(f_{g'(\alpha,\beta)}; d_t; K_1, \ldots, K_t).$$

Part (3) of 5.7 becomes

$$(3') \colon \forall^*_{\mathcal{V}'}(\alpha, \beta) < \boldsymbol{\delta}^1_3 \ \alpha = h(f_{g'(\alpha,\beta)}; \mathcal{C}; r(\alpha, \beta)), \beta = h(f_{g'(\alpha,\beta)}; \mathcal{C}; s(\alpha, \beta)).$$

Continuing as before leads to a contradiction.

Finally, we prove (3). Suppose $R \subseteq V(\alpha)$ is $\boldsymbol{\Sigma}^1_3$ and $\forall \beta < \boldsymbol{\delta}^1_3 \ \exists z \in R \ F_z(\alpha) > \beta$. We will produce a $\boldsymbol{\Sigma}^1_3$ well-founded relation of length $\boldsymbol{\delta}^1_3$, a contradiction. Define:

$$\begin{aligned}
z \prec w \Leftrightarrow\ & (z, w \in R) \wedge (F_z(\alpha) < F_w(\alpha)) \\
\Leftrightarrow\ & (z, w \in R) \wedge \exists n, m, x, y \exists (\delta_0, \delta_1) \in \alpha \times \alpha \exists (\epsilon_0, \epsilon_1) \in \alpha \times \alpha [(x \text{ codes a} \\
& \quad \text{function } f_x \colon \ <^{\mathcal{U}_n(z)}\!\!\to \alpha \text{ of type } \mathcal{U}_n(z)) \\
& \wedge (y \text{ codes a function } f_y \colon \ <^{\mathcal{U}_m(w)}\!\!\to \alpha \text{ of type } \mathcal{U}_m(w)) \\
& \wedge (f_x \text{ has range in } C_{\sigma_n(z)} \text{ almost everywhere}) \\
& \wedge (f_y \text{ has range in } C_{\sigma_m(w)} \text{ almost everywhere}) \\
& \wedge (\delta_0, \delta_1) \in D_n(z) \wedge (\epsilon_0, \epsilon_1) \in D_m(w) \wedge h(f_x; \mathcal{C}_n(z); \delta_0) = \alpha \\
& \wedge h(f_y; \mathcal{C}_m(w); \epsilon_0) = \alpha \wedge h(f_x; \mathcal{C}_n(z); \delta_1) < h(f_y; \mathcal{C}_m(w); \epsilon_1)].
\end{aligned}$$

Now, all of the conjuncts inside the square brackets except the last are easily seen to define $\boldsymbol{\Delta}^1_3$ sets. For the last conjunct (for fixed $\delta_1, \epsilon_1, x, y$) we have

$$\begin{aligned}
& h(f_x; \mathcal{C}_n(z); \delta_1) < h(f_y; \mathcal{C}_m(w); \epsilon_1) \Leftrightarrow \exists \zeta_1, \rho_1 < \sup_j j(\alpha) \ [\zeta_1 = h_1(f_x; \mathcal{C}_n(z); \delta_1) \wedge \\
& \rho_1 = h_1(f_y; \mathcal{C}_m(w); \epsilon_1) \wedge |T_{a_0} \restriction \sup_j j(\sup f_x)(\zeta_1)| < |T_{b_0} \restriction \sup_j j(\sup f_y)(\rho_1)|].
\end{aligned}$$

Here $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ are the reals appearing in $\mathcal{C}_n(z)$ and $\mathcal{C}_m(w)$, and $h_1(\cdots)$ is defined in the proof of theorem 6.2. Again, all the conjuncts except the last in this expression are easily $\boldsymbol{\Delta}^1_3$. Also, for any fixed $\theta_1, \ldots, \theta_4 < \boldsymbol{\delta}^1_3$, there is a

$\boldsymbol{\Sigma}^1_3$ relation $M(a,b)$ such that

$$M(a,b) \cap \{(a,b) \colon T_a \restriction (\theta_1)(\theta_2), T_b \restriction (\theta_3)(\theta_4) \text{ are well-founded } \}$$
$$= \{(a,b) \colon |T_a \restriction (\theta_1)(\theta_2)| < |T_b \restriction (\theta_3)(\theta_4)|\}.$$

This shows that $\prec$ is $\boldsymbol{\Sigma}^1_3$, a contradiction. This completes the proof of theorem 6.3. $\square$

From this point, the argument for getting the strong partition relation on $\boldsymbol{\delta}^1_3$, due to Martin (see [**5**]), is fairly standard. For the sake of completeness, we give a brief sketch.

THEOREM 6.4. $\boldsymbol{\delta}^1_3 \to (\boldsymbol{\delta}^1_3)^{\boldsymbol{\delta}^1_3}$.

PROOF. Given a partition $\mathcal{P} \colon (\boldsymbol{\delta}^1_3)^{\boldsymbol{\delta}^1_3} \to \{0,1\}$, consider the game G: I and II play reals $z, w \in \omega^\omega$ respectively. If there is a least $\alpha < \boldsymbol{\delta}^1_3$ such that $z \notin V(\alpha)$ or $w \notin V(\alpha)$, then two wins provided $z \notin V(\alpha)$. Suppose that $\forall \alpha < \boldsymbol{\delta}^1_3\ z, w \in V(\alpha)$. Hence z, w code functions F_z, $F_w : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ via our coding. In this case, let $h \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ be defined by $h(\alpha) = \sup_{\beta < \omega\cdot(\alpha+1)} \max\{F_z(\beta), F_w(\beta)\}$. II wins then if $\mathcal{P}(h) = 1$. Suppose without loss of generality that II wins, and fix a winning strategy σ for II. For $\alpha, \beta < \boldsymbol{\delta}^1_3$, let $S(\alpha,\beta) = \sigma'' W(\alpha,\beta)$. Thus, $S(\alpha,\beta) \in \boldsymbol{\Sigma}^1_3$. Note that $S(\alpha,\beta) \subseteq V(\alpha)$. From property (3) of the coding, $h(\alpha,\beta) \doteq \sup\{F_w(\alpha) \colon w \in S(\alpha,\beta)\} < \boldsymbol{\delta}^1_3$. Let $D \subseteq \boldsymbol{\delta}^1_3$ be c.u.b. and closed under h, and let C be the set of closure points of D. We claim that C is homogeneous for $\mathcal{P}$. For let $f \colon \boldsymbol{\delta}^1_3 \to C$ be of uniform cofinality ω. Let $f' \colon \boldsymbol{\delta}^1_3 \to D$ induce f, i.e., $f(\alpha) = \sup_{\beta < \omega\cdot(\alpha+1)} f'(\beta)$. Let z code f', that is $F_z = f'$, and let I play z against II's strategy σ to produce $w = \sigma(z)$. Since $z \in V(\alpha)$ for all $\alpha < \boldsymbol{\delta}^1_3$, $w \in V(\alpha)$ for all α as well. Thus F_w is a function from $\boldsymbol{\delta}^1_3$ to $\boldsymbol{\delta}^1_3$. From the definition of D we have $F_w(\omega\cdot\beta + n) < F_z(\omega\cdot\beta + n + 1)$. Hence,

$$h(\alpha) = \sup_{\beta < \omega\cdot(\alpha+1)} \max\{F_z(\beta), F_w(\beta)\} = \sup_{\beta < \omega\cdot(\alpha+1)} F_z(\beta) = f(\alpha).$$

Thus, $\mathcal{P}(f) = 1$, and we are done. $\square$

CHAPTER 7

The Lower Bound For $\boldsymbol{\delta}^1_5$

In this section we use the strong partition relation on $\boldsymbol{\delta}^1_3$ to establish the lower bound for $\boldsymbol{\delta}^1_5$. Our main tool is the following theorem of Martin:

THEOREM 7.1 (Martin). *Assume $\kappa \to (\kappa)^\kappa$. Then for any measure $\mathcal{V}$ on κ, the ultrapower $j_\mathcal{V}(\kappa)$ is a cardinal.*

PROOF. Fix $\mathcal{V}$ such that $j_\mathcal{V}(\kappa)$ is not a cardinal, and let $F : j_\mathcal{V}(\kappa) \to \lambda$ be a bijection, with $\lambda < j_\mathcal{V}(\kappa)$. Consider the partition $\mathcal{P}$: we partition $f, g \colon \kappa \to \kappa$ of uniform cofinality ω with $f(\alpha) < g(\alpha) < f(\alpha+1)$ according to whether $F([f]_\mathcal{V}) < F([g]_\mathcal{V})$. It is clear by well-foundedness that we can not have a c.u.b. set homogeneous for the contrary side of the partition. Let $C \subseteq \kappa$ be c.u.b. and homogeneous for $\mathcal{P}$. Fix an ordinal θ with $\lambda < \theta < j_\mathcal{V}(\kappa)$, and let $h \colon \kappa \to \kappa$ represent θ. Define $h' \colon \kappa \to C$ inductively by: $h'(\alpha) =$ the $\omega \cdot (h(\alpha)+1)^{st}$ element in C greater than $\sup\limits_{\beta<\alpha} h'(\beta)$. Since κ is regular, this is well defined. We then produce an order preserving map H from θ into λ, a contradiction. Namely, if $\delta < \theta$, let $f_\delta \colon \kappa \to \kappa$ represent δ with respect to $\mathcal{V}$. Define $f'_\delta \colon \kappa \to C$ by: $f'_\delta(\alpha) =$ the $\omega \cdot (f_\delta(\alpha)+1)^{st}$ element of C greater than $\sup\limits_{\beta<\alpha} h'(\beta)$ if $f_\delta(\alpha) < h(\alpha)$, and $f'_\delta(\alpha) =$ the ω^{th} element of C greater than $\sup\limits_{\beta<\alpha} h'(\beta)$ otherwise. Then set $H(\delta) = F([f'_\delta]_\mathcal{V})$. It is easy to see that this is well-defined and order-preserving from θ into λ. $\square$

The remainder of this section is devoted to a proof of the next theorem.

THEOREM 7.2. $\boldsymbol{\delta}^1_5 \geq (\aleph_{\omega^{\omega^\omega}})^+$.

Since $(\boldsymbol{\delta}^1_5)^- =$ the supremum of the ultrapowers of $\boldsymbol{\delta}^1_3$ by the measures on $\boldsymbol{\delta}^1_3$, we must show that there are at least ω^{ω^ω} distinct ultrapowers. Recall that if $<^\mathcal{V}$ is a well-ordering, and $f \colon \mathrm{dom}(<^\mathcal{V}) \to ON$, then we say f if of the correct type if f is order-preserving, of uniform cofinality ω, and non-continuous, that is, $f(\theta) > \sup\{f(\nu) \colon \nu <^\mathcal{V} \theta\}$ for all $\theta \in \mathrm{dom}(<^\mathcal{V})$ of limit rank. We say $f \colon \mathrm{dom}(<^\mathcal{V}) \to ON$ is of *continuous type* if f is order-preserving, continuous, and of uniform cofinality ω at points of non-limit rank. We will consider measures $\mathcal{V}$ which are induced from the weak partition relation on $\boldsymbol{\delta}^1_3$, functions $f \colon \mathrm{dom}(<^\mathcal{V}) \to \boldsymbol{\delta}^1_3$ of continuous type for some ordering $<^\mathcal{V}$ of order type $< \boldsymbol{\delta}^1_3$, and finitely many measures $v_i(\mathcal{V})$ on $\mathrm{dom}(<^\mathcal{V})$. We call a measure of this form an *order measure*. Note that a function $f \colon \mathrm{dom}(<^\mathcal{V}) \to \boldsymbol{\delta}^1_3$ of the correct type induces a $\tilde{f}$ of continuous type by $\tilde{f}(\theta) = \sup\{f(\nu) \colon \nu <^\mathcal{V} \theta\}$ for θ of limit rank. Using functions of continuous type allows us to state the basic lemmas of this section in terms of the normal measures on $\boldsymbol{\delta}^1_3$.

DEFINITION 7.1. Let $\mathcal{U}$, $\mathcal{V}$ be order measures on $\boldsymbol{\delta}^1_3$ corresponding to orderings $<^{\mathcal{U}}$, $<^{\mathcal{V}}$ and measures $v_i(\mathcal{U})$, $v_j(\mathcal{V})$. Then:

(1) $\mathcal{U} \oplus \mathcal{V}$ is the order measure defined as follows: let $<^{\mathcal{U}} \oplus <^{\mathcal{V}}$ be the lexicographic ordering on pairs (a, α), where $a = 0$ and $\alpha \in \mathrm{dom}(<^{\mathcal{U}})$ or $a = 1$ and $\alpha \in \mathrm{dom}(<^{\mathcal{V}})$. Then $\mathcal{U} \oplus \mathcal{V}$ corresponds to ordering $<^{\mathcal{U}} \oplus <^{\mathcal{V}}$ and measures $\{v_i(\mathcal{U})\} \cup \{v_j(\mathcal{V})\}$.

(2) $\mathcal{U} \otimes \mathcal{V}$ is the order measure defined as follows: let $<^{\mathcal{U}} \otimes <^{\mathcal{V}}$ be the lexicographic ordering on pairs (β, α) where $\alpha \in \mathrm{dom}(<^{\mathcal{U}})$, $\beta \in \mathrm{dom}(<^{\mathcal{V}})$. Then $\mathcal{U} \otimes \mathcal{V}$ corresponds to ordering $<^{\mathcal{U}} \otimes <^{\mathcal{V}}$ and measures $\{v_j(\mathcal{V}) \times v_i(\mathcal{U})\}$. Here $u \times v$ denotes the product measure.

Note that both these operations are associative.
For $\mathcal{V}$ a measure on $\boldsymbol{\delta}^1_3$, let $|\mathcal{V}| =$ the ordinal δ $(< \omega^{\omega^\omega})$ such that $j_{\mathcal{V}}(\boldsymbol{\delta}^1_3) = \aleph_{\omega+\delta}$.

DEFINITION 7.2. We say the order measure $\mathcal{V}$ on $\boldsymbol{\delta}^1_3$ has *true ultrapower* $\geq \theta$ if for any order measure $\mathcal{U}$ on $\boldsymbol{\delta}^1_3$, $|\mathcal{U} \oplus \mathcal{V}| \geq |\mathcal{U}| + \theta$. Let $|\mathcal{V}|_t =$ the largest θ such that $\mathcal{V}$ has true ultrapower $\geq \theta$.

Consider first $\mu_\omega =$ the ω-cofinal normal measure on $\boldsymbol{\delta}^1_3$. This may be represented as an order measure with $<^{\mathcal{V}}=$ order type of $\omega + 1$, and $v(\mathcal{V}) =$ the principal measure on $\{\omega\}$. It is clear from theorem 7.1 that μ_ω has true ultrapower ≥ 1 (since one can embed the $\mathcal{U}$ ultrapower into a proper initial segment of the $\mathcal{U} \oplus \mu_\omega$ ultrapower). Thus $\bigoplus_{i=1}^{n} \mu_\omega$ has true ultrapower $\geq n$. Since $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$, it follows that $(\boldsymbol{\delta}^1_5)^- \geq \aleph_{\omega \cdot 2}$.

Consider now $\mu_{\omega_1} =$ the ω_1-cofinal normal measure on $\boldsymbol{\delta}^1_3$. This may be represented as the order measure with ordering $<^{\omega_1}=$ order type of ω_1, with $v = W^1_1 =$ the normal measure on ω_1. The following lemma is analogous to those used in the embedding theorems (c.f. statement (B) of theorem 1.2).

LEMMA 7.1. *Fix $n < \omega$. Then there is a measure μ on ω_ω and a function H defined on* $\mathrm{dom}(\mu)$ *satisfying:*

(1) $\forall^*_\mu \alpha\ H(\alpha)\colon n \to \omega_1$ *is order-preserving.*

(2) *For any $A \subseteq \omega_1$ of W^1_1 measure one, $\forall^*_\mu \alpha\ H(\alpha)$ has range in A.*

The proof is trivial, letting $\mu = W^n_1 =$ the n-fold product of the normal measure on ω_1. Λ

LEMMA 7.2. *For any order measure $\mathcal{V}$ on $\boldsymbol{\delta}^1_3$, $|\mathcal{V} \otimes \mu_{\omega_1}|_t \geq |\mathcal{V}|_t \cdot \omega$.*

PROOF. Fix $n < \omega$. It is enough to show that for any order measure $\mathcal{U}$ that $|\mathcal{U} \oplus (\mathcal{V} \otimes \mu_{\omega_1})| \geq |\mathcal{U} \oplus (\bigoplus_{i=1}^{n} \mathcal{V})|$. We show that $|\mathcal{V} \otimes \mu_{\omega_1}| \geq |\bigoplus_{i=1}^{n} \mathcal{V}|$, the general case being essentially identical. We define an embedding π from the ultrapower of $\boldsymbol{\delta}^1_3$ by $\bigoplus_{i=1}^{n} \mathcal{V}$ into the ultrapower of $\boldsymbol{\delta}^1_3$ by $\mathcal{V} \otimes \mu_{\omega_1}$. Fix $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ representing an equivalence class with respect to $\bigoplus_{i=1}^{n} \mathcal{V}$. We may view F as defined on functions $f\colon \bigoplus_{i=1}^{n} <^{\mathcal{V}} \to \boldsymbol{\delta}^1_3$, hence f is defined on pairs (m, γ) where $1 \leq m \leq n$ and $\gamma \in \mathrm{dom}(<^{\mathcal{V}})$. We define a function G which will represent $\pi([F])$ with respect to $\mathcal{V} \otimes \mu_{\omega_1}$. Fix a $g\colon <^{\mathcal{V}} \otimes <^{\omega_1} \to \boldsymbol{\delta}^1_3$ of continuous type, and we define $G([g])$. We represent $G([g])$ with respect to $\mu = W^n_1$ by the function which assigns to $(\alpha_1, \ldots, \alpha_n) \in (\omega_1)^n$ the value $G([g])(\alpha_1, \ldots, \alpha_n) = F(f)$ where $f(m, \gamma) = g(H(\alpha_1, \ldots, \alpha_n)(m), \gamma) = g(\alpha_m, \gamma)$.

