SETS MEETING ISOMETRIC COPIES OF THE LATTICE Z^2 IN EXACTLY ONE POINT

STEVE JACKSON[1] AND R. DANIEL MAULDIN[2]

ABSTRACT. The construction of a subset S of $\mathbb{R}^2$ such that each isometric copy of $\mathbb{Z}^2$ (the lattice points in the plane) meets S in exactly one point is indicated. This provides a positive answer to a problem of H. Steinhaus.

1. Introduction

In the 1950's, Steinhaus posed the following problem. Is there a set S in the plane such that every set congruent to $\mathbb{Z}^2$ has exactly one point in common with S? The problem seems to have first appeared in a 1958 paper of Sierpiński [14]. Steinhaus also asked several related questions which have been stated and studied in [5, 13, 14]. This specific problem has been widely noted, see e.g. [3, 4], but has remained unsolved until now. Here using a combination of techniques from analysis, set theory, number theory and plane geometry we show the answer is in the affirmative:

Theorem 1.1 (ZFC). *There is a set $S \subseteq \mathbb{R}^2$ such that for every isometric copy L of the integer lattice $\mathbb{Z}^2$ we have $|S \cap L| = 1$.*

We call a set S as in theorem 1.1 a Steinhaus set and note that whether there can be a Lebesgue measurable Steinhaus set remains unsolved. This problem has been the origin of many papers including those of J. Beck [1], H. T. Croft [2], Komjáth [13], and Kolountzakis [11]. Kolountzakis and Wolff [12] showed that there is no measurable Steinhaus set for the higher dimensional version of Steinhaus' problem for the standard lattice [12]. Steinhaus' problem and variants were discussed in some detail by Croft [2] and have been updated in sections E10 and G9 of [3].

It is natural to attempt to construct a Steinhaus set by a direct transfinite induction on a well ordering of the lattices. However, as Croft noted in [2], this straightforward attack quickly runs into problems. Nevertheless, we succeed in carrying out a transfinite induction argument by embedding our well-ordering of the lattices into a structure with several additional properties. This allows us to generate a new family of lattices at each stage and, by using several geometric and arithmetic facts specific to this problem, to continually extend a partial Steinhaus set.

Let us say a lattice distance is a real number of the form $\sqrt{n^2+m^2}$ where $n, m \in \mathbb{Z}$. Our methods allow us to prove a strengthening of theorem 1.1:

---

[1]Research supported by NSF Grant DMS-0097181
[2]Research supported by NSF Grant DMS-0100078
2000 Mathematics Subject Classification: Primary MSC 04A20, Secondary 11H31 *Key words and phrases*: lattice points, Steinhaus set.
Date: April 28, 2003.

Theorem 1.2 (ZFC)**.** *There is a set $S \subseteq \mathbb{R}^2$ satisfying:*

(1) *For every isometric copy L of $\mathbb{Z}^2$ we have $S \cap L \neq \emptyset$.*
(2) *For all distinct $z_1, z_2 \in S$, $\rho(z_1, z_2)^2 \notin \mathbb{Z}$.*

The Steinhaus problem has a natural interpretation for smaller sets of lattices. Namely, given an arbitrary set $\mathcal{L}$ of lattices (each of which is an isometric copy of $\mathbb{Z}^2$), we may ask whether there is a set S satisfying (2) of theorem 1.2 and such that $S \cap L \neq \emptyset$ for all $L \in \mathcal{L}$. We call a set $S \subseteq \mathbb{R}^2$ satisfying (2) a *partial Steinhaus set.*

Our first step toward proving theorem 1.2 is establishing this restricted version of the problem for the case where $\mathcal{L}$ is the (countable) family of rational translations of $\mathbb{Z}^2$. The proof involves only elementary number theory and combinatorics.