From (2) of lemma 7.1 it follows that G depends only on $[g]$. It also follows easily that $\pi(F)$ depends only on $[F]$, and that π is an embedding. $\square$

Applying the lemma immediately gives that $|\mathcal{V} \bigotimes_{i=1}^{n} \mu_{\omega_1}|_t \geq |\mathcal{V}|_t \cdot \omega^n$.

Consider next μ_{ω_2} = the ω_2 cofinal normal measure on $\boldsymbol{\delta}^1_3$. This may be represented as the order measure with ordering $<^{\omega_2}$= order type of ω_2, and measure $v = S^1_1$ = the ω-cofinal normal measure on ω_2.

Lemma 7.3. *Fix $n < \omega$. Then there is a measure μ on ω_ω and a function H defined on* $\mathrm{dom}(\mu)$ *satisfying:*

(1) *$\forall^*_\mu \alpha$ $H(\alpha)$ is an ordinal representable with respect to W^n_1 by an order-preserving function $h\colon \bigotimes_{i=1}^{n} <^{\omega_1} \to <^{\omega_2}$.*
(2) *For every $A \subseteq \omega_2$ of v measure one, $\forall^*_\mu \alpha$ $H(\alpha)$ is representable by h having range in A.*

Proof. Let $\mu = S^{n+1}_1$, a measure on ω_{n+2}. Recall that this measure is induced by the strong partition relation on ω_1, functions $f\colon \mathrm{dom}(<_{n+1}) \to \omega_1$ of the correct type, and W^{n+1}_1. Here $<_{n+1}$ is the order on $n+1$-tuples $(\alpha_1, \dots, \alpha_{n+1}) <_{n+1} (\beta_1, \dots, \beta_{n+1})$ iff $(\alpha_{n+1}, \alpha_1, \dots, \alpha_n) <^{lex} (\beta_{n+1}, \beta_1, \dots, \beta_n)$. Let $f\colon \mathrm{dom}(<_{n+1}) \to \omega_1$ of the correct type represent $\alpha < \omega_{n+2}$. Let $\alpha_1 < \dots < \alpha_n < \omega_1$. Then $H(\alpha)$ is represented by the function whose value at $(\alpha_1, \dots, \alpha_n)$ is $H(\alpha)(\alpha_1, \dots, \alpha_n) = [g]_{W^1_1}$, where $g\colon \omega_1 \to \omega_1$ is defined by: $g(\beta) = f(\alpha_1, \dots, \alpha_n, \beta)$. It is easy to check that this is well-defined, and that (1) and (2) above are satisfied. $\square$

Lemma 7.4. *For any $n < \omega$ and any order measure $\mathcal{V}$ on $\boldsymbol{\delta}^1_3$, $|\mathcal{V} \otimes \mu_{\omega_2}|_t \geq |\mathcal{V} \otimes \bigotimes_{i=1}^{n} \mu_{\omega_1}|_t$.*

Proof. A straightforward modification of lemma 7.2 using lemma 7.3. $\square$

Applying lemma 7.4 gives that $|\mathcal{V} \bigotimes_{i=1}^{n} \mu_{\omega_2}|_t \geq |\mathcal{V}|_t \cdot \omega^{\omega \cdot n}$.

Finally, we consider the order measures $\mathcal{V}_m$ on $\boldsymbol{\delta}^1_3$ corresponding to the ordering $<^{\mathcal{V}_m}$= order type of ω_{m+1}, and measure $v = S^m_1$ on ω_{m+1}.

Lemma 7.5. *Fix $n, m < \omega$. There is a measure μ on ω_ω and a function H defined on* $\mathrm{dom}(\mu)$ *satisfying:*

(1) *$\forall^*_\mu \alpha$ $H(\alpha)$ is representable with respect to the n-fold product of S^m_1 by an order-preserving function $h\colon \bigotimes_{i=1}^{n} <^{\mathcal{V}_m} \to <^{\mathcal{V}_{m+1}}$ of the correct type.*
(2) *For any $A \subseteq \omega_{m+2}$ of S^{m+1}_1 measure one, $\forall^*_\mu \alpha$ $H(\alpha)$ is represented by h having range almost everywhere in A.*

Proof. Note that the n-fold product of S^m_1 is the measure induced by the strong partition relation on ω_1, functions $f\colon \mathrm{dom}(<_{n,m}) \to \omega_1$ of the correct type, and W^m_1, where $<_{n,m}$ is the ordering on tuples $(k, \alpha_1, \dots, \alpha_m)$ for $1 \leq k \leq n$ and $\alpha_1 < \dots < \alpha_m < \omega_1$ defined by: $(i, \alpha_1, \dots, \alpha_m) <_{n,m} (j, \beta_1, \dots, \beta_m)$ iff $(\alpha_m, i, \alpha_1, \dots, \alpha_{m-1}) <^{lex} (\beta_m, j, \beta_1, \dots, \beta_{m-1})$. Let $\mu = S^{n+m+1}_1$. Fix $f\colon \mathrm{dom}(<^{n+m+1}) \to \omega_1$ of the correct type representing $\alpha < \omega_{n+m+2}$ and we define $H(\alpha)$. Fix $g\colon \mathrm{dom}(<_{n,m}) \to \omega_1$ of the correct type representing $(\beta_1, \dots, \beta_n) \in (\omega_{m+1})^n$. We then set $H(\alpha)(\beta_1, \dots, \beta_m) = [h]_{W^{m+1}_1}$, where h is defined by:

$h(\gamma_1, \ldots, \gamma_{m+1}) = f(\gamma_1, \ldots, \gamma_m, g(1, \gamma_1, \ldots, \gamma_m), \ldots, g(n, \gamma_1, \ldots, \gamma_m), \gamma_{m+1})$. It is again easy to see that this is well-defined and that (1) and (2) are satisfied. $\square$

As before, a straightforward modification of the proof of lemma 7.2 using lemma 7.5 now shows:

LEMMA 7.6. *For any $n, m < \omega$ and any order measure $\mathcal{V}$ on $\boldsymbol{\delta}^1_3$, $|\mathcal{V} \otimes \mathcal{V}_{m+1}|_t \geq |\mathcal{V} \otimes \bigotimes_{i=1}^{n} \mathcal{V}_m|_t$.*

Noting that $\mu_{\omega_2} = \mathcal{V}_1$, and applying the above lemma for $m = 1, 2, \ldots$ successively then shows that $|\mathcal{V} \otimes \bigotimes_{i=1}^{n} \mathcal{V}_m|_t \geq |\mathcal{V}|_t \cdot \omega^{\omega^m \cdot n}$. From this it follows that $|\mathcal{V}_m|_t \geq \omega^{\omega^m}$. Theorem 7.2 now follows.

CHAPTER 8

Analysis of Measures on $(\boldsymbol{\delta}^1_5)^-$

Our goal in this section is to analyze the measures on $(\boldsymbol{\delta}^1_5)^-$. The arguments are similar in outline to the analysis of measures on $\boldsymbol{\delta}^1_3$ of §5. Our first main task is to extend the notions of a type-2 tree of uniform cofinalities and a type-2 complex to the third level. This will require first analyzing the possible uniform cofinalities for a measure $M^{\mathcal{C}}$, where $\mathcal{C}$ is a type ≤ 2 tree of uniform cofinalities (defined below). Again, the methods are similar to those at the previous level, but descriptions will now play a more central role.

We first extend the notion of a type-2 tree of uniform cofinalities slightly.

DEFINITION 8.1. A *type ≤ 2 tree of uniform cofinalities* is a triple of the form $\mathcal{C} = \langle m;\mathcal{R};\mathcal{U}\rangle$ where $m < \omega$, $\mathcal{R}$ is a type-1 tree of uniform cofinalities, and $\mathcal{U}$ is a type-2 tree of uniform cofinalities.

To each type ≤ 2 tree of uniform cofinalities $\mathcal{C}$ we associate the measure $M^{\mathcal{C}} = W_1^m \times M^{\mathcal{R}} \times M^{\mathcal{U}}$, where we recall $M^{\mathcal{R}}$ is the measure on $(\aleph_\omega)^{<\omega}$ corresponding to $\mathcal{R}$, and $M^{\mathcal{U}}$ is the measure on $(\boldsymbol{\delta}^1_3)^{<\omega}$ corresponding to $\mathcal{U}$ (c.f. definitions 4.5, 5.4).

Thus, $M^{\mathcal{C}}$ is a measure on tuples

$$\vec{\theta} = (\alpha_1, \dots, \alpha_m, \dots, \beta^{\vec{r}}, \gamma^{\vec{s}}, \dots)_{\vec{r}\in \mathrm{dom}(\mathcal{R}), \vec{s}\in \mathrm{dom}(\mathcal{U})}.$$

We say a tuple $\vec{\theta}$ as above is of type $\mathcal{C}$ if $\alpha_1 < \dots < \alpha_m$ and there are functions $g\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$, $f\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of types $\mathcal{R}$, $\mathcal{U}$ respectively such that g, f represent the ordinals $\vec{\beta}$ and $\vec{\gamma}$ respectively. Thus, the set of $\vec{\theta}$ of type $\mathcal{C}$ has measure one with respect to $M^{\mathcal{C}}$. To conserve notation, we will frequently express $\vec{\theta}$ as $\vec{\theta} = (\vec{\alpha}, \beta^{\vec{r}}, \gamma^{\vec{s}})$.

We say the type ≤ 2 tree $\mathcal{C}_2 = \langle m_2;\mathcal{R}_2;\mathcal{U}_2\rangle$ *extends* $\mathcal{C} = \langle m_1;\mathcal{R}_1;\mathcal{U}_1\rangle$ if $m_2 \geq m_1$, $\mathcal{R}_2$ extends $\mathcal{R}_1$, and $\mathcal{U}_2$ extends $\mathcal{U}_1$ (we allow $\mathcal{R}_2 = \mathcal{R}_1$, $\mathcal{U}_2 = \mathcal{U}_1$). We say $\mathcal{C}_2$ is an *immediate extension* of $\mathcal{C}_1$ if $\mathcal{C}_2 \neq \mathcal{C}_1$ and there are no extensions of $\mathcal{C}_1$ properly between $\mathcal{C}_1$ and $\mathcal{C}_2$. This means that either $m_2 = m_1 + 1$ and $\mathcal{R}_2 = \mathcal{R}_1$, $\mathcal{U}_2 = \mathcal{U}_1$, or else $m_2 = m_1$ and there is a unique sequence not ending in 0 in $\mathrm{dom}(\mathcal{R}_2) - \mathrm{dom}(\mathcal{R}_1)$, or a unique sequence not ending in 0 in $\mathrm{dom}(\mathcal{U}_2) - \mathrm{dom}(\mathcal{U}_1)$ (and the other components being equal).

As before, if $\mathcal{C}_2$ extends $\mathcal{C}_1$ and we are given an increasing map from m_1 to m_2, and a way of identifying $\mathrm{dom}(\mathcal{R}_1)$ with a subset of $\mathrm{dom}(\mathcal{R}_2)$, and similarly for $\mathrm{dom}(\mathcal{U}_1)$, $\mathrm{dom}(\mathcal{U}_2)$, then the measure $M^{\mathcal{C}_2}$ projects to the measure $M^{\mathcal{C}_1}$ in a natural manner.

Also as before, a *partial* type ≤ 2 tree of uniform cofinalities is a triple $\mathcal{C}' = \langle m';\mathcal{R}';\mathcal{U}'\rangle$ where at most one of $\mathcal{R}'$, $\mathcal{U}'$ is a partial tree. Recall this means that there is a unique sequence $\vec{r} \in \mathrm{dom}(\mathcal{R})$ (or $\vec{s} \in \mathrm{dom}(\mathcal{U})$) for which $\mathcal{R}(\vec{r})$ (or $\mathcal{U}(\vec{s})$) is undefined, and this sequence does not end in 0 and is maximal in $\mathrm{dom}(\mathcal{R})$ (or $\mathrm{dom}(\mathcal{U})$). Likewise, $\mathcal{C}' = \langle m';\mathcal{R}';\mathcal{U}'\rangle$ is a *partial extension* of $\mathcal{C} = \langle m;\mathcal{R};\mathcal{U}\rangle$ if either $m' = m+1$, $\mathcal{R}' = \mathcal{R}$, and $\mathcal{U}' = \mathcal{U}$, or $m' = m$ and the distinguished sequence in $\mathrm{dom}(\mathcal{R}') \cup \mathrm{dom}(\mathcal{U}')$ is the only sequence in $\mathrm{dom}(\mathcal{C}') - \mathrm{dom}(\mathcal{C})$ (where again we have implicitly a way of identifying $\mathrm{dom}(\mathcal{R})$ with a subset of $\mathrm{dom}(\mathcal{U}')$, etc.). In the first case, by the "extra sequence" in $\mathrm{dom}(\mathcal{C}')$ we mean the integer $\leq m+1$ not identified with an integer $\leq m$ in $\mathrm{dom}(\mathcal{C})$. We say $\vec{\theta}'$ is of semi-type $\mathcal{C}'$ if the representing functions g', f' are of semi-types $\mathcal{R}'$, $\mathcal{U}'$ respectively (if, say the distinguished sequence $\vec{s}$ is in $\mathrm{dom}(\mathcal{U}') - \mathrm{dom}(\mathcal{U})$, then recall this means that the uniform cofinality of the "component function" $f^{\vec{s}}$ of f is not restricted). If $\vec{\theta}$ is of type $\mathcal{C}$, and $\vec{\theta}'$ is of semi-type $\mathcal{C}'$, we say $\vec{\theta}'$ extends $\vec{\theta}$ if $\vec{\theta}' \restriction (\mathrm{dom}(\mathcal{R}) \cup \mathrm{dom}(\mathcal{U})) = \vec{\theta}$. Thus, $\vec{\theta}'$ consists of $\vec{\theta}$ together with one extra ordinal corresponding to the extra sequence in $\mathrm{dom}(\mathcal{C}')$.

The next theorem analyzes uniform cofinalities of functions $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to the measures $M^{\mathcal{C}}$. It is analogous to theorem 4.2.

THEOREM 8.1. *Let $\mathcal{C} = \langle m, \mathcal{R}, \mathcal{U}\rangle$ be a type ≤ 2 tree of uniform cofinalities, and $M^{\mathcal{C}}$ the associated measure. Suppose $F\colon \mathrm{dom}(M^{\mathcal{C}}) \to \boldsymbol{\delta}^1_3$ has range in the limit ordinals. Then F has uniform cofinality with respect to $M^{\mathcal{C}}$ described by one of the following cases.*

(1) *$\forall^*_{M^{\mathcal{C}}} \vec{\theta}\ F(\vec{\theta})$ has uniform cofinality ω.*

(2) *There is a partial extension $\mathcal{C}'$ of $\mathcal{C}$ such that $\forall^*_{M^{\mathcal{C}}} \vec{\theta}\ F(\vec{\theta})$ has uniform cofinality the order type of $\{\vec{\theta}' \colon \vec{\theta}'$ is of semi-type $\mathcal{C}'$ and extends $\vec{\theta}\}$.*

PROOF. Fix $\mathcal{C}$ and $F\colon \mathrm{dom}(M^{\mathcal{C}}) \to \boldsymbol{\delta}^1_3$. Let $r\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ be such that $r(\alpha)$ is greater than the supremum of the $F(\vec{\theta})$ for $\vec{\theta} = (\vec{\alpha}, \beta^{\vec{r}}, \gamma^{\vec{s}})$ of type $\mathcal{C}$ and the $\gamma^{\vec{s}}$ representable by a $f\colon \mathrm{dom}(<^{\mathcal{U}}) \to \alpha$. Let T_x be well-founded and such that $|T_x \restriction \sup_j j(\alpha)| > r(\alpha)$ on a c.u.b. set. This produces a function $S\colon \mathrm{dom}(M^{\mathcal{C}}) \to \boldsymbol{\delta}^1_3$ and an $l\colon \{(\vec{\theta}, \beta)\colon \beta < S(\vec{\theta})\} \to \boldsymbol{\delta}^1_3$ such that $\forall^*_{W^m_1} \vec{\alpha}\ \forall^*_{M^{\mathcal{R}}} \beta^{\vec{r}}\ \forall^* f\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ we have (1) $F(\vec{\theta}) = \sup_{\beta < S(\vec{\theta})} l(\vec{\theta}, \beta)$ and (2) $S(\vec{\theta}) \leq \sup_j j(\sup f)$, where $\vec{\theta} = (\vec{\alpha}; \beta^{\vec{r}}; [f])$. If equality holds in (2), and there is no $\tilde{S}$ strictly smaller than S on a $M^{\mathcal{C}}$ measure one set for which (1) holds (for some $\tilde{l}$), then clearly F has uniform cofinality ω. So, assume strict inequality holds in (2).

For tuples $\vec{u} = (d; K_1, \dots, K_a)$, $\vec{v} = (d'; K'_1, \dots, K'_b)$, where d, d' are generalized type-2 descriptions defined relative to $\mathcal{U}$ and the $\vec{K}$, $\vec{K}'$ respectively, define $\vec{u} \prec \vec{v}$ iff $\forall^* f\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$, $h(f; d; \vec{K}) < h(f; d'; \vec{K}')$ (c.f. definition 5.5 and the paragraph immediately following). Let now $\vec{u} = (d; K_1, \dots, K_a)$ be $\prec$ minimal such that there exists a function $S\colon \mathrm{dom}(M^{\mathcal{C}}) \to \boldsymbol{\delta}^1_3$ and an $l\colon \{(\vec{\theta}, \beta)\colon \beta < S(\vec{\theta})\} \to \boldsymbol{\delta}^1_3$ such that $\forall^*_{W^m_1} \vec{\alpha}\ \forall^*_{M^{\mathcal{R}}} \beta^{\vec{r}}\ \forall^* f\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ we have (1) $F(\vec{\theta}) = \sup_{\beta < S(\vec{\theta})} l(\vec{\theta}, \beta)$ and (2) $S(\vec{\theta}) \leq h(f; d; K_1, \dots, K_a)$, where again $\vec{\theta} = (\vec{\alpha}; \beta^{\vec{r}}; [f])$.