Lemma 1.3 (A)**.** *Let $\mathcal{L}_{\mathbb{Q}}$ denote the set of rational translations of $\mathbb{Z}^2$, that is, lattices of the form $\mathbb{Z}^2 + (r, s)$ where $r, s \in \mathbb{Q}$. Then there is a set $S \subseteq \mathbb{R}^2$ satisfying the following.*

(1) *For every lattice $L \in \mathcal{L}_{\mathbb{Q}}$, $S \cap L \neq \emptyset$.*
(2) *For all distinct $z_1, z_2 \in S$, $\rho(z_1, z_2)^2 \notin \mathbb{Z}$.*

Proof. We wish to show there is a map $f : (\mathbb{Q}/\mathbb{Z}) \times (\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q} \times \mathbb{Q}$ such that $\rho^2(f(x), f(y)) \notin \mathbb{Z}$ holds for $x \neq y$ and f is a selector: $f([r], [s]) \in [r] \times [s]$. The set S is the image set of the selector.

For this we define a graph on the Abelian group $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$ by joining x with y if and only if there are elements $g(x) \in x + (\mathbb{Z} \times \mathbb{Z})$ and $g(y) \in y + (\mathbb{Z} \times \mathbb{Z})$ such that the square of the distance between $g(x)$ and $g(y)$ is an integer. When constructing the selector we only have to worry about points that are joined. Let H be the subgroup of $\mathbb{Q} \times \mathbb{Q}$ consisting of elements with denominators only divisible by primes of the form $4k + 1$. Note that if $(r, s) \in \mathbb{Q} \times \mathbb{Q}$ and $r^2 + s^2 \in \mathbb{Z}$, then $(r, s) \in H$. For suppose $r = \frac{a}{b}$, $s = \frac{c}{d}$ (written in lowest terms), and $r^2 + s^2 = e \in \mathbb{Z}$. Suppose $p = 2$ or p is a prime congruent to 3 mod 4, and p divides b or d. Then the exact power of p dividing b, say p^k, must be the exact power dividing d, as otherwise multiplying through by the square of the least common multiple m of b and d would give $r^2m^2 + s^2m^2 = em^2$ where the right-hand side and exactly one of the left-hand side terms is divisible by p, a contradiction. Consider the case $p \equiv 3$ mod 4. Since these exact powers are the same, neither term on the left is divisible by p, and both are non-zero mod p. This would give that -1 is a square mod p, a contradiction as $p \equiv 3 \mod 4$. The case $p = 2$ is left to the reader. Thus, if $[x], [y]$ are joined then $x - y \in H$, that is, no edges go between distinct cosets of H, so it suffices to make the construction per cosets, and this again reduces to making the construction on H.

To motivate the following argument, let $x = (\frac{i_1}{d}, \frac{j_1}{d})$, $y = (\frac{i_2}{d}, \frac{j_2}{d})$, where d is only divisible by primes congruent to 1 mod 4, and suppose $f([x]) = x + (k_1, l_1)$, $f([y]) = y + (k_2, l_2)$ and $\rho(f([x]), f([y]))^2 \in \mathbb{Z}$. Multiplying through by d^2 this becomes

$$(1) \qquad (i_1 - i_2)^2 + (j_1 - j_2)^2 + 2d(i_1 - i_2)(k_1 - k_2) + 2d(j_1 - j_2)(l_1 - l_2) \in d^2\mathbb{Z}.$$