We may assume that equality holds in (2) almost everywhere and there does not exist a $\tilde{S} < S$ almost everywhere also satisfying (1) (for some $\tilde{l}$). For otherwise, fix $\tilde{S}$, $\tilde{l}$ satisfying (1) and $\forall^*_{W^m_1} \vec{\alpha}\ \forall^*_{M^{\mathcal{R}}} \beta^{\vec{r}}\ \forall^* f\ \tilde{S}(\vec{\theta}) < h(f; d; K_1, \dots, K_a)$. If $\mathcal{L}(d)$ is not defined, then by theorem 5.1 it follows that $\exists \delta < \boldsymbol{\delta}^1_3\ \forall^* \vec{\theta}\ \tilde{S}(\vec{\theta}) < \delta$. Thus there is a measure one set on which $\tilde{S}$ depends only on $\vec{\alpha}, \beta^{\vec{r}}$. Theorem 4.2 now finishes the argument. Assume next that $\mathcal{L}(d)$ is defined. Then there is a $C \subseteq \boldsymbol{\delta}^1_3$

such that $\forall^* \vec{\alpha}\ \forall^* \beta^{\vec{r}}\ \forall^* f\ \tilde{S}(\vec{\theta}) < h(N_C; f; \mathcal{L}(d); K_1, \dots, K_a)$. Let x be such that T_x is well-founded and $|T_x \restriction \sup_j j(\alpha)| > N_C(\alpha)$ on a c.u.b. set. This produces an S', l' satisfying (1) and (2) with the bound $h(\sup_j j; f; \mathcal{L}(d); K_1, \dots, K_a)$. Either F has uniform cofinality ω almost everywhere, or there is a K_{a+1} such that $\forall^* \vec{\alpha}$ $\forall^* \beta^{\vec{r}}\ \forall^* f\ S'(\vec{\theta}) \le h(f; \mathcal{L}(d); K_1, \dots, K_a, K_{a+1})$. The latter alternative violates the minimality of $(d; K_1, \dots, K_a)$.

Thus, we are in the case where there is some $(d; K_1, \dots, K_a)$ such that $\forall^* \alpha\ \forall^* \beta^{\vec{r}}$ $\forall^* f\ F(\vec{\theta})$ has uniform cofinality $h(f; d; K_1, \dots, K_a)$, and $\mathcal{L}(d)$ is defined. Say $\mathcal{L}(d) = \langle j_1, d_1, \dots, d_{l-1}, j_l \rangle$ has type $\langle j_1, q_1, \dots, q_{l-1}, j_l \rangle \in \operatorname{dom}(\mathcal{U})$. Let $\mathcal{U}'$ be the partial extension of $\mathcal{U}$ having the extra tuple $\langle j_1, q_1, \dots, j_{l-1}, q_{l-1}, k+1 \rangle$ in its domain, where k is maximal such that $\langle j_1, q_1, \dots, j_{l-1}, q_{l-1}, k \rangle \in \operatorname{dom}(\mathcal{U})$, and where we identify $\langle j_1, q_1, \dots, j_{l-1}, q_{l-1}, j \rangle \in \operatorname{dom}(\mathcal{U})$ with $\langle j_1, q_1, \dots, j_{l-1}, q_{l-1}, j \rangle \in \operatorname{dom}(\mathcal{U}')$ for $j \le j_l$, and with $\langle j_1, q_1, \dots, j_{l-1}, q_{l-1}, j+1 \rangle \in \operatorname{dom}(\mathcal{U}')$ for $j > j_l$. From theorem 5.1 it follows that for almost all $f \colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ that $h(f; d; K_1, \dots, K_a)$ is the supremum of the $[f'^{\langle j_1, q_1, \dots, q_{l-1}, j_l+1 \rangle}]$ for $f' \colon \operatorname{dom}(\mathcal{U}') \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{U}'$ with $[f']$ extending $[f]$. Letting $\mathcal{C}' = \langle m, \mathcal{R}, \mathcal{U}' \rangle$ then verifies (2) of the theorem. $\square$

We now define another collection of measures on $(\boldsymbol{\delta}^1_3)^{<\omega}$. Let $\vec{t}$ be a tuple of the form $\vec{t} = (f_m; K_1, \dots, K_t)$ where $K_i \in W_1 \cup S_1$, and f_m is a formal symbol where $m \in \omega$. We say that a tuple of the form $\vec{u} = (d; K_{t+1}, \dots, K_a)$ *represents an ordinal with respect to* $\vec{t}$ provided $K_{t+1}, \dots, K_a \in W_1 \cup S_1$ and either $d = d^{(f_m)} \in \bar{\mathcal{D}}$ is defined and satisfies condition D relative to $K_1, \dots, K_t, K_{t+1}, \dots, K_a$, or else $d \in \mathcal{D}$ and satisfies C relative to this sequence of measures. Recall that if $f \colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type and $h_1, \dots, h_t$ are given, then $h(f; d; h_1, \dots, h_t, K_{t+1}, \dots, K_a)$ is represented with respect to K_{t+1} by $[h_{t+1}] \to h(f; d; h_1, \dots, h_t, h_{t+1}, K_{t+2}, \dots, K_a)$, etc., and where $h(f; d; h_1, \dots, h_t, \dots, h_a) = f(h(d_i; h_1, \dots, h_a))$.

Definition 8.2. Let $\vec{t} = (f_m; K_1, \dots, K_t)$, and suppose $\vec{u}_1, \dots, \vec{u}_k$ represent ordinals with respect to $\vec{t}$, where say $\vec{u}_i = (d_i; K^i_{t+1}, \dots, K^i_{a_i})$. Then $v^{(\vec{t}, \vec{u}_1, \dots, \vec{u}_k)}$ is the measure on $(\boldsymbol{\delta}^1_3)^k$ defined as follows. We define A to have $v^{(\vec{t}, \vec{u}_1, \dots, \vec{u}_k)}$ measure one if there is a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $f \colon \omega_{m+1} \to C$ of the correct type, for almost all $h_1, \dots, h_t$ with respect to $K_1, \dots, K_t$, $(\alpha_1, \dots, \alpha_k) \in A$, where $\alpha_i = h(f; d_i; h_1, \dots, h_t, K^i_{t+1}, \dots, K^i_{a_i})$.

We refer to measures of this kind as *description measures.*

We let $h(f; \vec{u}_i; h_1, \dots, h_t)$ abbreviate $h(f; d_i; h_1, \dots, h_t, K^i_{t+1}, \dots, K^i_{a_i})$ for $d_i \in \bar{\mathcal{D}}$, and $h(\vec{u}_i; h_1, \dots, h_t)$ abbreviate $h(d_i; h_1, \dots, h_t, K^i_{t+1}, \dots, K^i_{a_i})$ for $d_i \in \mathcal{D}$. We frequently just write $h(f; \vec{u}_i; \vec{h})$ or $h(\vec{u}_i; \vec{h})$.

We say the description measure $v' = v^{(\vec{t}', \vec{u}'_1, \dots, \vec{u}'_l)}$ *extends* the description measure $v^{(\vec{t}, \vec{u}_1, \dots, \vec{u}_k)}$ provided $\vec{t}'$ extends $\vec{t}$ as sequences, $l \ge k$, and $\vec{u}'_i = \vec{u}_i$ for $i \le k$. In this case, the measure v' project to the measure v (by deleting the last $l - k$ components of the tuple).

Note that it is possible for the same measure to be represented as a description measure in different ways.

We consider now the possible uniform cofinalities for functions $F \colon (\boldsymbol{\delta}^1_3)^k \to \boldsymbol{\delta}^1_3$ with respect to description measures.

THEOREM 8.2. *Let* $v = v^{(\vec{t}, \vec{u}_1, \dots, \vec{u}_k)}$ *be a description measure, where* $\vec{t} = (f_m; K_1, \dots, K_t)$. *Suppose* $F \colon (\boldsymbol{\delta}^1_3)^k \to \boldsymbol{\delta}^1_3$. *Then there is a tuple* $\vec{u} = (d; K_{t+1}, \dots, K_a)$ *defining an ordinal with respect to* $\vec{t}$ *such that* $\forall^* f\ \forall^* h_1, \dots, h_t$ $F(\alpha_1, \dots, \alpha_k)$ *has uniform cofinality* $h(f; \vec{u}; \vec{h})$ *(or* $h(\vec{u}; \vec{h})$ *if* $d \in \bar{\mathcal{D}}$*), where* $\alpha_i = h(f; \vec{u}_i; h_1, \dots, h_t)$.

PROOF. The proof is very similar to that of theorem 8.1. Say $\vec{u}_i = (d_i; K^i_{t+1}, \dots, K^i_{a_i})$. As in theorem 8.1, we get an $S \colon \mathrm{dom}(v) \to \boldsymbol{\delta}^1_3$, an $l \colon \{(\alpha_1, \dots, \alpha_k, \beta) \colon \beta < S(\alpha_1, \dots, \alpha_k)\} \to \boldsymbol{\delta}^1_3$, and a $\vec{u} = (d; K_{t+1}, \dots, K_a)$ such that:

(1) $\forall^*_v \alpha_1, \dots, \alpha_k\ F(\vec{\alpha}) = \sup_{\beta < S(\vec{\alpha})} l(\vec{\alpha}, \beta)$.

(2) $\forall^* f\ \forall^* h_1, \dots, h_t\ S(\alpha_1, \dots, \alpha_k) \leq h(f; \vec{u}; h_1, \dots, h_t)$, where again $\alpha_i = h(f; \vec{u}_i; h_1, \dots, h_t)$.

We may assume $[S]_v$ is chosen minimal. We may also assume $\vec{u}$ is chosen minimal, subject to satisfying (2), in the ordering $\vec{u}_1 \prec \vec{u}_2$ iff $\forall^* f\ \forall^* h_1, \dots, h_t$ $h(f; \vec{u}_1; \vec{h}) < h(f; \vec{u}_2; \vec{h})$. If equality holds almost everywhere in (2), we are done, so assume strict inequality holds almost everywhere.

First assume $d \in \bar{\mathcal{D}}$ and $\bar{\mathcal{L}}(d)$ is defined. By theorem 3.1, $\exists C \subseteq \boldsymbol{\delta}^1_3$ such that $\forall^* f\ \forall^* h_1, \dots, h_t\ S(\alpha_1, \dots, \alpha_k) < h(N_C; f; \bar{\mathcal{L}}(d); h_1, \dots, h_t, \vec{K})$. Fix a real x such that T_x is well-founded and $|T_x \restriction \sup_j j(\beta)| > N_C(\beta)$ on a c.u.b. set. For v almost all $\vec{\alpha}$, let $\tilde{S}(\vec{\alpha})$ be the least ordinal δ such that $\forall^* h_{t+1}, \dots, h_a\ |T_x \restriction \delta(\vec{h})| > S(\vec{\alpha})(\vec{h})$. Thus, $\forall^* f\ \forall^* h_1, \dots, h_t\ \tilde{S}(\vec{\alpha}) \leq h(\sup_j j; f; \mathcal{L}(d); \vec{h})$, where $\alpha_i = h(f; \vec{u}_i; \vec{h})$. Now for v almost all $\vec{\alpha}$ and $\beta < \tilde{S}(\vec{\alpha})$, define $\tilde{l}(\vec{\alpha}, \beta) = l(\vec{\alpha}, \gamma)$ if $\gamma < S(\vec{\alpha})$, and $= 0$ otherwise, where γ is defined by:

$$\forall^* h_{t+1}, \dots, h_t\ \gamma(\vec{h}) = |T_x \restriction (\tilde{S}(\vec{\alpha})(\vec{h}))(\beta(\vec{h}))|.$$

Thus $\tilde{S}$, $\tilde{l}$ is also a "lifting" of F (i.e., satisfies (1)), and so by minimality $\tilde{S} \leq S$ almost everywhere. If $\tilde{S} = S$ almost everywhere and $\forall^* f\ \forall^* h_1, \dots, h_t\ \tilde{S}(\vec{\alpha}) = h(\sup_j j; f; \mathcal{L}(d); \vec{h})$, then $\tilde{S}$ has uniform cofinality ω almost everywhere, and thus so does F. Otherwise we must have $\forall^* f\ \forall^* h_1, \dots, h_t\ S(\vec{\alpha}) < h(\sup_j j; f; \mathcal{L}(d); \vec{h})$, and thus for some $\tilde{K}$ we have $\forall^* f\ \forall^* h_1, \dots, h_t\ S(\vec{\alpha}) < h(f; \mathcal{L}(d); \vec{h}; K_{t+1}, \dots, K_a, \tilde{K})$, violating the minimality of $\vec{u}$.

If $d \in \bar{\mathcal{D}}$ and $\bar{\mathcal{L}}(d)$ is not defined, or if $d \in \mathcal{D}$, then by theorem 3.1 (or trivially) $\exists \delta < \boldsymbol{\delta}^1_3\ \forall^*_v \alpha_1, \dots, \alpha_k\ S(\alpha_1, \dots, \alpha_k) < \delta$. By taking a well-ordering of $\aleph_\omega$ of length $> \delta$ we may assume $S(\vec{\alpha}) \leq \aleph_\omega$ almost everywhere. Thus, either F has uniform cofinality ω, or for some least m, $S(\vec{\alpha}) < \aleph_{m+1}$ almost everywhere. In this case, we assume $\vec{u}$ is minimal such that $\forall^* f\ \forall^* h_1, \dots, h_t\ S(\alpha_1, \dots, \alpha_k) \leq h(\vec{u}; h_1, \dots, h_t)$ (and thus $d = d^{(f_m)}$). If equality holds almost everywhere, we are done. If strict inequality holds almost everywhere and $\mathcal{L}(d)$ is defined, we again violate the minimality of d, using theorem 3.2. Otherwise, by theorem 3.2 again, $\exists \delta < \omega_1\ \forall^*_v \alpha_1, \dots, \alpha_k$ $S(\vec{\alpha}) < \delta$. Easily, F then has uniform cofinality ω. $\square$

The next several definitions generalize the corresponding notions which were used in the analysis of measures on $\boldsymbol{\delta}^1_3$.

DEFINITION 8.3. A type-3 *tree of uniform cofinalities* is a function $\mathcal{T}$ satisfying the following:

(1) For each minimal (non-empty) type ≤ 2 tree of uniform cofinalities p_1 of the form $p_1 = \langle 0, \emptyset, \mathcal{U} \rangle$, $\langle p_1, i_1 \rangle \in \operatorname{dom}(\mathcal{T})$ for $0 \leq i_1 \leq a$, for some integer a depending on p_1.
(2) In general, if $\vec{t} = \langle p_1, i_1, \dots, i_{m-1}, p_m, i_m \rangle \in \operatorname{dom}(\mathcal{T})$, then $\mathcal{T}(\vec{t})$ is either the symbol (s), the symbol (ω), or a partial extension of the type ≤ 2 tree of uniform cofinalities p_m. Also, $\mathcal{T}(\vec{t}) = (s)$ iff $i_m = 0$. In the first two cases, $\vec{t}$ is maximal in $\operatorname{dom}(\mathcal{T})$. In the third case, for each type ≤ 2 tree p_{m+1} which completes the partial tree $\mathcal{T}(\vec{t})$, we have $\langle p_1, i_1, \dots, i_m, p_{m+1}, i_{m+1} \rangle \in \operatorname{dom}(\mathcal{T})$ for $0 \leq i_{m+1} \leq a$, for some integer $a \geq 0$ (which depends on $\vec{t}$, p_{m+1}).

REMARK 8.1. The minimal type ≤ 2 trees of the form $p_1 = \langle 0, \emptyset, \mathcal{U} \rangle$ correspond to the three possible normal measures on $\boldsymbol{\delta}^1_3$. In general, we think of p_m as specifying a possible "type" for an m-tuple of ordinals $< \boldsymbol{\delta}^1_3$.

Suppose $\mathcal{T}$ is a type-3 tree of uniform cofinalities, and $\vec{t} = \langle p_1, i_1, \dots, p_m, i_m \rangle \in \operatorname{dom}(\mathcal{T})$. Recall M^{p_m} is a measure on tuples $\vec{\theta}$ of ordinals $< \boldsymbol{\delta}^1_3$ of size m. The sequence $p_1, \dots, p_m$ determines an order of enumeration for the components of these tuples, and thus we may view M^{p_m} as a measure on sequences $(\alpha_1, \dots, \alpha_m)$. We say $\vec{\alpha}$ is of type $\vec{t}$ if $\vec{\theta}$ is of type p_m (that is, if $p_m = \langle l, \mathcal{R}, \mathcal{U} \rangle$, then $\vec{\theta} = (\delta_1, \dots, \delta_l, \beta^{\vec{r}}, \gamma^{\vec{s}})_{\vec{r} \in \operatorname{dom}(\mathcal{R}), \vec{s} \in \operatorname{dom}(\mathcal{U})}$ is represented by $g\colon \operatorname{dom}(<^{\mathcal{R}}) \to \omega_1$ and $f\colon \operatorname{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of types $\mathcal{R}$, $\mathcal{U}$ respectively).

To each type-3 tree of uniform cofinalities $\mathcal{T}$ we associate an ordering $<^{\mathcal{T}}$.

DEFINITION 8.4. Let $\mathcal{T}$ be a type-3 tree of uniform cofinalities. Then $<^{\mathcal{T}}$ is lexicographic ordering on sequences $\langle \alpha_1, i_1, \dots, \alpha_m, i_m \rangle$ such that for some $\langle p_1, i_1, \dots, p_m, i_m \rangle \in \operatorname{dom}(\mathcal{T})$, $(\alpha_1, \dots, \alpha_m)$ has type $\langle p_1, i_1, \dots, p_m, i_m \rangle$.

Note that $<^{\mathcal{T}}$ has order-type $\boldsymbol{\delta}^1_3$.