Suppose p^a is one of the prime components of d, and p^e, p^f are the exact powers of p dividing $i_1 - i_2$, $j_1 - j_2$ respectively. If $e \geq a$, then easily $f \geq a$ as well, and conversely. If $e < a$, then easily $e = f$. This gives $(\frac{i_1 - i_2}{p^e})^2 + (\frac{j_1 - j_2}{p^e})^2 \equiv 0$ mod p^{a-e}. Recall that for each prime power p^a, where $p \equiv 1 \mod 4$, there are exactly two square roots, say λ_{p^a}, μ_{p^a}, of -1 mod p^a, and for any $a' < a$, λ_{p^a} mod $p^{a'}$, μ_{p^a} mod $p^{a'}$ are the roots mod $p^{a'}$. So we must have $\frac{j_1-j_2}{p^e} \equiv \lambda_{p^a}\frac{i_1-i_2}{p^e}$ mod p^{a-e}, where λ_{p^a} denotes one of the roots. So, $j_1 - j_2 \equiv \lambda_{p^a}(i_1 - i_2) \mod p^a$ holds in either case ($e \geq a$ or $e < a$). Since this holds for each prime power p^a of d, we conclude that for $\rho(f([x]), f([y]))^2 \in \mathbb{Z}$ to hold, we must have $j_1 - j_2 \equiv \lambda(i_1 - i_2)$ mod d, where $\lambda^2 \equiv -1 \mod d$. To further motivate the argument, suppose now that $d = p^a$ is a prime power and $j_1 - j_2 = \lambda(i_1 - i_2)$ where λ is chosen so that $\lambda^2 \equiv -1 \mod p^{2a}$. Substituting in equation (1) and dividing through by $2d(i_1 - i_2)$ gives $k_1 + \lambda l_1 \equiv k_2 + \lambda l_2 \mod p^{a-b}$, where p^b is the exact power dividing $i_1 - i_2$. Note, for example, that if $a = 1$ (i.e., d is a prime) then this equation would not hold if we had $k_1 = k_2 = 0$ and $l_1 = i_1$, $l_2 = i_2$. This suggests we somehow write each $x = (\frac{i}{d}, \frac{j}{d})$ in terms of the "basis elements" $(1, \lambda_{p^a})$, $(1, \mu_{p^a})$ for each p^a dividing d, and use this to define the k, l values (where $f([x]) = x + (k, l)$). We now carry out the details of this argument.

Let $P_1, P_2, \cdots$ enumerate the positive powers of primes of the form $4k+1$ such that if P_i divides P_j then $i \leq j$. By recursion, for each i fix the distinct mod P_i natural numbers $\lambda_i, \mu_i > 0$ such that $P_i^2 | \lambda_i^2 + 1$, $\mu_i^2 + 1$ and $P_i^2 | \lambda_j - \lambda_i, \mu_j - \mu_i$, where P_j is the next member of the sequence which is a power of the same prime as P_i. Note that λ_i and $\mu_i > 0$ are the distinct square roots of -1 mod P_i^2. Let $B(P_i)$ be an integer divisible by every $P_1, \ldots, P_{i-1}$ but not by P_i, and, if $P_i = p^n$, then let $A(P_i)$ be an integer divisible by each of $P_1, \ldots, P_{i-1}$ which are not powers of p, but $(A(P_i), P_i) = 1$. If (x, y) is a pair with x, y rational numbers, $0 \leq x, y < 1$, and the denominators of x, y are only divisible by primes of the form $4k+1$, then (x, y) we claim can uniquely be written mod 1 as

$$\sum \frac{A(P_i)}{P_i} \left(\alpha_i(1, \lambda_i) + \beta_i(1, \mu_i)\right), \tag{2}$$

with $0 \leq \alpha_i, \beta_i < p$, here p is the prime whose power P_i is. To see this, first write $(x, y) = \frac{(e_1, f_1)}{p_1^{a_1}} + \cdots + \frac{(e_k, f_k)}{p_k^{a_k}} \mod 1$, where the e_i, f_i are integers and $d = p_1^{a_1} \ldots p_k^{a_k}$. Each term in this sum is of the form $\frac{(e,f)}{P_j}$ for some P_j, and it is enough to show that this term can be written as $\sum \frac{A(P_i)}{P_i} (\alpha_i(1, \lambda_i) + \beta_i(1, \mu_i))$ where the P_i range over the divisors of P_j, and $0 \leq \alpha_i, \beta_i < p$. Find α_j, β_j in the desired range with $A(P_j)(\alpha_j(1, \lambda_j) + \beta_j(1, \mu_j)) \equiv (e, f) \mod p$, where P_j is a power of p. This is possible as $(A(P_j), p) = 1$, and the two by two system is non-singular mod p. Then $\frac{(e,f)}{P_j} - \frac{A(P_j)}{P_j}(\alpha_j(1, \lambda_j) + \beta_j(1, \mu_j))$ is of the form $\frac{(e', f')}{P_k}$ where $P_j = P_k p$. Continuing we finish. The uniqueness part of the claim is easily checked. We now add $(0, \sum t_i B(P_i))$ to the point defined by (2), where $t_i = (\alpha_i + \beta_i)$. This will be $f([x], [y])$.