DEFINITION 8.5. Let $\mathcal{T}$ be a type-3 tree of uniform cofinalities. We say a function $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ is of type $\mathcal{T}$ if F satisfies:

(1) F is order preserving with respect to $<^{\mathcal{T}}$.
(2) If $\vec{\rho} = \langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m \rangle$ is of limit rank in $<^{\mathcal{T}}$ then $F(\vec{\rho}) = \sup\{F(\vec{\rho}\,') : \vec{\rho}\,' <^{\mathcal{T}} \vec{\rho}\}$. Otherwise, $F(\langle \alpha_1, i_1, \dots, \alpha_m, i_m \rangle)$ has uniform cofinality ω.

Using the strong partition relation on $\boldsymbol{\delta}^1_3$ we now associate to each type-3 tree of uniform cofinalities $\mathcal{T}$ a measure $M^{\mathcal{T}}$ in the usual manner.

DEFINITION 8.6. Let $\mathcal{T}$ be a type-3 tree of uniform cofinalities. Then A has $M^{\mathcal{T}}$ measure one iff $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ $\forall F\colon \operatorname{dom}(\mathcal{T}) \to C$ of type $\mathcal{T}$, $(\dots, \delta^{\vec{t}}, \dots)_{\vec{t} \in \operatorname{dom}(\mathcal{T})} \in A$, where for $\vec{t} = \langle p_1, i_1, \dots, p_m, i_m \rangle \in \operatorname{dom}(\mathcal{T})$, $\delta^{\vec{t}}$ is represented with respect to M^{p_m} by $F^{\vec{t}}$ defined by: $F^{\vec{t}}(\alpha_1, \dots, \alpha_m) = F(\langle \alpha_1, i_1, \dots, \alpha_m, i_m \rangle)$.

Each $M^{\mathcal{T}}$ is thus a measure on $({\boldsymbol{\delta}^1_5}^-)^{<\omega}$.

As with type-1 and type-2 trees, we defined a *pruned subtree* $\mathcal{T}'$ of the type-3 tree $\mathcal{T}$ to be the function obtained by deleting some (possibly all) sequences $\vec{t}$

ending in 0 from $\mathrm{dom}(\mathcal{T})$. We again define $F\colon \mathrm{dom}(\mathcal{T}') \to \boldsymbol{\delta}^1_3$ having type $\mathcal{T}'$ in the natural manner. That is, if $\vec{t} = \langle p_1, i_1, \dots, p_m, i_m\rangle \in \mathrm{dom}(\mathcal{T}')$ is maximal, and $\mathcal{T}'(\vec{t}) = p'0$, then for for $\vec{\rho} = \langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle$ of type p_m, $F(\vec{\rho})$ has uniform cofinality $\{\beta\colon (\alpha_1, \dots, \alpha_m, \beta)$ has semi-type $p'\}$, if this set has limit order-type, and otherwise $F(\vec{\rho})$ has uniform cofinality ω. Also, $F(\vec{\rho}) > \sup\{F(\vec{\rho}')\colon \vec{\rho}' <^{\mathcal{T}} \vec{\rho}\}$.

The next lemma shows, as usual, that the requirement that $\langle p_1, i_1, \dots, p_m, i_m\rangle$ be non-maximal in $\mathrm{dom}(\mathcal{T})$ for $i_m > 0$ and $\mathcal{T}(\langle p_1, i_1, \dots, p_m, i_m\rangle) \neq (\omega)$ causes no loss of generality.

LEMMA 8.1. *Let $\mathcal{T}$ be a type-3 tree of uniform cofinalities, and $\mathcal{T}'$ a pruned subtree of $\mathcal{T}$. Suppose $F'\colon \mathrm{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}^1_3$ is of type $\mathcal{T}'$ and has range in the additively closed ordinals. Then there is a $F\colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ which induces F' by restriction. Furthermore, $[F]$ is unique in the sense that if G of type $\mathcal{T}$ also induces F', then for all $\vec{t} = \langle p_1, i_1, \dots, p_m, 0\rangle \in \mathrm{dom}(\mathcal{T})$ and M^{p_m} almost all $(\alpha_1, \dots, \alpha_m)$ we have $F^{\vec{t}}(\vec{\alpha}) = G^{\vec{t}}(\vec{\alpha})$.*

PROOF. The existence of F follows immediately from the assumed uniform cofinality of $F'^{\vec{t}}$ for $\vec{t} = \langle p_1, i_1, \dots, p_m, i_m\rangle \in \mathrm{dom}(\mathcal{T})$ when $\vec{t}$ is maximal in $\mathrm{dom}(\mathcal{T})$ and $i_m > 0$. Suppose then that $G\colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ is another such function. Let now $\vec{t} = \langle p_1, i_1, \dots, p_m, i_m, p_{m+1}, 0\rangle \in \mathrm{dom}(\mathcal{T})$. Suppose $H\colon \mathrm{dom}(M^{p_{m+1}}) \to \boldsymbol{\delta}^1_3$ is such that $H(\alpha_1, \dots, \alpha_m, \beta) < F^{\vec{t}}(\alpha_1, \dots, \alpha_m, \beta)$ for $M^{p_{m+1}}$ almost all $(\vec{\alpha}, \beta)$. Consider the case where $p_m = (l; \mathcal{R}; \mathcal{U})$, $p_{m+1} = (l; \mathcal{R}; \mathcal{U}_2)$ where say $\vec{s} = \langle j_1, q_1, \dots, q_{n-1}, j_n\rangle \in \mathrm{dom}(\mathcal{U}_2) - \mathrm{dom}(\mathcal{U})$, the other cases being similar. For fixed $\epsilon_1 < \cdots < \epsilon_l$, and $g\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$, we partition $f\colon \mathrm{dom}(<^{\mathcal{U}_2}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}_2$ with the extra values $h(\langle j_1, \gamma_1, \dots, \gamma_{n-1}, j_n\rangle)$ of uniform cofinality ω inserted between $\eta = f(\langle j_1, \gamma_1, \dots, \gamma_{n-1}, j_n - 1\rangle)$ and $N_f(\eta)$. We partition them according to whether $H(\alpha_1, \dots, \alpha_m, \beta) < F^{\vec{t}}(\alpha_1, \dots, \alpha_m, \beta')$, where $(\vec{\alpha}, \beta) = \vec{\epsilon} \cup [g] \cup [f]$, and $\beta' = [h]_{M^{\gamma_{n-1}}}$. A routine argument using the sliding lemmas for type-2 trees shows that for almost all $\vec{\epsilon}$ and f that on the homogeneous side of the partition this property holds. It then follows that there is a $M^{p_{m+1}}$ measure one set A such that for $(\vec{\alpha}, \beta)$, $(\vec{\alpha}, \beta') \in A$, we have $H(\vec{\alpha}, \beta) = H(\vec{\alpha}, \beta')$. This shows that $[F^{\vec{t}}]$ is determined from $[F']$, and thus $[F^{\vec{t}}] = [G^{\vec{t}}]$. □

The next lemma is our sliding lemma for type-3 trees. It is analogous to lemmas 4.3, 5.2.

LEMMA 8.2. *Let $\mathcal{T}$ be a type-3 tree of uniform cofinalities, $\langle p_1, i_1, \dots, i_{m-1}, p_m, i_m\rangle \in \mathrm{dom}(\mathcal{T})$, and $a \in \omega$ be maximal such that $\langle p_1, i_1, \dots, p_m, a\rangle \in \mathrm{dom}(\mathcal{T})$. Let $F\colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ be of type $\mathcal{T}$, and let $G\colon \mathrm{dom}(M^{p_m}) \to \boldsymbol{\delta}^1_3$ be given.*

(1) *Suppose that $\forall^*_{M^{p_m}} \alpha_1, \dots, \alpha_m\ G(\alpha_1, \dots, \alpha_m) < F(\langle \alpha_1, i_1, \dots, i_{m-1}\rangle)$. Let $\mathcal{T}'$ be the partial extension of $\mathcal{T}$ with extra sequence $\langle p_1, i_1, \dots, i_{m-1}, p_m, a+1\rangle$ in its domain. Then there is an $F'\colon \mathrm{dom}(<^{\mathcal{T}'}) \to \mathrm{ran}(F)$ of semi-type $\mathcal{T}'$ inducing $F_2\colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ with $[F_2] = [F]$ and such that $\forall^*_{M^{p_m}} \alpha_1, \dots, \alpha_m$ $G(\alpha_1, \dots, \alpha_m) < F'(\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, a+1\rangle)$.*

(2) *Suppose that $\forall^*_{M^{p_m}} \alpha_1, \dots, \alpha_m\ G(\alpha_1, \dots, \alpha_m) < F(\langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle)$, and $i_m > 0$. Let $\mathcal{T}'$ be as above except we now identify the sequence $\langle p_1, i_1, \dots, i_{m-1}, p_m, j\rangle \in \mathrm{dom}(\mathcal{T})$ with $\langle p_1, i_1, \dots, i_{m-1}, p_m, j\rangle \in \mathrm{dom}(\mathcal{T}')$ for $j < i_m$, and with $\langle p_1, i_1, \dots, i_{m-1}, p_m, j+1\rangle \in \mathrm{dom}(\mathcal{T}')$ for $j \geq i_m$.*

Then there is an $F'\colon \mathrm{dom}(<^{\mathcal{T}'}) \to \mathrm{ran}(F)$ *of semi-type* $\mathcal{T}'$ *inducing* F_2 *with* $[F_2] = [F]$ *and such that* $\forall^*_{M^{p_m}} \alpha_1, \dots, \alpha_m\ G(\alpha_1, \dots, \alpha_m) < F'(\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m \rangle)$.

PROOF. We consider (1), the proof of (2) being similar. Fix $F\colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ and G as in (1). Define H on a M^{p_m} measure one set by $H(\alpha_1, \dots, \alpha_m) =$ the least β so that $F(\langle \alpha_1, i_1, \dots, i_{m-1}, \beta, 0 \rangle) > G(\alpha_1, \dots, \alpha_m)$. The other cases being similar but easier, we assume p_{m-1} is of the form $p_{m-1} = \langle l; \mathcal{R}; \tilde{\mathcal{U}} \rangle$, and $p_m = \langle l; \mathcal{R}; \mathcal{U} \rangle$, that is, $\mathcal{U}$ is an immediate extension of $\tilde{\mathcal{U}}$ with say $\langle j_1, q_1, \dots, q_{k-1}, j_k \rangle$ the extra sequence (not ending in 0) in $\mathrm{dom}(\mathcal{U}) - \mathrm{dom}(\tilde{\mathcal{U}})$. The other case being similar, suppose $j_k = s + 1$ where s is maximal so that $\langle j_1, q_1, \dots, q_{k-1}, s \rangle \in \mathrm{dom}(\tilde{\mathcal{U}})$, thus $\langle j_1, q_1, \dots, q_{k-1}, j \rangle \in \mathrm{dom}(\tilde{\mathcal{U}})$ is identified with $\langle j_1, q_1, \dots, q_{k-1}, j \rangle \in \mathrm{dom}(\mathcal{U})$ for $j < j_k$.

Fix $f_0\colon l \to \omega_1$, $f_1\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ such that for almost all $f_2\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$, $H(\alpha_1, \dots, \alpha_m)$ is defined, where $f = f_0 \cup f_1 \cup f_2\colon \mathrm{dom}(<^{p_m}) \to \boldsymbol{\delta}^1_3$ represents $\alpha_1, \dots, \alpha_m$.

Consider the partition $\mathcal{P}(f_0, f_1)$: we partition functions $f_2\colon \mathrm{dom}(<^{\mathcal{U}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}$ with the extra value $h(\langle j_1, \gamma_1, \dots, \gamma_{k-1}, j_k \rangle)$ of uniform cofinality ω inserted between $\eta = f_2(\langle j_1, \gamma_1, \dots, \gamma_{k-1}, j_k \rangle)$ and $N_{f_2}(\eta)$ according to whether $H(\alpha_1, \dots, \alpha_m) < \beta$, where $f = f_0 \cup f_1 \cup f_2$ represents $\alpha_1, \dots, \alpha_m$, and $\beta = [h]_{M^{q_{k-1}}}$. Lemma 5.2 shows that on the homogeneous side of the partition the stated property holds. An easy additivity argument then shows that there is a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ homogeneous for $\mathcal{P}(f_0, f_1)$ for almost all $f_0, [f_1]$ (with respect to $W^l_1 \times M^{\mathcal{R}}$). Let $C' =$ closure points of C. Define then F' as follows. If $\langle \alpha'_1, i'_1, \dots, \alpha'_{m'}, i'_{m'} \rangle$ of type $\langle p'_1, i'_1, \dots, p'_{m'}, i'_{m'} \rangle \in \mathrm{dom}(\mathcal{T})$ is represented by $f'\colon \mathrm{dom}(M^{p'_{m'}}) \to \boldsymbol{\delta}^1_3$ of type $p'_{m'}$, then $F'(\langle \alpha'_1, i'_1, \dots, \alpha'_{m'}, i'_{m'} \rangle) = F(\langle \alpha''_1, i'_1, \dots, \alpha''_{m'}, i'_{m'} \rangle)$ where $\langle \alpha''_1, i'_1, \dots, \alpha''_{m'}, i'_{m'} \rangle$ is represented with respect to $M^{p'_{m'}}$ by $f'' = l \circ f'$, where $l(\alpha) \doteq$ the $\omega \cdot \alpha^{th}$ element of C'. It is not difficult to check that F' is well-defined, order preserving, $[F'] = [F]$, and F' bounds G as required. Also, F' is of type $\mathcal{T}$ except for the requirement that $F(\vec{\rho})$ have uniform cofinality ω for $\vec{\rho}$ in certain measure zero sets with respect to the $M^{p'_{m'}}$. It is trivial to redefine F on these measure zero sets to obtain also this last condition. □

Next we define the notion of a generalized description of type-3, defined relative to a type-3 tree of uniform cofinalities $\mathcal{T}$. A type-3 generalized description d will have an index of the form $\vec{t} = (f_n; K_1, \dots, K_t)$ associated to it.

DEFINITION 8.7. A *type-3 generalized description* relative to the type-3 tree of uniform cofinalities $\mathcal{T}$ is an object of the form $d = d^{(f_n; K_1, \dots, K_t)} = \langle d_1, i_1, \dots, d_m, i_m \rangle$ satisfying:

1. Either $d_i \in \bar{\mathcal{D}}$ is a description defined and satisfying condition D relative to $K_1, \dots, K_t, K^i_{t+1}, \dots, K^i_{a_i}$ for some sequence $K^i_{t+1}, \dots, K^i_{a_i}$ of measures in $W_1 \cup S_1$, or $d_i \in \mathcal{D}$ satisfies C for some such sequence.
2. Let $\vec{u}_i = (d_i; K^i_{t+1}, \dots, K^i_{a_i})$, and $v(d)$ be the description measure $v(d) = v^{(\vec{t}; \vec{u}_1, \dots, \vec{u}_m)}$, where $\vec{t} = (f_n; K_1, \dots, K_t)$. Then for some (necessarily unique) $\langle p_1, i_1, \dots, i_{m-1}, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{T})$, $v(d) = M^{p_m}$. In other words, $\forall^* f\ \forall^* h_1, \dots, h_t\ (\alpha_1, \dots, \alpha_m)$ has type p_m, where $\alpha_i = h(f; \vec{u}_i; h_1, \dots, h_t)$.

We define the interpretation of a generalized type-3 description. Suppose d as above is defined relative to $\mathcal{T}$, and suppose $F\colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ is given. We define $h(F;d) \doteq h(F;d;W^3_n, K_1, \dots, K_t)$ to be the ordinal which is represented with respect to the measure W^n_3 by the function which assigns to $[f_n]_{S^n_1}$, for $f_n\colon \omega_{n+1} \to \boldsymbol{\delta}^1_3$ of the correct type, the value $h(F;d;f_n)$. This, in turn, is represented with respect to $K_1, \dots, K_t$ in the usual manner by $h(F;d;f_n;h_1,\dots,h_t)$, where we set $h(F;d;f_n;h_1,\dots,h_t) = F(\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m\rangle)$ where $\alpha_i = h(f_n;\vec{u}_i;h_1,\dots,h_t)$ if $d_i \in \bar{\mathcal{D}}$ and $\alpha_i = h(\vec{u}_i;h_1,\dots,h_t)$ if $d_i \in \mathcal{D}$. Likewise, given $g\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ we define $h(g;F;d)$, where we set $h(g;F;d;f_n;h_1,\dots,h_t) = g(h(F;d;f_n;h_1,\dots,h_t))$. Also, we order the type-3 generalized descriptions defined relative to $\mathcal{T}$ by: $d_1 < d_2$ iff for almost all $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$, $h(F;d_1) < h(F;d_2)$.

As we did previously for type-2 descriptions, we extend the $\mathcal{L}$ operation to type-3 descriptions. Suppose $\mathcal{T}$ is a type-3 tree of uniform cofinalities, and $d = d^{(f_n;K_1,\dots,K_t)}$ is a generalized type-3 description defined relative to $\mathcal{T}$. Say $d = \langle d_1, i_1, \dots, i_{m-1}, d_m, i_m\rangle$. Let $K^i_{t+1}, \dots, K^i_{a_i}$ be the sequence of measures associated to d_i, and $\vec{u}_i = (d_i; K^i_{t+1}, \dots, K^i_{a_i})$ the corresponding tuple. We define $\mathcal{L}(d)$ through the following cases.

- **(i):** If $i_m = 0$, and there is a least integer $e \geq 1$ such that $\langle d_1, i_1, \dots, i_{m-1}, \mathcal{L}^{(e)}(d_m), 0\rangle$ has type $\langle p_1, i_1, \dots, i_{m-1}, q, 0\rangle$ for some $\langle p_1, i_1, \dots, i_{m-1}, q, 0\rangle \in \operatorname{dom}(\mathcal{T})$, then we set $\mathcal{L}(d) = \langle d_1, i_1, \dots, i_{m-1}, \mathcal{L}^{(e)}(d_m), k\rangle$ where k is maximal such that $\langle p_1, i_1, \dots, i_{m-1}, q, k\rangle \in \operatorname{dom}(\mathcal{T})$. Here, $\mathcal{L}^{(e)}(d_m)$ denotes the e-fold iterate of $\mathcal{L}$ computed relative to the measures $K_1, \dots, K_t, K^m_{t+1}, \dots, K^m_{a_m}$ associated to d_m.
- **(ii):** If $i_m = 0$, and an e as above does not exist, then we set $\mathcal{L}(d) = \langle d_1, i_1, \dots, i_{m-1} - 1\rangle$ provided $m > 1$. If $m = 1$, we leave $\mathcal{L}(d)$ undefined.
- **(iii):** If $i_m > 0$ and $\mathcal{T}(\langle p_1, i_1, \dots, p_m, i_m\rangle) = \omega$, then $\mathcal{L}(d) = \langle d_1, i_1, \dots, d_m, i_m - 1\rangle$.
- **(iv):** Suppose $i_m > 0$ and $\mathcal{T}(\langle p_1, i_1, \dots, p_m, i_m\rangle) \neq \omega$. Let $\vec{u} = (d^\star; K_{t+1}, \dots, K_a)$ be as in theorem 8.2; thus for $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$, $\forall^* f\ \forall^* h_1, \dots, h_t\ F(\langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle)$ has uniform cofinality $h(f; d^\star; h_1, \dots, h_t, K_{t+1}, \dots, K_a)$, where $\alpha_i = h(f; d_i; h_1, \dots, h_t, K^i_{t+1}, \dots, K^i_{a_i})$. Suppose there is a least integer $e \geq 1$ such that if $\mathcal{L}^{(e)}(\vec{u}) = \mathcal{L}^{(e)}(d^\star)$ computed relative to the sequence of measures $K_1, \dots, K_a$, then $\upsilon^{(\vec{t};\vec{u}_1,\dots,\vec{u}_m,\mathcal{L}^{(e)}(\vec{u}))} = M^q$ for some $\langle p_1, i_1, \dots, p_m, i_m, q, 0\rangle \in \operatorname{dom}(\mathcal{T})$. We set $\mathcal{L}(d) = \langle d_1, i_1, \dots, d_m, i_m, \mathcal{L}^{(e)}(d^\star), k\rangle$, where k is maximal such that $\langle p_1, i_1, \dots, p_m, i_m, q, k\rangle \in \operatorname{dom}(\mathcal{T})$.
- **(v):** Suppose $i_m > 0$, $\mathcal{T}(\langle d_1, i_1, \dots, d_m, i_m\rangle) \neq \omega$, and an e as above does not exist. We set $\mathcal{L}(d) = \langle d_1, i_1, \dots, d_m, i_m - 1\rangle$.

The following theorem is the level three analog of theorem 5.1.

LEMMA 8.3. *Let $\mathcal{T}$ be a type-3 tree of uniform cofinalities, and let $d = d^{(f_n;K_1,\dots,K_t)}$ be a generalized type-3 description defined relative to $\mathcal{T}$. Say $d = \langle d_1, i_1, \dots, i_{m-1}, d_m, i_m\rangle$.*

(1) *Suppose $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ is of type $\mathcal{T}$, $\Omega \in ON$, and $\forall^* f\ \forall^* h_1, \dots, h_t$ $\Omega(f, h_1, \dots, h_t) < h(F; d; f; h_1, \dots, h_t)$. If d is not minimal with respect to $\mathcal{L}$, then there is an $F'\colon \operatorname{dom}(<^{\mathcal{T}}) \to \operatorname{ran}(F)$ of type $\mathcal{T}$ such that $[F'] = [F]$ and $\forall^* f\ \forall^* h_1, \dots, h_t\ \Omega(f, h_1, \dots, h_t) < h(N_{F'}; F'; \mathcal{L}(d); f; h_1, \dots, h_t)$ $\doteq N_{F'}(h(F'; \mathcal{L}(d); f; h_1, \dots, h_t))$. If d is minimal with respect to $\mathcal{L}$ (that is, $\mathcal{L}(d)$ is not defined), then $\Omega < \boldsymbol{\delta}^1_3$.*

(2) *Suppose* $\Omega \in ON$ *and* $\forall^* F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ *of type* $\mathcal{T}$ $\forall^* f$ $\forall^* h_1, \dots, h_t$ $\Omega(F, f, h_1, \dots, h_t) < h(F; d; f, h_1, \dots, h_t)$. *If* d *is not minimal with respect to* $\mathcal{L}$, *then there is a* $g\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ *such that* $\forall^* F$ $\forall^* f$ $\forall^* h_1, \dots, h_t$ $\Omega(F, f, h_1, \dots, h_t) < h(g; F; d; f; h_1, \dots, h_t) \doteq N_g(h(F; d; f; h_1, \dots, h_t))$. *If* d *is minimal with respect to* $\mathcal{L}$, *then* $\Omega < \boldsymbol{\delta}^1_3$.

PROOF. (2) easily implies (1) by a straightforward partition argument using the strong partition relation on $\boldsymbol{\delta}^1_3$. To show (1), fix $\mathcal{T}$, $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$, $\Omega \in ON$, and $d = \langle d_1, i_1, \dots, i_{m-1}, d_m, i_m\rangle$ of type $\langle p_1, i_1, \dots, p_m, i_m\rangle$ as in the statement of the lemma. We consider the case $i_m > 0$ and $\mathcal{T}(\langle p_1, i_1, \dots, p_m, i_m\rangle) \neq \omega$, the other cases being similar. Let $c : \operatorname{dom}(M^{p_m}) \to \boldsymbol{\delta}^1_3$ represent the uniform cofinality of $F^{\langle p_1, i_1, \dots, p_m, i_m\rangle}$ with respect to M^{p_m}. Let $\vec{t} = (f_n; K_1, \dots, K_t)$, let $\vec{u}_i = (d_i; K^i_{t+1}, \dots, K^i_{a_i})$ be the tuple corresponding to d_i, and let $\vec{u} = (d^\star; K_{t+1}, \dots, K_a)$ be as in theorem 8.2. Thus,

$$\forall^* f_n\ \forall^* h_1, \dots, h_t\ c(\alpha_1, \dots, \alpha_m) = h(f_n; d^\star; h_1, \dots, h_t, K_{t+1}, \dots, K_a),$$

where $\alpha_i = h(f_n; d_i; h_1, \dots, h_t; K^i_{t+1}, \dots, K^i_{t_i})$. We thus have that

$$\forall^* f_n\ \forall^* h_1, \dots, h_t\ \exists \gamma < h(f_n; d^\star; h_1, \dots, h_t; K_{t+1}, \dots, K_a)$$
$$[\Omega(f_n, h_1, \dots, h_t) < F(\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m, \gamma, 0\rangle)].$$

Let $\mathcal{C}'$ be the partial extension of p_m giving the uniform cofinality c as in theorem 8.1. Let $p_m = \langle l; \mathcal{R}; \mathcal{U}\rangle$, and suppose $\mathcal{C}' = \langle l; \mathcal{R}; \mathcal{U}'\rangle$, the other cases being similar but easier. It follows that $d^\star \in \bar{\mathcal{D}}$, as $c(\alpha_1, \dots, \alpha_m) > \aleph_\omega$ almost everywhere. If $\bar{\mathcal{L}}(d^\star)$ is defined, from theorem 3.1 it then follows that there is a $r\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ such that

$$\forall^* f_n\ \forall^* h_1, \dots, h_t\ \Omega(f_n, h_1, \dots, h_t) <$$
$$F(\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m, h(r; f_n; \bar{\mathcal{L}}(d); h_1, \dots, h_t; K_{t+1}, \dots, K_a), 0\rangle).$$

In this case, let $\tilde{v}$ be the description measure $v^{(\vec{t}, \vec{u}_1, \dots, \vec{u}_m, \bar{\mathcal{L}}(\vec{u}))}$, where $\mathcal{L}(\vec{u})$ denotes $(\mathcal{L}(d^\star); K_{t+1}, \dots, K_a)$ and $\mathcal{L}(d^\star)$ is computed relative to the sequence $K_1, \dots, K_t, \dots, K_{t+1}, \dots, K_a$.

Assume first that there is a G with $G(\alpha_1, \dots, \alpha_m) < c(\alpha_1, \dots, \alpha_m)$ almost everywhere with respect to M^{p_m} such that for $\tilde{v}$ almost all $(\alpha_1, \dots, \alpha_m, \gamma)$ we have $\gamma < G(\alpha_1, \dots, \alpha_m)$. From lemma 8.2, there is an $F'\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ such that $[F'] = [F]$ and

$$\forall^*_{M^{p_m}} \alpha_1, \dots, \alpha_m\ N_{F'}(F'(\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m - 1\rangle) >$$
$$F(\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m, G(\alpha_1, \dots, \alpha_m), 0\rangle),$$

and we are done. If $\bar{\mathcal{L}}(d^\star)$ is not defined, then there is also a $G < c$ almost everywhere such that $\forall^* f$ $\forall^* h_1, \dots, h_t$ $\Omega(f, \vec{h}) < F(\langle \alpha_1, i_1, \dots, \alpha_m, i_m, G(\vec{\alpha}), 0\rangle)$, where $\alpha_i = h(f_n; d_i; \vec{h})$. We can get an F' as above also in this case, and we are done.

Assume now there is no such function G. Let $H\colon \operatorname{dom}(\tilde{v}) \to \boldsymbol{\delta}^1_3$ represent the least equivalence class with respect to $\tilde{v}$ which has a representative satisfying:

(1) There is a measure one set A with respect to $\tilde{v}$ such that if $(\alpha_1, \dots, \alpha_m, \gamma)$, $(\alpha_1, \dots, \alpha_m, \gamma') \in A$ and $\gamma \leq \gamma'$, then $H(\vec{\alpha}, \gamma) \leq H(\vec{\alpha}, \gamma')$.

(2) There does not exist a measure one set A with respect to $\tilde{v}$ such that if $(\alpha_1, \dots, \alpha_m, \gamma)$, $(\alpha_1, \dots, \alpha_m, \gamma') \in A$ then $H(\vec{\alpha}, \gamma) = H(\vec{\alpha}, \gamma')$.
(3) $\tilde{v}$ almost all $\alpha_1, \dots, \alpha_m, \gamma$ are of semi-type $\mathcal{C}'$.

The reader will note the similarity between the definition of H and the argument used in the analysis of measures on $\boldsymbol{\delta}^1_3$.

Note that H exists, since if we define $H(\alpha_1, \dots, \alpha_m, \gamma) =$ the least $\delta \geq \gamma$ such that $(\alpha_1, \dots, \alpha_m, \delta)$ is of semi-type $\mathcal{C}'$, then (1)-(3) are easily seen to be satisfied, (2) following from the nonexistence of G.

Let $v'' = \tilde{H}(\tilde{v})$, where $\tilde{H}$ is the function $\tilde{H}(\alpha_1, \dots, \alpha_m, \gamma) = (\alpha_1, \dots, \alpha_m, H(\alpha_1, \dots, \alpha_m))$. We first claim that there is an immediate extension p_{m+1} of p_m extending $\mathcal{C}'$ such that $v'' = M^{p_{m+1}}$. By countable additivity of v'' it is enough to show that for any c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ that for v'' almost all $\alpha_1, \dots, \alpha_m, \gamma$ there are $f_0 \colon l \to \omega_1$, $f_1 \colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$, and an $f'_2 \colon \mathrm{dom}(<^{\mathcal{U}'}) \to C$ of semi-type $\mathcal{U}'$ with $f' = f_0 \cup f_1 \cup f'_2$ representing $(\alpha_1, \dots, \alpha_m, \gamma)$. Note that for any c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ that $\forall^*_{v''} \alpha_1, \dots, \alpha_m, \gamma \; \exists f_0, f_1 \; \exists f'_2 \colon \mathrm{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{U}'$ representing $(\alpha_1, \dots, \alpha_m, \gamma)$ and such that f'_2 induces $f_2 \colon \mathrm{dom}(<^{\mathcal{U}}) \to C$. Suppose the claim fails, fix a c.u.b. C to the contrary, and let C' be the set of closure points of C. Let $\langle j_1, q_1, \dots, q_{k-1}, j_k \rangle$ be the unique sequence not ending in 0 in $\mathrm{dom}(\mathcal{U}') - \mathrm{dom}(\mathcal{U})$. For $(\alpha_1, \dots, \alpha_m, \gamma)$ of semi-type $\mathcal{C}'$ represented by f_0, f_1, f'_2 with f'_2 inducing $f_2 \colon \mathrm{dom}(<^{\mathcal{U}}) \to C'$ of type $\mathcal{C}$, we define $\tilde{f}_2 \colon \mathrm{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ as follows: $\tilde{f}_2$ also induces f_2, and for $\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k \rangle$ of type $\langle j_1, q_1, \dots, q_{k-1}, j_k \rangle$ we set $\tilde{f}_2(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k \rangle) =$ the largest element of C in the interval $(f'_2(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k - 1 \rangle), f'_2(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k \rangle)]$, if there is an element of C in this interval, and otherwise $= f'_2(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k \rangle)$. Thus, f'_2 is of semi-type $\mathcal{C}'$. Let A have measure one with respect to v'' and such that for $(\alpha_1, \dots, \alpha_m, \gamma) \in A$ and f_0, f_1, f'_2 representing $(\alpha_1, \dots, \alpha_m, \gamma)$, $f'_2(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k \rangle) > N_C(f(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k - 1 \rangle))$ almost everywhere (with respect to $M^{q_{k-1}}$). This is possible from the nonexistence of G. Consider the map H' which assigns to $(\alpha_1, \dots, \alpha_m, \gamma)$ the value δ, where for f_0, f_1, f'_2 representing $(\alpha_1, \dots, \alpha_m, \gamma)$, f_0, f_1, $\tilde{f}_2$ represent $(\alpha_1, \dots, \alpha_m, \delta)$. Then $H' < H$ on A. Clearly H' satisfies (1) and (3), and the nonexistence of G also easily shows that (2) is satisfied. This contradicts the minimality of H.

Fix an immediate extension p_{m+1} of p_m extending $\mathcal{C}'$ so that $v'' = M^{p_{m+1}}$. A routine partition argument [partitioning functions $f'_2 \colon \mathrm{dom}(<^{p_{m+1}}) \to \boldsymbol{\delta}^1_3$ of type p_{m+1} with the extra value $g(\langle j_1, \beta_1, \dots, j_k \rangle)$ of uniform cofinality ω inserted after $f'_2(\langle j_1, \beta_1, \dots, j_k \rangle)$] shows that there is a function $s \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ such that for almost all f_0, f_1, f'_2 of type p_{m+1} representing $(\alpha_1, \dots, \alpha_m, \gamma)$,

$$\gamma' \geq \sup\{l_r(\epsilon) \colon H(\alpha_1, \dots, \alpha_m, \epsilon) \leq \gamma\},$$

where γ' is represented with respect to $M^{q_{k-1}}$ by

$$(\beta_1, \dots, \beta_{k-1}) \to N_s(f'_2(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k \rangle)),$$

and $l_r(\epsilon)$ is defined by:

$$\forall^*_{K_{t+1}} h_{t+1} \dots \forall^*_{K_a} h_a \; l_r(\epsilon)(h_{t+1}, \dots, h_a) = N_r(\epsilon(h_{t+1}, \dots, h_a)).$$

Lemma 8.2 then shows that there is an $F' \colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ with $[F'] = [F]$ and such that for almost all f_0, f_1, f'_2 of type p_{m+1} representing $(\alpha_1, \dots, \alpha_m, \gamma)$,

$$N_{F'}(F'(\langle \alpha_1, i_1, \dots, \alpha_m, i_m, \gamma, k \rangle)) \geq F(\langle \alpha_1, i_1, \dots, \alpha_m, i_m, \gamma', 0 \rangle),$$

where k is maximal such that $\langle p_1, i_1, \dots, p_m, i_m, p_{m+1}, k\rangle \in \mathrm{dom}(\mathcal{T})$, and γ' is as above. Thus we have produced a function F' and an H satisfying (1)-(3) such that $[F'] = [F]$ and

$$\forall^* f_n\ \forall^* h_1, \dots, h_t\ \Omega(f_n, h_1, \dots, h_t) < N_{F'}(F'(\langle \alpha_1, i_1, \dots, \alpha_m, i_m, H(\alpha_1, \dots, \alpha_m, \gamma, k)\rangle)),$$

where

$$\alpha_i = h(f_n; d_i; \vec{h}; K^i_{t+1}, \dots, K^i_{t_2}), \text{ and } \gamma = h(f_n; \bar{\mathcal{L}}(d^\star); \vec{h}; K_{t+1}, \dots, K_a).$$

It is enough to show then that for some $e \in \omega$ that $\forall^* f_n\ \forall^* h_1, \dots, h_t$ $H(\alpha_1, \dots, \alpha_m, \gamma) = \delta$, with γ as above and $\delta = h(f_n; \bar{\mathcal{L}}^{(e)}(d^\star); \vec{h}; K_{t+1}, \dots, K_a)$. To see this, let $e \geq 1$ be maximal such that $\forall^* f_n\ \forall^* h_1, \dots, h_t\ \delta \geq H(\alpha_1, \dots, \alpha_m, \gamma)$, with γ, δ as above. Assume strict inequality holds almost everywhere. From theorem 3.1 it follows that there is a function $g\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ such that $\forall^* f_n\ \forall^* h_1, \dots, h_t\ H(\alpha_1, \dots, \alpha_m, \gamma) < l_g(\epsilon)$, with γ as above, and $\epsilon = h(f_n; \bar{\mathcal{L}}^{(e+1)}(d^\star); h_1, \dots, h_t; K_{t+1}, \dots, K_a)$. Note that $\bar{\mathcal{L}}^{(e+1)}(d^\star)$ is well-defined, as otherwise $M^{p_{m+1}}$ concentrates on tuples $(\alpha_1, \dots, \alpha_m, \gamma)$ for $\gamma \leq \gamma_0$ for some fixed $\gamma_0 < \boldsymbol{\delta}^1_3$, a contradiction. For fixed f_0, $f_1\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$, consider the partition $\mathcal{P}(f_0, f_1)$: we partition $f_2'\colon \mathrm{dom}(<^{\mathcal{U}'}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{U}'$ with the extra value $\theta(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k\rangle)$ of uniform cofinality ω inserted between $f_2'(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k\rangle)$ and $N_{f_2'}(f_2'(\langle j_1, \beta_1, \dots, \beta_{k-1}, j_k\rangle))$ according to whether $[\theta]_{M^{q_{k-1}}} > l_g([f_2'^{\langle j_1, \dots, j_k\rangle}]_{M^{q_{k-1}}})$. On the homogeneous side of the partition the stated property holds, and in fact there is a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ homogeneous for $\mathcal{P}(f_0, f_1)$ for almost all f_0, f_1. Now, $\forall^* f_n\ \forall^* h_1, \dots, h_t\ H(\alpha_1, \dots, \alpha_m, \gamma) = [f_2'^{\langle j_1, \dots, j_k\rangle}]_{M^{q_{k-1}}}$ for some $f_2'\colon \mathrm{dom}(<^{\mathcal{U}'}) \to C'$ of type p_{m+1}. However, $\epsilon = h(f_n; \bar{\mathcal{L}}^{(e+1)}(d^\star); h_1, \dots, h_t; K_{t+1}, \dots, K_a) < H(\alpha_1, \dots, \alpha_m, \gamma)$. Since f_2' has range in C', it follows from the definition of C and monotonicity of l_g that $l_g(\epsilon) < H(\alpha_1, \dots, \alpha_m, \gamma)$ as well, a contradiction.