Assume that the square of the Euclidean norm of the difference of two such points is an integer. The difference of the two points is of the form

$$\sum \frac{A(P_i)}{P_i} \left(u_i(1, \lambda_i) + v_i(1, \mu_i)\right) + (0, S), \tag{3}$$

where $S = \sum (u_i + v_i) B(P_i)$ with $-(p-1) \leq u_i, v_i \leq (p-1)$ for $P_i = p^n$ and the sum here is taken over all i such that not both u_i and v_i are zero.

If the point given by (3) is $(\frac{a}{d}, \frac{b}{c} + S)$ where $(a, d) = (b, c) = 1$, then $d = c$, as otherwise the square of the norm could not be an integer. From this we get that

$b \equiv \lambda a \pmod d$ where $\lambda^2+1 \equiv 0 \pmod{d^2}$. Next, note that all the P_i which occur in the sum in (3) divide d. Thus, for every i, either $\lambda \equiv \lambda_i \pmod{P_i^2}$ or $\lambda \equiv \mu_i \pmod{P_i^2}$ and the same case must hold for powers of the same prime. By renaming, if needed, we assume that $\lambda \equiv \lambda_i \pmod{P_i^2}$ holds for every i.

Since $b \equiv \lambda a \pmod d$ and $\lambda \equiv \lambda_i \pmod{P_i^2}$, we have

$$\sum \frac{A(P_i)v_i(\lambda_i - \mu_i)}{P_i} \equiv 0 \pmod 1$$

holds. Notice that p does not divide $\lambda_i - \mu_i$ where P_i is a power of p, and from this we get by backward induction on i that $v_i = 0$ holds for every i.

What we have now is that

$$\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d} + S\right)^2$$

is an integer, where

$$\frac{a}{d} = \sum \frac{u_i A(P_i)}{P_i},$$
$$\frac{b}{d} = \sum \frac{u_i \lambda_i A(P_i)}{P_i},$$

and

$$S = \sum u_i B(P_i).$$

Let i be the first index such that $u_i \neq 0$ and let $P_i = p^n$. Then the first nonzero term of S, that is, $u_i B(P_i)$ is divisible by p^{n-1} but not by p^n and all later terms are divisible by p^n (because of the factors $B(P_j)$). So S is divisible by p^{n-1} but not by p^n. If we replace $\frac{b}{d}$ by $\frac{\lambda a}{d}$ then we can easily conclude, using $\lambda^2 \equiv -1 \pmod{d^2}$,

$$\left(\frac{a}{d}\right)^2 + \left(\frac{\lambda a}{d} + S\right)^2 \equiv \frac{2\lambda a}{d} S \pmod 1$$

which is certainly not an integer as d is divisible by p^n while S is not.

We now argue that the same holds for $(\frac{a}{d}, \frac{b}{d})$ instead of $(\frac{a}{d}, \frac{\lambda a}{d})$. To this end, it suffices to show that the integer $\frac{\lambda a}{d} - \frac{b}{d}$ is divisible by p^n. Indeed, if X is that last difference, then we can repeat the above argument with $S - X$ in place of S.