$\square$

Finally, we extend to the current setting the notion of complex (of type 3). As before, the complexes will generate arbitrary measures, this time on $(\boldsymbol{\delta}^1_5)^-$.

Definition 8.8. A *type-3 complex* is an object $\mathcal{E}$ of the form

$$\mathcal{E} = \langle (f_n; K_1, \dots, K_i; K_{i+1}, \dots, K_t), \mathcal{T}, (d_i, \dots, d_t), (x_i, \dots, x_{t-1})\rangle$$

satisfying:

(1) $n \in \omega$, $K_1, \dots, K_t \in W_1 \cup S_1$.
(2) $\mathcal{T}$ is a type-3 tree of uniform cofinalities.
(3) $d_i = d_i^{(f_n; K_1, \dots, K_i)}, \dots, d_t = d_t^{(f_n; K_1, \dots, K_t)}$ are generalized type-3 descriptions defined relative to $\mathcal{T}$. Also, $d_i > \cdots > d_t$.
(4) $T_{x_i}, \dots, T_{x_t}$ are well-founded where T is the Martin tree of §2.

For $\mathcal{E}$ a complex (with the above notation), and $F\colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type T, we define an ordinal $h(F; \mathcal{E})$. This is represented with respect to W^n_3 by the function which assigns to $[f_n]$, for $f_n\colon \omega_{n+1} \to \boldsymbol{\delta}^1_3$ of the correct type, the value $h(F; \mathcal{E}; f_n)$. This, in turn is represented with respect to $K_1, \dots, K_i$ in the usual manner by $h(F; \mathcal{E}; f_n; h_1, \dots, h_i)$. Then, $h(F; \mathcal{E}; f_n; h_1, \dots, h_i) = |T_{x_i} \restriction \sup_j j(h(F; d_i; f_n; h_1, \dots, h_i))(\alpha_{i+1})|$, where α_{i+1} is represented with respect to K_{i+1}

by $\alpha_{i+1}(h_{i+1}) = |T_{x_{i+1}} \restriction \sup_j j(h(F;d_{i+1};f_n;h_1,\dots,h_{i+1}))(\alpha_{i+2})|,\dots$, and where $\alpha_t(h_t) = h(F;d_t;f_n;h_1,\dots,h_t)$.

If $\mathcal{E}$ is a type-3 complex, $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ is of type $\mathcal{T}$, and $\delta \in ON$ with $\delta < h(F;d_t)$, we also define the ordinal $h(F;\mathcal{E};\delta)$. This is defined just as $h(F;\mathcal{E})$ except we use $\alpha_t(h_t) = \delta(f_n,h_1,\dots,h_t)$ in the above definition.

A *partial type-3 complex* $\mathcal{E}'$ is defined just as a type-3 complex except the underlying tree $\mathcal{T}'$ is now a partial type-3 tree of uniform cofinalities. For $F\colon \operatorname{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}^1_3$, we define $h(F;\mathcal{E}')$ and $h(F;\mathcal{E}';\delta)$ just as for complexes.

For each type-3 complex $\mathcal{E}$ and measure μ on $\boldsymbol{\delta}^1_3$ we define a measure $\mathcal{V}^{\mathcal{E},\mu}$ on $(\boldsymbol{\delta}^1_5)^-$.

DEFINITION 8.9. Let $\mathcal{E}$ be a type-3 complex and μ a measure on $\boldsymbol{\delta}^1_3$. We define A to have $\mathcal{V}^{\mathcal{E},\mu}$ measure one if $\forall^*_\mu \delta\ \exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3\ \forall F\colon \operatorname{dom}(<^{\mathcal{T}}) \to C$ of type $\mathcal{T}$, $h(F;\mathcal{E};\delta) \in A$.

We now state the main result of this section.

THEOREM 8.3. *For any measure $\mathcal{V}$ on $(\boldsymbol{\delta}^1_5)^-$ there is a type-3 complex $\mathcal{E} = \langle (f_n;K_1,\dots,K_i;K_{i+1},\dots,K_t),\mathcal{T},(d_i,\dots,d_t),(x_i,\dots,x_{t-1})\rangle$ and a measure μ on $\boldsymbol{\delta}^1_3$ such that $\mathcal{V} = \mathcal{V}^{\mathcal{E},\mu}$.*

Fix the measure $\mathcal{V}$ for the remainder of the argument. We give the first step of the analysis directly first. Let $\theta < (\boldsymbol{\delta}^1_5)^-$ be the least ordinal such that $\mathcal{V}(\theta) = 1$. We may assume $\theta > \boldsymbol{\delta}^1_3$. Suppose first that $\theta = (id;d';K_1,\dots,K_i)$ for some description d' and measures $K_1,\dots,K_i \in W_1 \cup S_1$. Theorem 3.1 then gives a description $d = d^{(f_n}$ such that d is defined and satisfies condition D relative to $K_1,\dots,K_i$, and for all $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ we have that $h(F;d;K_1,\dots,K_i) < \theta$, and also $\forall \alpha < \theta\ \exists F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with $\alpha < h(F;d;K_1,\dots,K_i)$. Let p_1 be the minimal non-empty type ≤ 2 tree of uniform cofinalities corresponding to the cofinality κ ($= \omega$, ω_1, or ω_2) such that $\forall^* f_n\ \forall^* h_1,\dots,h_i\ h(f_n;d;h_1,\dots,h_i)$ has cofinality κ. Let $\mathcal{T}'$ be the partial tree of uniform cofinalities with domain $\{\langle p_1,1\rangle\} \cup \bigcup_{p_a} \{\langle p_a,0\rangle\}$,that is, $\mathcal{T}'$ has $\langle p_1,1\rangle$ as the only sequence not ending in 0 in its domain (and $\mathcal{T}'(\langle p_1,0\rangle) =$ "undefined"). Note that an equivalence class of a function $F\colon \operatorname{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{T}'$ may be identified with the equivalence class of a function $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to the κ-cofinal normal measure on $\boldsymbol{\delta}^1_3$. Let $d_i = \langle d,1\rangle$, and let $\mathcal{E}'$ be the partial complex $\langle (f_n;K_1,\dots,K_i),\mathcal{T}';(d_i),\emptyset\rangle$. Note that $\forall^* f_n\ \forall^* h_1,\dots,h_i\ h(F;\mathcal{E}')(f_n,h_1,\dots,h_i) = F(\langle h(f_n;d;h_1,\dots,h_i),1\rangle)$. Thus, $\forall \alpha < \theta\ \exists F\ \alpha < h(F;\mathcal{E}')$, and for all F, $h(F;\mathcal{E}') < \theta$.

Suppose now that θ is not of the above form. In particular, θ is not a cardinal. Let d', $K_1,\dots,K_i$ be such that $(id;d';K_1,\dots,K_i) = |\theta|^+$. Then by theorem 3.1 again, there is a description d, a p_1 as above, and an $F_1\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ such that $\forall \alpha < \theta$ $\exists F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with $F < F_1$ almost everywhere with respect to M^{p_1} = the κ-cofinal normal measure on $\boldsymbol{\delta}^1_3$ such that $\alpha < h(F;d;K_1,\dots,K_i)$. Fix such an F_1, and fix x_i such that T_{x_i} is well-founded and $|T_{x_i} \restriction \sup_j j(\alpha)| > F_1(\alpha)$ on a c.u.b. set. For $\alpha < \theta$, define $Q(\alpha) < (\sup_j j;d;K_1,\dots,K_i)$ by:

$$\forall^* f_n\ \forall^* h_1,\dots,h_i\ \alpha(f_n,\vec{h}) = |T_{x_i} \restriction \sup_j j(h(f_n;d;\vec{h}))(Q(\alpha)(f_n,\vec{h}))|.$$

Let $\mathcal{V}_2 = Q(\mathcal{V})$, and by countable additivity let K_{i+1} be such that $\mathcal{V}_2$ concentrates on $(id; d; K_1, \ldots, K_i, K_{i+1})$. If $\mathcal{V}_2$ concentrates on a smaller ordinal, we repeat the argument working with $\mathcal{V}_2$. Continuing, by well-foundedness we eventually produce reals $(x_i, \ldots, x_{j-1})$, descriptions $(d'_i, \ldots, d'_j)$, measures $(K_i, \ldots, K_j)$, and a minimal type ≤ 2 tree of uniform cofinalities which we now call p_1 (corresponding to the κ-cofinal normal measure on $\boldsymbol{\delta}^1_3$) such that

(i) $\forall^* f_n\ \forall^* h_1, \ldots, h_i, \ldots, h_j\ h(f_n; d'_j; h_1, \ldots, h_i, \ldots, h_j)$ has cofinality κ.
(ii) $\forall^*_{\mathcal{V}} \alpha < \theta\ \exists F \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3\ \alpha < h(F; \mathcal{E}'_j)$.
(iii) $\forall F \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3\ \neg \forall^*_{\mathcal{V}} \alpha < \theta\ \exists [F']_{M^{p_1}} \leq [F]_{M^{p_1}}\ \alpha < h(F'; \mathcal{E}'_j)$.

where $\mathcal{E}'_j = \langle (f_n; K_1, \ldots, K_i; K_{i+1}, \ldots, K_j), \mathcal{T}', (d_i, \ldots, d_j), (x_i, \ldots, x_{j-1}) \rangle$, $\mathcal{T}'$ is the partial type-3 tree having $\langle p_1, 1 \rangle$ as the only sequence not ending in 0 in its domain, $d_k = \langle d'_k, 0 \rangle$ for $k = 1, \ldots, j-1$, and $d_j = \langle d'_j, 1 \rangle$. Note that for $k = 1, \ldots, j-1$, $h(F; f_n; d_k; h_1, \ldots, h_k) = h(f_n; d'_k; h_1, \ldots, h_k)$ for any $F \colon \operatorname{dom}(<^{\mathcal{T}'})\ \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{T}'$, since $F(\langle \alpha, 0 \rangle) = \alpha$ almost everywhere. For $\mathcal{V}$ almost all α, let $r_j(\alpha)$ be such that $\exists F \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3\ r_j(\alpha) \leq h(F; d'_j; K_1, \ldots, K_j)$ and also $\alpha = h(F; \mathcal{E}'_j; r_j(\alpha))$. [Note that $h(F; \mathcal{E}'_j; r_j(\alpha))$ does not depend on F; this last equation is equivalent to $\forall^* f_n\ \forall^* h_1, \ldots, h_i\ \alpha(f_n, h_1, \ldots, h_i) = |T_{x_i} \restriction \sup_j j(h(F; d'_i; h_1, \ldots, h_i))(\alpha_{i+1})|$, where α_{i+1} is represented with respect to K_{i+1} by $\alpha_{i+1}(h_{i+1}) = |T_{x_{i+1}} \restriction \sup_j j(h(F; d'_{i+1} h_1, \ldots, h_{i+1}))(\alpha_{i+2})|$, etc., and $\alpha_j(h_j) = r_j(\alpha)(f_n, h_1, \ldots, h_j)$.]

Since the first case above is an instance of the second case, we adopt the notation of the second case. Let now $g : \theta \to ON$ represent the least $\mathcal{V}$ equivalence class which has a representative satisfying:

(1) For $\alpha < \theta$, $g(\alpha)$ is represented by a function $F \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to $M^{p_1} =$ the κ-cofinal normal measure on $\boldsymbol{\delta}^1_3$ (we may also regard F as being a function from $\operatorname{dom}(<^{\mathcal{T}'})$ into $\boldsymbol{\delta}^1_3$ of semi-type $\mathcal{T}'$).
(2) There is a $\mathcal{V}$ measure one set $A \subseteq \theta$ such that for $\alpha, \beta \in A$, if $r_j(\alpha) \leq r_j(\beta)$, then $g(\alpha) \leq g(\beta)$.
(3) There does not exist a $\mathcal{V}$ measure one set A and an F_0 such that $g(\alpha) \leq [F_0]$ for $\alpha \in A$.

Such a g exists since the function $\tilde{g}$ satisfies (1)-(3), where $\tilde{g}(\alpha) =$ least equivalence class of a function F such that $r_j(\alpha) \leq h(F; d_j; K_1, \ldots, K_j)$. By countable additivity we may fix a uniform cofinality c with respect to M^{p_1} such that $\forall^*_{\mathcal{V}} \alpha < \theta$ if $F \colon \operatorname{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{T}'$ represents $g(\alpha)$, then F has uniform cofinality c almost everywhere with respect to M^{p_1}. Let $\mathcal{T}$ be the tree of uniform cofinalities which extends $\mathcal{T}'$ by $\mathcal{T}(\langle p_1, 1 \rangle) = c$. Let A be a measure one set on which (2) holds. Consider then the following partition $\mathcal{P}$: we partition $F, G \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with $F(\alpha) < G(\alpha) < F(\alpha + 1)$ for all α, and F of uniform cofinality c, G of uniform cofinality ω, according to whether $h(G; d_j; K_1, \ldots, K_j) > \sup\{r_j(\alpha) \colon \alpha \in A \wedge g(\alpha) \leq [F]\}$. By (2), (3) and standard arguments, it follows that on the homogeneous side of the partition the property stated in $\mathcal{P}$ holds. Fix a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ homogeneous for $\mathcal{P}$, and let x_j be such that T_{x_j} is well-founded and $|T_{x_j} \restriction \sup_j j(\alpha)| > N_C(\alpha)$ holds on a c.u.b. set. By countable additivity we may fix a measure $K_{j+1} \in W_1 \cup S_1$ such that $\forall^*_{\mathcal{V}} \alpha < \theta\ \exists \beta \in ON$ satisfying:

(a) $\beta < h(F_{g(\alpha)}; d_j; K_1, \ldots, K_i, \ldots, K_j, K_{j+1})$, where $F_{g(\alpha)}\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ represents $g(\alpha)$. That is,

$$\forall^* f_n\ \forall^* h_1, \ldots, h_i, \ldots, h_j\ \beta(f_n, h_1, \ldots, h_j) < j_{K_{j+1}}(F(\langle \alpha_1, 1\rangle)),$$

where $\alpha_1 = h(f_n; d'_j; h_1, \ldots, h_j)$.

(b) $\alpha = h(F_{g(\alpha)}; \mathcal{E}; \beta)$, where

$$\mathcal{E} = \langle (f_n; K_1, \ldots, K_i; K_{i+1}, \ldots, K_{j+1}), \mathcal{T}, (d_i, \ldots, d_j, d_{j+1}), (x_i, \ldots, x_j)\rangle,$$

where $d_{j+1} = d_j = \langle d'_j, 1\rangle$.

Let $r(\alpha)$ be the unique value of β satisfying (a) and (b) above. Thus we have specified a complex $\mathcal{E}$ and functions $g, r\colon \theta \to ON$ satisfying the following:

(1) $g(\mathcal{V}) = M^{\mathcal{T}}$ = the measure induced by the strong partition relation on $\boldsymbol{\delta}^1_3$ and functions $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$.

(2) $\forall^*_{\mathcal{V}} \alpha < \theta\ r(\alpha) < h(F_{g(\alpha)}; d_{j+1}; K_1, \ldots, K_{j+1})$, where $F_{g(\alpha)}\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ represents $g(\alpha)$.

(3) $\forall^*_{\mathcal{V}} \alpha < \theta\ \alpha = h(F_{g(\alpha)}; \mathcal{E}; r(\alpha))$.

This again leads us to the definition of a situation.

DEFINITION 8.10. A *situation* for the measure $\mathcal{V}$ is a triple of the form $\mathcal{S} = (\mathcal{E}, g, r)$ where $\mathcal{E}$ is a type-3 complex,

$$\mathcal{E} = \langle (f_n; K_1, \ldots, K_i; K_{i+1}, \ldots, K_t), \mathcal{T}, (d_i, \ldots, d_t), (x_i, \ldots, x_{t-1})\rangle,$$

$g, r\colon \theta \to ON$, and:

(1) $g(\mathcal{V}) = M^{\mathcal{T}}$ = the measure induced from the strong partition relation on $\boldsymbol{\delta}^1_3$ and functions $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ (and the measures M^{p_m} for $\langle p_1, i_1, \ldots, i_{m-1}, p_m, i_m\rangle \in \operatorname{dom}(\mathcal{T})$).