To show the last claim decompose it as

$$\frac{\lambda a}{d} - \frac{b}{d} = \sum \frac{u_j(\lambda - \lambda_j)A(P_j)}{P_j} = \sum{}' + \sum{}''$$

where $\sum'$ contains the terms with $p|P_j$ and $\sum''$ contains the other terms. In the first sum, using the fact that $\lambda \equiv \lambda_j \pmod{P_j^2}$, every term is an integer divisible by p^n. The second sum is an integer of the form $\frac{B}{C}$ where B is divisible by p^n (because of the factors $A(P_j)$) but C is not divisible by p. □

Note that for any choice of the $A(P_i)$, $B(P_i)$ we have $f([0],[0]) = (0,0)$ in the above construction. However, for any $(e,f) \in \mathbb{Z}^2$ we could add (e,f) to the value of $f([x],[y])$ for all x, y and still keep the squared distances between distinct points in the range of f non-integer. Thus, we are free to make $f([0],[0])$ any point in $\mathbb{Z}^2$.

Actually, we require a slightly stronger form of the lemma 1.3. To state it we call sets of the form $x + (\mathbb{Z}\times\mathbb{Z})$, $(x \in \mathbb{Q}\times\mathbb{Q})$ *$\mathbb{Z}\times\mathbb{Z}$-subsets of $\mathbb{Q}\times\mathbb{Q}$*. Also, a subset $E \subseteq \mathbb{Q}\times\mathbb{Q}$ is *small*, if for every $\mathbb{Z}\times\mathbb{Z}$-subset D, $D \cap E$ is contained in the union of finitely many lines. If L is isometric to $\mathbb{Z}^2$ and Q is the set of points having rational coordinates with respect to L, then we define in an analogous manner the notion of E being small relative to L.

Let d be a positive integer. Let R_d be the subgroup of $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$ of points $([x],[y])$ where x, y can be written with denominator d. Let $H_d = H \cap R_d$. If $d = p_1^{a_1} \dots p_k^{a_k} q_1^{b_1} \dots q_l^{b_l}$ where each $p_i \equiv 1 \pmod 4$ and each $q_j = 2$ or $q_j \equiv 3$ mod 4, then H_d are those $([x],[y])$ where x, y can be written with denominator $p_1^{a_1} \dots p_k^{a_k}$. Note that R_d is isomorphic to the direct sum $H_d \oplus K_d$, where K_d is the subgroup of $([x],[y])$ where x, y can be written with denominator $q_1^{b_1} \dots q_l^{b_l}$. In particular, the distinct cosets of H_d in R_d are given by $([x],[y]) + H_d$, where $([x],[y]) \in K_d$.

Suppose now f is a selector on R_d. We say f is *good* if on each coset of H_d in R_d, f is defined as in the construction of lemma 1.3. More precisely, we mean the following. Let $d = p_1^{a_1} \dots p_k^{a_k} q_1^{b_1} \dots q_l^{b_l}$ as above (if $d = 1$, we declare f to be good). Let $\bar{d} = p_1^{a_1} \dots p_k^{a_k}$. Then for each coset $([x],[y]) + H_d$, where $([x],[y]) \in K_d$, there is a sequence of prime powers P_1, P_2, $\dots$, P_{i_0} which "build up" $\bar{d}$ (that is $\bar{d}$ is the product of the P_i which do not divide another term in the sequence) and integers $A(P_i)$, $B(P_i)$, $i \le i_0$, as in the proof of lemma 1.3, and an initial translation (e, f) such that for all $([r],[s]) \in H_d$, $f(([x],[y]) + ([r],[s])) = (e, f) + f'([r],[s])$, where f' is defined on H_d as in lemma 1.3 using the P_i, $A(P_i)$, and $B(P_i)$.

It was shown in the proof of lemma 1.3 that if f is a good selector on R_d, then the range of f is a partial Steinhaus set.