(2) $\forall^*_{\mathcal{V}} \alpha < \theta\ r(\alpha) < h(F_{g(\alpha)}; d_t; K_1, \ldots, K_t)$, where $F_{g(\alpha)}\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ represents $g(\alpha)$.

(3) $\forall^*_{\mathcal{V}} \alpha < \theta\ \alpha = h(F_{g(\alpha)}; \mathcal{E}; r(\alpha))$.

Among all situations for the measure $\mathcal{V}$ we now select one such that the function $\alpha \to h(F_{g(\alpha)}; d_t; K_1, \ldots, K_t)$ is minimal. We denote this situation by $(\mathcal{E}, g, r)$ with $\mathcal{E}$ as in the previous definition.

CLAIM 3. $\forall^*_{\mathcal{V}} \alpha < \theta\ r(\alpha) < \boldsymbol{\delta}^1_3$.

Granting this claim, we finish the proof of theorem 8.3 as follows. Let $\mu = r(\mathcal{V})$, so μ is a measure on $\boldsymbol{\delta}^1_3$. We then claim that $\mathcal{V} = \mathcal{V}^{\mathcal{E}, \mu}$, that is, $\mathcal{V}(A) = 1$ iff $\forall^*_\mu \delta < \boldsymbol{\delta}^1_3\ \exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3\ \forall F\colon \operatorname{dom}(<^{\mathcal{T}}) \to C$ of type $\mathcal{T}$ $[h(F; \mathcal{E}; \delta) \in A]$. If not, then fix $A \subseteq \theta$ with $\mathcal{V}(A) = 1$ and $D \subseteq \boldsymbol{\delta}^1_3$ with $\mu(D) = 1$ such that $\forall \delta \in D$ $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3\ \forall F\colon \operatorname{dom}(<^{\mathcal{T}}) \to C$ of type $\mathcal{T}$ $h(F; \mathcal{E}; \delta) \notin A$. Consider then the partition $\mathcal{P}$: we partition $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ according to whether $\forall \delta < \inf(F)\ [(\delta \in D) \to h(F; \mathcal{E}; \delta) \notin A]$. It is not difficult to check that on the homogeneous side of the partition the stated property holds. Fix a c.u.b. C homogeneous for $\mathcal{P}$. Since $g(\mathcal{V}) = M^{\mathcal{T}}$ and $r(\mathcal{V}) = \mu$, $\exists \alpha \in A\ [(r(\alpha) \in D) \wedge (g(\alpha)$ is represented by a function $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to C$ of type $\mathcal{T})]$. Fix such α, F. So, $\alpha = h(F; \mathcal{E}; r(\alpha)) \in A$. However, an easy special case of lemma 8.2 produces an $F'\colon \operatorname{dom}(<^{\mathcal{T}}) \to C$ of type $\mathcal{T}$ with $[F'] = [F]$ and $\inf(\operatorname{ran} F') > r(\alpha)$. Hence $h(F; \mathcal{E}; r(\alpha)) = h(F'; \mathcal{E}; r(\alpha)) \notin A$, a contradiction.

The remainder of the proof consists of establishing the above claim. Using lemmas 8.2 and 8.3 the arguments are quite similar to those of theorem 5.2.

If $\mathcal{L}(d_t)$ is not defined, it follows immediately from lemma 8.3 that $\forall^*_{\mathcal{V}}\alpha\; r(\alpha) < \boldsymbol{\delta}^1_3$. So, assume $\mathcal{L}(d_t)$ is defined. To fix notation, say $d_t = \langle d^t_1, i_1, \dots, d^t_m, i_m\rangle$ has type $\langle p_1, i_1, \dots, p_m, i_m\rangle \in \mathrm{dom}(\mathcal{T})$.

We define a partial extension $\mathcal{T}'$ of $\mathcal{T}$ as follows (we refer the reader to the definition of $\mathcal{L}(d)$ before lemma 8.3). If $i_m = 0$ and $\mathcal{L}(d_t) = \langle d^t_1, i_1, \dots, i_{m-1}, \mathcal{L}^{(e)}(d^t_m), k\rangle$, then $\mathcal{T}'$ has the extra sequence $\vec{s} = \langle p_1, i_1, \dots, i_{m-1}, q, k+1\rangle$ in its domain, where $\mathcal{L}(d_t)$ has type $\langle p_1, i_1, \dots, i_{m-1}, q, k\rangle$. If $i_m = 0$ and $\mathcal{L}(d_t) = \langle d^t_1, i_1, \dots, i_{m-1} - 1\rangle$, then $\mathcal{T}'$ has the extra sequence $\langle p_1, i_1, \dots, i_{m-1}\rangle$ in its domain inserted between $\langle p_1, i_1, \dots, i_{m-1} - 1\rangle \in \mathrm{dom}(\mathcal{T})$ and $\langle p_1, i_1, \dots, i_{m-1}\rangle \in \mathrm{dom}(\mathcal{T})$. Similarly, if $\mathcal{L}(d_t) = \langle d^t_1, i_1, \dots, i_m - 1\rangle$, then $\mathcal{T}'$ has the extra sequence $\langle p_1, i_1, \dots, i_m\rangle$ in its domain inserted between $\langle p_1, i_1, \dots, i_m - 1\rangle \in \mathrm{dom}(\mathcal{T})$ and $\langle p_1, i_1, \dots, i_m\rangle \in \mathrm{dom}(\mathcal{T})$. Finally, if $\mathcal{L}(d_t) = \langle d^t_1, i_1, \dots, i_m, \mathcal{L}^{(e)}(d^\star), k\rangle$, then $\mathcal{T}'$ has the extra sequence $\vec{s} = \langle p_1, i_1, \dots, i_m, q, k+1\rangle$ in its domain, where $\mathcal{L}(d_t)$ has type $\langle p_1, i_1, \dots, i_m, q, k\rangle$.

Let $\tilde{d}_t$ be the generalized type-3 description defined as follows. If $\mathcal{L}(d_t) = \langle d^t_1, i_1, \dots, i_{m-1}, \mathcal{L}^{(e)}(d^t_m), k\rangle$, then $\tilde{d}_t = \langle d^t_1, i_1, \dots, i_{m-1}, \mathcal{L}^{(e)}(d^t_m), k+1\rangle$. If $\mathcal{L}(d_t) = \langle d^t_1, i_1, \dots, i_{m-1} - 1\rangle$, then $\tilde{d}_t = \langle d^t_1, i_1, \dots, i_{m-1}\rangle$. If $\mathcal{L}(d_t) = \langle d^t_1, i_1, \dots, i_m - 1\rangle$, $\tilde{d}_t = \langle d^t_1, i_1, \dots, i_m\rangle$. Finally, if $\mathcal{L}(d_t) = \langle d^t_1, i_1, \dots, i_m, \mathcal{L}^{(e)}(d^\star), k\rangle$, then $\tilde{d}_t = \langle d^t_1, i_1, \dots, i_m, \mathcal{L}^{(e)}(d^\star), k+1\rangle$. Thus, in all cases $\tilde{d}_t$ corresponds to the extra sequence in $\mathrm{dom}(\mathcal{T}') - \mathrm{dom}(\mathcal{T})$.

To unify notation, let $\vec{s} = \langle p'_1, i'_1, \dots, p'_l, i'_l\rangle$ be the unique sequence not ending in 0 in $\mathrm{dom}(\mathcal{T}') - \mathrm{dom}(\mathcal{T})$. Note that in all cases $\mathcal{L}(d_t)$ has type $\vec{t} = \langle p'_1, i'_1, \dots, p'_l, i'_l - 1\rangle$.

Now, for $\mathcal{V}$ almost all $\alpha < \theta$, $g(\alpha)$ is represented by a function $F_{g(\alpha)} \colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$, and $r(\alpha) < h(F_{g(\alpha)}; d_t; K_1, \dots, K_t)$. Recall this means that

$$
\begin{aligned}
\forall^* f_n\; \forall^* h_1, \dots, h_t\; r(\alpha)(f_n, h_1, \dots, h_t) &< h(F_{g(\alpha)}; d_t; f_n; h_1, \dots, h_t) \\
&= F_{g(\alpha)}(\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m\rangle)
\end{aligned}
$$

where $\alpha_i = h(d_i; h_1, \dots, h_t; K^i_{t+1}, \dots, K^i_{a_i})$. We apply lemma 8.3 with $\Omega = r(\alpha)$.

From lemmas 8.2, 8.3 it follows that $\forall^*_{\mathcal{V}}\alpha\; \exists F' \colon \mathrm{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{T}'$ inducing $F \colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ representing $g(\alpha)$ and such that $r(\alpha) < h(F'; \tilde{d}_t)$.

<u>Case I</u>. There is a measure one set $A \subseteq \theta$ with respect to $\mathcal{V}$ such that for $\alpha \in A$, there is an $F' \colon \mathrm{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{T}'$ inducing $F \colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ representing $g(\alpha)$, and such that for all $\beta \in A$ with $g(\beta) = [F]$, $r(\beta) < h(F'; \tilde{d}_t)$.

Fix such a measure one set A. We consider the partition $\mathcal{P}$: we partition $F' \colon \mathrm{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}^1_3$ of semi-type $\mathcal{T}'$ with $F^{\vec{s}}$ of uniform cofinality ω, according to whether $\sup\{r(\beta) \colon g(\beta) \le [F] \wedge \beta \in A\} < h(F'; \tilde{d}_t) = h(N_{F'}; F; \mathcal{L}(d_t))$, where F' induces $F \colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$. By hypotheses and lemma 8.2, on the homogeneous side of the partition the stated property holds. Let C be homogeneous for $\mathcal{P}$, and C' the set of limit points of C. Thus, for almost all $\alpha < \theta$ and $F_{g(\alpha)} \colon \mathrm{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ representing $g(\alpha)$ we have $r(\alpha) < h(N_{C'}; F_{g(\alpha)}; \mathcal{L}(d_t))$. Let x_t be a real with T_{x_t} well-founded and $|T_{x_t} \restriction \sup_j j(\gamma)| > N_{C'}(\gamma)$ on a c.u.b. set. It follows that for almost all $\alpha < \theta$ and $F_{g(\alpha)}$ representing $g(\alpha)$, there is an ordinal $r'(\alpha)$ with $r'(\alpha) < h(\sup_j j; F_{g(\alpha)}; \mathcal{L}(d_t))$ and such that $\forall^* f_n \; \forall^* h_1, \ldots, h_t$ $r(\alpha)(f_n, \vec{h}) = |T_{x_t} \restriction \sup_j j(h(F_{g(\alpha)}; \mathcal{L}(d_t); f_n; \vec{h}))(r'(\alpha)(f_n, \vec{h}))|$. Fix K_{t+1} such that $r'(\alpha) < h(F_{g(\alpha)}; \mathcal{L}(d_t); W_n^3, K_1, \ldots, K_t, K_{t+1})$ almost everywhere. Thus,

$$\forall^*_{\mathcal{V}} \alpha < \theta \; \forall^* f_n \; \forall^* h_1, \ldots, h_i \; \alpha(f_n, h_1, \ldots, h_i) = \\ |T_{x_i} \restriction \sup_j j(h(F_{g(\alpha)}; d_i; f_n; h_1, \ldots, h_i))(\alpha_{i+1})|,$$

..., etc., and

$$\alpha_{t-1}(h_{t-1}) = |T_{x_{t-1}} \restriction \sup_j j(h(F_{g(\alpha)}; d_{t-1}; f_n; h_1, \ldots, h_{t-1}))(\alpha_t)|,$$

$$\alpha_t(h_t) = |T_{x_t} \restriction (h(F_{g(\alpha)}; \mathcal{L}(d_t); f_n; h_1, \ldots, h_t))(\alpha_{t+1})|,$$

where $\alpha_{t+1}(h_{t+1}) = r'(\alpha)(f_n, h_1, \ldots, h_t, h_{t+1}))$. If we let

$$\mathcal{E}' = \langle (f_n; K_1, \ldots, K_i; K_{i+1}, \ldots, K_t, K_{t+1}), \mathcal{T}, \\ (d_1, \ldots, d_{t-1}, \mathcal{L}(d_t), \mathcal{L}(d_t)), (x_i, \ldots, x_{t-1}, x_t) \rangle,$$

then the situation $\mathcal{S}' = (\mathcal{E}', g, r')$ violates the minimality of $\mathcal{S}$.

Case II. There does not exist a measure one set $A \subseteq \theta$ as in case (I).

Note that this is equivalent to assuming that for any $\mathcal{V}$ measure one set $A \subseteq \theta$, for $\mathcal{V}$ almost all $\alpha \in A$ if $F'\colon \operatorname{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}_3^1$ of semi-type $\mathcal{T}'$ induces $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}_3^1$ of type $\mathcal{T}$ representing $g(\alpha)$, then $\exists \alpha' \in A$ such that $g(\alpha') = g(\alpha)$ and $r(\alpha') > h(F'; \tilde{d}_t)$.

Let $g'\colon \theta \to ON$ represent the least $\mathcal{V}$ equivalence class having a representative satisfying:

1. (1) $\forall^*_{\mathcal{V}} \alpha < \theta \; g'(\alpha)$ is represented by a function $F'\colon \operatorname{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}_3^1$ of semi-type $\mathcal{T}'$ inducing $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}_3^1$ of type $\mathcal{T}$ representing $g(\alpha)$.
2. (2) There is a $\mathcal{V}$ measure one set $A \subseteq \theta$ such that for $\alpha_1, \alpha_2 \in A$ with $g(\alpha_1) = g(\alpha_2)$, if $r(\alpha_1) \le r(\alpha_2)$ then $g'(\alpha_1) \le g'(\alpha_2)$.
3. (3) There does not exist a $\mathcal{V}$ measure one set $A \subseteq \theta$ such that $\forall^*_{\mathcal{V}} \alpha \in A$ $\exists F'\colon \operatorname{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}_3^1$ of semi-type $\mathcal{T}'$ inducing $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}_3^1$ of type $\mathcal{T}$ representing $g(\alpha)$ and such that $\sup\{g'(\alpha') : \alpha' \in A \wedge g(\alpha') \le g(\alpha)\} < [F']$.

We easily have that g' exists since if we define $\tilde{g}(\alpha) =$ the equivalence class of the least $F'\colon \operatorname{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}_3^1$ of semi-type $\mathcal{T}'$ inducing $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}_3^1$ of type $\mathcal{T}$ representing $g(\alpha)$ and such that $r(\alpha) < h(F'; \tilde{d}_t)$, then $\tilde{g}$ satisfies (1)-(3).

We claim that for any c.u.b. $C \subseteq \boldsymbol{\delta}_3^1$ that $\forall^*_{\mathcal{V}} \alpha < \theta \; g'(\alpha)$ is represented by a function $F'\colon \operatorname{dom}(<^{\mathcal{T}'}) \to C$ of semi-type $\mathcal{T}'$. If not, fix C to the contrary, and let C' be the set of closure points of C. Let $A \subseteq \theta$ be the measure one set of α such that for $\alpha \in A$, $g'(\alpha)$ is represented by a function $F'\colon \operatorname{dom}(<^{\mathcal{T}'}) \to \boldsymbol{\delta}_3^1$ of semi-type $\mathcal{T}'$ inducing $F\colon \operatorname{dom}(<^{\mathcal{T}}) \to C'$ of type $\mathcal{T}$, and for almost all $(\alpha_1, \ldots, \alpha_l)$ with respect to $M^{p'_l}$ we have $F'(\langle \alpha_1, i'_1, \ldots, \alpha_l, i'_l \rangle \notin C$ (recall $\vec{s} = \langle p'_1, i'_1, \ldots, p'_l, i'_l \rangle$ is the extra sequence not ending in 0 in $\operatorname{dom}(\mathcal{T}') - \operatorname{dom}(\mathcal{T})$). Let $A' \subseteq A$ be those $\alpha \in A$ such that if F' represents $g'(\alpha)$, then $\forall^*_{M^{p'_l}} \alpha_1, \ldots, \alpha_l \; F'(\langle \alpha_1, i'_1, \ldots, \alpha_l, i'_l \rangle) > N_C(F(\langle \alpha_1, i'_1, \ldots, \alpha_l, i'_l - 1 \rangle))$. That A' has measure one follows easily from (3). For $\alpha' \in A'$ and representing function F' as above, define $\tilde{F}\colon \operatorname{dom}(<^{\mathcal{T}'}) \to C$ of semi-type $\mathcal{T}'$ by: $\tilde{F}$ also induces F, and $\forall^*_{M^{p'_l}} \alpha_1, \ldots, \alpha_l \; \tilde{F}(\langle \alpha_1, i'_1, \ldots, \alpha_l, i'_l \rangle) =$ the largest element of C less than $F'(\langle\alpha_1, i'_1, \ldots, \alpha_l, i'_l\rangle)$. It follows easily that $\tilde{g}$ defined by $\tilde{g}(\alpha) = [\tilde{F}]$ is well-defined, satisfies (1)-(3) above, and $\tilde{g} < g'$ almost everywhere, violating the definition of g'.