Lemma 1.4 (A$'$). *Let d be a positive integer and f be a good selector on R_d. Suppose $d|d'$ and E is a small set missing $ran(f)$. Then f may be extended to a good selector f' on $R_{d'}$ whose range also misses E.*

Proof. Using the fact that the constructions on the different cosets of H are independent (c.f. the first paragraph in the proof of lemma 1.3) and also that a rational translation of a small set is small, it is enough to show the following (changing somewhat the notation from the statement of the lemma). Let d be a product of positive powers of primes congruent to 1 mod 4. Let $P_1, P_2, \dots, P_{i-1}$ build up d (as defined above). Let $A(P_1)$, $B(P_1)$, $\dots A(P_{i-1})$, $B(P_{i-1})$ satisfy the requirements given in lemma 1.3. Let f be the selector on H_d defined from these quantities as in lemma 1.3. Suppose finally that E is a small set missing $\mathrm{ran}(f)$, and P_i is given with $P_1, \dots P_i$ building up d' say. Then we show that there are $A(P_i)$, $B(P_i)$ so that if f' is defined using these extended sequences, then the range of f' misses E. The point is we have enough freedom in choosing the values of $A(P_i)$, $B(P_i)$. We are assuming $f([x],[y])$ has been defined for all (x, y) having a representation as in (2) with the sum ranging over prime powers in the list $P_1, \dots, P_{i-1}$. Let A, B, satisfy the requirements for $A(P_i)$, $B(P_i)$ given in lemma 1.3. Let U be the product of the P_j with $j < i$ and P_j, P_i relatively prime. Then $A' = A + KP_iU$, $B' = B + LP_iU$ also satisfy these requirements for any $K, L \in \mathbb{Z}$. Let f' be as constructed in lemma 1.3 using A', B'. For any fixed x, $y \in R_{d'} - R_d$, a computation shows that the corresponding values of $f'([x],[y])$ will have the form $(x' + KUa, y' + KUb + LUP_ia)$ for some fixed $x', y' \in \mathbb{Q}$, $a, b \in \mathbb{Z}$ with $a, b \neq 0$. Since E is small, for each $[x]$, $[y]$ the requirement that $f([x],[y]) \notin E$ rules out only the K, L lying on finitely many lines in $\mathbb{Z} \times \mathbb{Z}$. As there are only finitely many $[x]$, $[y]$ to consider at stage i, it is clear that we can choose K, L so that f' misses E. $\square$

We thank one of the referees for pointing out the proofs of the preceding lemmas. These proofs are based on and motivated by the more complicated proofs that we give in [9] for stronger results. To state one of these results we adopt the following terminology. For rationals r, s, let $L_{r,s} = \mathbb{Z}^2 + (r, s)$ be the rational translation of $\mathbb{Z}^2$ by (r, s). Let $R = \mathbb{Q}^2 \cap ([0, 1) \times [0, 1))$. For each positive integer d, let $R_d \subseteq R$ be defined by $R_d = \{(\frac{i}{d}, \frac{j}{d}) : 0 \leq i, j < d\}$. We prove the following in [9].

Lemma 1.5. *Let $d > 1$ and suppose functions k, l mapping R_d into $\mathbb{Z}$ have been defined such that setting $S_d = \{(r + k(r, s), s + l(r, s)) : r, s \in R_d\}$ we have:*

$$(*)_d\text{: for all distinct } z_1, z_2 \in S_d,\ \rho(z_1, z_2)^2 \notin \mathbb{Z}.$$

Then for any d' with $d|d'$, the functions k, l may be extended to $R_{d'}$ so as to satisfy $(*)_{d'}$.

This last lemma assures us not only that there is a set S satisfying lemma A, but also that any partial Steinhaus set defined for the translates in R_d may be extended. We also prove similar extension theorems where we must avoid small obstruction sets. These extension lemmas are much stronger than is required for the proof of our main theorem but we feel that they may be useful in studying analogous problems for other lattices, other dimensions and other groups of isometries.

By a rational translation of a lattice L we mean a lattice of the form $L + r\vec{u} + s\vec{v}$ where $r, s \in \mathbb{Q}$, and $\vec{u}$, $\vec{v}$ are the unit basis vectors of L.

Definition 1.6. Two lattices are equivalent $L_1 \sim L_2$, if L_2 can be obtained from L_1 by rational rotations and translations.