Let $\mathcal{T}''$ be the type-3 tree of uniform cofinalities completing $\mathcal{T}'$ (so $\mathcal{T}''$ is an immediate extension of $\mathcal{T}$) such that $\forall^*_{\mathcal{V}}\alpha < \theta\; g'(\alpha)$ is represented by $F'\colon \operatorname{dom}(<^{\mathcal{T}''}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}''$. Let $A \subseteq \theta$ be a $\mathcal{V}$ measure set on which (1) and (2) hold. Thus, $g'(\mathcal{V}) = M^{\mathcal{T}''}$. Consider then the partition $\mathcal{P}$: we partition F' of type $\mathcal{T}''$ with the extra values $G(\langle\alpha_1, i'_1, \ldots, \alpha_l, i'_l\rangle)$ of uniform cofinality ω inserted after $F'(\langle\alpha_1, i'_1, \ldots, \alpha_l, i'_l\rangle)$ according to whether $\sup\{r(\alpha)\colon \alpha \in A \wedge (g'(\alpha) \leq [F'])\} \leq h(N_G; F'; \tilde{d}_t; K_1, \ldots, K_t)$. Lemma 8.2, property (2) for g', and the hypothesis for this case show that on the homogeneous side of the partition the stated property holds. Let $C \subseteq \boldsymbol{\delta}^1_3$ be a c.u.b. set homogeneous for $\mathcal{P}$, and let x_t be a real with $|T_{x_t} \restriction \sup_j j(\gamma)| > N_C(\gamma)$ on a c.u.b. set. It follows that $\forall^*_{\mathcal{V}}\alpha < \theta\; \exists r'(\alpha) \in ON$ such that if F' of type $\mathcal{T}''$ represents $g'(\alpha)$, then $r'(\alpha) < h(\sup_j j; F'; \tilde{d}_t; K_1, \ldots, K_t)$ and

$$\forall^* f_n\; \forall^* h_1, \ldots, h_t\; r(\alpha)(f_n, h_1, \ldots, h_t) = \\ |T_{x_t} \restriction \sup_j j(h(F'; \tilde{d}_t; h_1, \ldots, h_t))(r'(\alpha)(f_n, h_1, \ldots, h_t))|.$$

Let K_{t+1} be such that $\forall^*_{\mathcal{V}}\alpha < \theta\; r'(\alpha) < h(F'; \tilde{d}_t; K_1, \ldots, K_t, K_{t+1})$. If we then set $\mathcal{E}' =$

$$\langle (f_n; K_1, \ldots, K_i; K_{i+1}, \ldots, K_t, K_{t+1}), \mathcal{T}''; (d_i, \ldots, d_{t-1}, \tilde{d}_t, \tilde{d}_t); (x_i, \ldots, x_{t-1}, x_t)\rangle,$$

it follows that $\mathcal{S}' = (\mathcal{E}'; g', r')$ violates the minimality of $\mathcal{S}$.

This completes the proof of theorem 8.3. Λ

CHAPTER 9

The Weak Partition Relation on $\boldsymbol{\delta}^1_5$

We use the analysis of measures on $(\boldsymbol{\delta}^1_5)^-$, i.e. theorem 8.3, to obtain a coding of the subsets of $(\boldsymbol{\delta}^1_5)^-$ sufficiently good to obtain the weak partition relation on $(\boldsymbol{\delta}^1_5)^-$. The arguments are very similar to those of §6. Again, the idea for converting an analysis of measures to a coding of subsets is due to Kunen. The weak partition relation on $(\boldsymbol{\delta}^1_5)^-$ then easily follows; it is slightly easier here in fact since we do not need to modify the coding of subsets to get the desired coding of functions.

Recall from theorem 8.3 that every measure $\mathcal{V}$ on $(\boldsymbol{\delta}^1_5)^-$ is of the form $\mathcal{V} = \mathcal{V}^{\mathcal{E},\mu}$ for some type-3 complex

$$\mathcal{E} = \langle (f_n; K_1, \dots, K_i; K_{i+1}, \dots, K_t), \mathcal{T}, (d_i, \dots, d_t), (x_i, \dots, x_{t-1}) \rangle$$

and measure μ on $\boldsymbol{\delta}^1_3$. Recall that $\mathcal{V}^{\mathcal{E},\mu}(A) = 1$ iff $\forall^*_\mu \delta < \boldsymbol{\delta}^1_3$ $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ $\forall F\colon \operatorname{dom}(<^{\mathcal{T}}) \to C$ of type $\mathcal{T}$ $h(F; \mathcal{E}, \delta) \in A$, with $h(F; \mathcal{E}, \delta)$ defined after definition 8.8. We adopt the convention that all c.u.b. sets consist of points closed under all the embeddings j corresponding to measures in $W_1 \cup S_1$.

Recall also from definition 6.1 the notions of good ordinal, good real σ, and the coding C_σ of c.u.b. sets. In particular, $C_\sigma = GO' \cap \{\beta < \boldsymbol{\delta}^1_3 \colon \forall \gamma < \beta\ |K \restriction \gamma| < \beta\}$, where $K \subseteq (\omega \times \boldsymbol{\delta}^1_3)^{<\omega}$ is the Kunen tree (defined before definition 6.1 Throughout this chapter, T will denote the Martin tree of §2.

Definition 9.1. A set $S \subseteq (\boldsymbol{\delta}^1_5)^-$ is *simple* with respect to the complex $\mathcal{E}$ if $\exists$ (a good real σ) $\exists D \subseteq \boldsymbol{\delta}^1_3$ $S = \{h(F; \mathcal{E}, \delta) \colon (\delta \in D) \wedge (F\colon \operatorname{dom}(<^{\mathcal{T}}) \to C_\sigma$ is of type $\mathcal{T})\}$. We say S is simple if it simple with respect to some complex $\mathcal{E}$.

The Kunen argument, exactly as in theorem 6.1, shows that any $A \subseteq (\boldsymbol{\delta}^1_5)^-$ is a countable union of simple sets.

A simple set is thus determined by a set $D \subseteq \boldsymbol{\delta}^1_3$, a complex $\mathcal{E}$, and a good real σ coding a c.u.b. set. We denote the corresponding simple set by $S_{D,\mathcal{E},\sigma}$. Complexes $\mathcal{E}$ are clearly codable by reals in a straightforward manner, and we may code sets $D \subseteq \boldsymbol{\delta}^1_3$ by reals using the coding of §6, although any reasonable coding would suffice here. Thus, every real x is viewed as coding a set $D_x \subseteq \boldsymbol{\delta}^1_3$.

To each real z we associate a set $A_z \subseteq (\boldsymbol{\delta}^1_5)^-$ as follows. First, decode z into an infinite sequence of triples (x_n, y_n, σ_n) in some standard manner. Let $D_n = D_{x_n} \subseteq \boldsymbol{\delta}^1_3$ be the set coded by x_n. If for some n, σ_n is not good (i.e. K_{σ_n} is ill-founded) set $S_n = \emptyset$, otherwise let $C_n = C_{\sigma_n}$ be the c.u.b. set coded by σ_n. If for some n, y_n does not have the correct "syntactical form" to code a complex, set $S_n = \emptyset$. Otherwise, decode y_n into $\mathcal{E}_n = \langle (f_j; K_1, \dots, K_i; K_{i+1}, \dots, K_t), \mathcal{T}, (d_i, \dots, d_t), (w_i, \dots, w_{t-1}) \rangle$. If for some n one of the trees T_{w_k} is ill-founded, set $S_n = \emptyset$. Otherwise, set $S_n = S_{D_n, \mathcal{E}_n, \sigma_n}$. Finally, let $A_z = \bigcup_n S_n$.

The next theorem shows our coding is reasonable. It is analogous to theorem 6.2.

THEOREM 9.1.

(a) $\forall A \subseteq (\boldsymbol{\delta}^1_5)^- \; \exists z \in \omega^\omega \; (A = A_z)$. *Thus, every set is coded.*

(b) $\forall \alpha < (\boldsymbol{\delta}^1_5)^- \; \{z \colon \alpha \in A_z\} \in \boldsymbol{\Delta}^1_5$.

PROOF. (a) is immediate from the above discussion. So fix $\alpha < (\boldsymbol{\delta}^1_5)^-$ and we show that $B_\alpha \doteq \{z \colon \alpha \in S_{D_{x(z)}, \mathcal{E}_{y(z)}, \sigma(z)}\} \in \boldsymbol{\Delta}^1_5$, where z codes the triple $(x(z), y(z), \sigma(z))$, from which (b) immediately follows. Let D_z, $\mathcal{E}_z$, σ_z abbreviate $D_{x(z)}$, $\mathcal{E}_{y(z)}$, $\sigma(z)$. It suffices to fix the initial segment $\tilde{\mathcal{E}} = \langle (f_n; K_1, \dots, K_i; K_{i+1}, \dots, K_t), \mathcal{T}, (d_i, \dots, d_t) \rangle$ and show $C_\alpha = \{z \colon (\mathcal{E}_z$ extends $\tilde{\mathcal{E}}) \wedge \alpha \in S_{D_z, \mathcal{E}_z, \sigma_z}\} \in \boldsymbol{\Delta}^1_5$. Since $\tilde{\mathcal{E}}$ is fixed, we may view C_α as a relation on tuples $(x, \vec{w}, \sigma)$, that is, $C_\alpha = \{(x, \vec{w}, \sigma) \colon \alpha \in S_{D_x, \mathcal{E}, \sigma}$ where $\mathcal{E} = \tilde{\mathcal{E}}^\frown \vec{w}\}$. We let $h(F; \vec{w}, \delta)$ abbreviate $h(F; \tilde{\mathcal{E}}^\frown \vec{w}, \delta)$, which only depends on $[F]$.

We have:

$$\begin{aligned} C_\alpha(x, \vec{w}, \sigma) \Leftrightarrow {} & [K_\sigma \text{ is well-founded }] \wedge [\forall j \; T_{w_j} \text{ is well-founded }] \\ & \wedge \exists \vec{\eta} \in ({\boldsymbol{\delta}^1_5}^-)^{<\omega} \; [(\vec{\eta} = [F] \text{ for some } F \colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3 \text{ of type } \mathcal{T} \\ & \text{having range almost everywhere in } C_\sigma) \\ & \wedge \exists \delta < \boldsymbol{\delta}^1_3 \; (\delta \in D_x \wedge h(F; \vec{w}, \delta) = \alpha)]. \end{aligned}$$

The first two conjuncts are easily $\boldsymbol{\Pi}^1_4$. Since $\boldsymbol{\Delta}^1_5$ is closed under $< \boldsymbol{\delta}^1_5$ length unions and intersections, it suffices to to show, for fixed $\vec{\eta}$, that the expression in square brackets in the third conjunct defines a $\boldsymbol{\Delta}^1_5$ set. We may assume $\vec{\eta}$ is representable by an $F \colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$, as otherwise the set is empty. Since $\vec{\eta}$ is fixed, we may fix a function $F \colon \operatorname{dom}(<^{\mathcal{T}}) \to \boldsymbol{\delta}^1_3$ of type $\mathcal{T}$ representing $\vec{\eta}$. Thus, for such fixed α, $\vec{\eta}$, and F, we must show that $D \in \boldsymbol{\Delta}^1_5$, where:

$$\begin{aligned} D(x, \vec{w}, \sigma) \Leftrightarrow {} & [(F \text{ has range almost everywhere in } C_\sigma) \\ & \wedge \exists \delta < \boldsymbol{\delta}^1_3 \; (\delta \in D_x \wedge h(F; \vec{w}, \delta) = \alpha)]. \end{aligned}$$

With the analysis of measures on $\boldsymbol{\delta}^1_3$, theorem 5.2, the proof of lemma 6.2 easily generalizes to:

LEMMA 9.1. *Let μ be a measure on $\boldsymbol{\delta}^1_3$, and suppose sets $R_\beta(x) \in \boldsymbol{\Delta}^1_5$ are given for $\beta < \boldsymbol{\delta}^1_3$. Then $S(x)$ defined by $S(x) \Leftrightarrow \forall^*_\mu \beta \; R_\beta(x)$ is $\boldsymbol{\Delta}^1_5$.* Λ

It follows that the first conjunct in the expression for D defines a $\boldsymbol{\Delta}^1_5$ set. Since "$\delta \in D_x$" defines a $\boldsymbol{\Delta}^1_3$ set for any fixed $\delta < \boldsymbol{\delta}^1_3$, it then suffices to show that for fixed $\alpha, \delta, \vec{\eta}, F$ that $E \in \boldsymbol{\Delta}^1_5$, where $E(\vec{w}) \Leftrightarrow (h(F; \vec{w}; \delta) = \alpha)$.

Let $H \colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ represent α with respect to W^n_3 (recall $\tilde{\mathcal{E}} = \langle (f_n; \dots) \dots \rangle$). As usual, we let $H(f)$ abbreviate $H([f])$, etc. We then have:

$$\begin{aligned} E(\vec{w}) \Leftrightarrow & \exists u \in \omega^\omega \; [(u \text{ is good }) \wedge \forall \nu < \boldsymbol{\delta}^1_3 \; \{(\nu \text{ is representable by a function} \\ & f \colon \omega_{n+1} \to \boldsymbol{\delta}^1_3 \text{ of the correct type with range in } C_u \text{ almost everywhere }) \\ & \to (h(F; \vec{w}; \delta; f) = H(f))\}]. \end{aligned}$$

Here $h(F; \vec{w}; \delta; f)$ represents $h(F; \vec{w}; \delta)$ with respect to W^n_3. Also:

$$\begin{aligned}E(\vec{w}) \Leftrightarrow & \forall u \in \omega^\omega\ [(u \text{ is good }) \to \exists \nu < \boldsymbol{\delta}^1_3\ \{(\nu \text{ is representable by a function}\\ & f\colon \omega_{n+1} \to \boldsymbol{\delta}^1_3 \text{ of the correct type with range in } C_u \text{ almost everywhere })\\ & \wedge h(F;\vec{w};\delta;f) = H(f)\}].\end{aligned}$$

Since "u is good" is $\boldsymbol{\Pi}^1_4$, and for fixed f, "f has range in C_u almost everywhere" defines a $\boldsymbol{\Delta}^1_3$ set, it suffices to show that for fixed δ, F, f that the set $F_0(\vec{w}) \Leftrightarrow [h(F;\vec{w};\delta;f) = H(f)]$ is $\boldsymbol{\Delta}^1_5$. In fact, $F_0 \in \boldsymbol{\Delta}^1_3$. This follows easily from lemma 6.2 and standard tree computations. For example, the next step is to fix a function $[h_1] \to H(f,h_1)$ representing $H(f)$ with respect to K_1. Then $F_0(\vec{w}) \Leftrightarrow \forall^* h_1\ [h(F;\vec{w};\delta;f,h_1) = H(f,h_1)]$, so by lemma 6.2 it suffices to show that for fixed h_1 that $F_1(\vec{w}) \Leftrightarrow [h(F;\vec{w};\delta;f,h_1) = H(f,h_1)]$ is $\boldsymbol{\Delta}^1_3$. At step i we must show, for fixed $\delta, F, f, h_1, \dots, h_i)$, that $F_i(\vec{w}) \Leftrightarrow [h(F;\vec{w};\delta;f,h_1,\dots,h_i) = H(f,h_1,\dots,h_i)]$ is $\boldsymbol{\Delta}^1_3$. Now,

$$\begin{aligned}F_i(\vec{w}) \Leftrightarrow \exists \delta_1, \delta_2 < H(f,h_1,\dots,h_i)\ & [\delta_1 = h(F;d_i;f;h_1,\dots,h_i) \wedge\\ & \delta_2 = \alpha_{i+1}(F;\vec{w};\delta;f,h_1,\dots,h_i)\\ & \wedge |T_{x_i} \restriction (\sup_j j(\delta_1))(\delta_2)| = H(f,h_1,\dots,h_i)].\end{aligned}$$

Here, $\alpha_{i+1}(\dots)$ is as in the definition of $h(F;\mathcal{E},\delta)$. The third conjunct easily defines a $\boldsymbol{\Delta}^1_3$ set. For the first conjunct, where say $d_i = \langle d^i_1, i_1, \dots, d^i_m, i_m\rangle$, we have:

$$\begin{aligned}[\delta_1 = h(F;d_i;f,h_1,\dots,h_i)] \Leftrightarrow \exists \epsilon_1,\dots,\epsilon_m < \delta_1\ & [\epsilon_1 = h(f;d^i_1;\vec{h};K^1_{i+1},\dots,K^1_{a_1})\\ & \wedge \dots \wedge\ \epsilon_m = h(f;d^i_m;\vec{h};K^m_{i+1},\dots,K^m_{a_m})\\ & \wedge \delta_1 = F(\langle \epsilon_1, i_1, \dots, \epsilon_m, i_m\rangle)].\end{aligned}$$

It follows readily that this defines a $\boldsymbol{\Delta}^1_3$ set as well. Proceeding inductively with the second conjunct (which defines the set $F_{i+1}(\vec{w})$) shows that $F(\vec{w}) \in \boldsymbol{\Delta}^1_3$. This completes the proof of theorem 9.1. $\square$

It is straightforward at this point to obtain the weak partition on $\boldsymbol{\delta}^1_5$, as in Kunen's proof of the weak partition relation on $\boldsymbol{\delta}^1_3$. For the sake of completeness we give a brief sketch.

THEOREM 9.2. *For all $\lambda < \boldsymbol{\delta}^1_5$ we have $\boldsymbol{\delta}^1_5 \to (\boldsymbol{\delta}^1_5)^\lambda$.*

PROOF. Fix λ, and a well-ordering $\prec$ of $(\boldsymbol{\delta}^1_5)^-$ of length λ. Code functions $f\colon \lambda \to \boldsymbol{\delta}^1_5$ via relations $R \subseteq ({\boldsymbol{\delta}^1_5}^-)^3$. Namely, if R is such that for all $\alpha < (\boldsymbol{\delta}^1_5)^-$ R_α is well-founded, where $R_\alpha = \{(\beta,\gamma)\colon R(\alpha,\beta,\gamma)\}$, then R codes the function f_R defined by $f_R(|\alpha|_\prec) = |R_\alpha|$. One then plays the Martin game (c.f. the proof of theorem 9.2) using this coding, and our coding of subsets of $(\boldsymbol{\delta}^1_5)^-$ as in theorem 6.4. The following facts are used:

(1) $\forall \alpha < \lambda\ \forall \beta < \boldsymbol{\delta}^1_5\ A_{\alpha,\beta} = \{x\colon x \text{ codes a function } f_x \text{ such that } \forall \alpha' < \alpha\ f_x(\alpha') \text{ is defined and } < \beta\} \in \boldsymbol{\Delta}^1_5$.
(2) If $\alpha < \lambda$, $B \in \boldsymbol{\Sigma}^1_5$, and $\forall x \in B\ f_x(\alpha)$ is defined, then $\sup_{x \in B} f_x(\alpha) < \boldsymbol{\delta}^1_5$.

Both facts follow easily from the closure properties of $\boldsymbol{\Delta}^1_5$ and theorem 9.1. $\square$

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