In other words, $L_1 \sim L_2$ if and only if all of the points of L_2 have rational coordinates with respect to the coordinate system determined by L_1 (and vice-versa). This is easily an equivalence relation and each equivalence class is countable. We also note that if two lattices are not equivalent then there can be at most one point with rational coordinates with respect to both of them.

An important aspect of the construction is that if one has a Steinhaus set for all the rational translates of a given lattice L, then it is a Steinhaus set for the equivalence class of L:

Lemma 1.7. *Let L_1 be a lattice and suppose $S \subseteq \mathbb{R}^2$ satisfies the following:*

(1) *For every lattice L which is a rational translation of L_1, $S \cap L \neq \emptyset$.*
(2) *For all distinct z_1, $z_2 \in S$, $\rho(z_1, z_2)^2 \notin \mathbb{Z}$.*

Then for every lattice L' which is equivalent to L_1, we have $S \cap L' \neq \emptyset$.

Proof. Without loss of generality we may assume $L_1 = \mathbb{Z}^2$, and thus S meets every rational translation of $\mathbb{Z}^2$. It is enough to consider a rational rotation L_2 of $\mathbb{Z}^2$, and show that S meets every rational translation of L_2. Let $\vec{u} = (\frac{a}{d}, \frac{b}{d})$ and $\vec{v} = (-\frac{b}{d}, \frac{a}{d})$ be the unit basis vectors of L_2, where $a, b, d \in \mathbb{Z}$. Note that every rational translation of L_2 may be expressed in the form $L_2(\frac{i}{d'}, \frac{j}{d'}) \doteq L_2 + \frac{i}{d'}\vec{u} + \frac{j}{d'}\vec{v}$, where $d|d'$. Fix such a d'. Let $M = \bigcup_{i,j} L_2(\frac{i}{d'}, \frac{j}{d'})$, so M is isometric with the scaled lattice $\frac{1}{d'}\mathbb{Z} \times \frac{1}{d'}\mathbb{Z}$. We claim there are exactly ${d'}^2$ points of M in the unit square $R = \{(x, y) : 0 \leq x, y < 1\}$. To see this first note that the scaled unit vectors $v_1 = (d', 0)$ and $v_2 = (0, d')$ in Z^2 have integer coordinates with respect to L_2. Next, note that $d'(M \cap R)$ consists of all the lattice points of L_2 in the fundamental domain determined by v_1 and v_2. Recalling the well-known general fact about standard lattices in $\mathbb{R}^n$: if $v_1, \ldots, v_n \in \mathbb{Z}^n$ and $D = |\det(v_1, \ldots, v_n)| \neq 0$, then there are exactly D many points of $\mathbb{Z}^n$ in the fundamental region $\{t_1 v_1 + \cdots + t_n v_n : 0 \leq t_1, \ldots, t_n < 1\}$, we see that $A = M \cap R$ consists of exactly d'^2 points. For each $p \in A$, let p' be the unique point in S with $p - p' \in \mathbb{Z}^2$. Let $A' = \{p' : p \in A\}$. Clearly $A' \subseteq S$ and $|A'| = d'^2$. Since $d|d'$, each point of A' has coordinates in the L_2 lattice of the form $(\frac{i}{d'}, \frac{j}{d'})$. For each $p' \in A'$, let p'' be the unique point of the form $p' + a\vec{u} + b\vec{v}$ with $a, b \in \mathbb{Z}$ which lies in the fundamental domain of L_2. The map $p' \to p''$ is one-to-one on A' as otherwise $\rho(z_1, z_2)^2 \in \mathbb{Z}$ for distinct $z_1, z_2 \in A' \subseteq S$. Thus, if $A'' = \{p'' : p' \in A'\}$, then $|A''| = d'^2$ as well. This shows that S meets every lattice of the form $L_2(\frac{i}{d'}, \frac{j}{d'})$. □

Naturally, this last observation suggests constructing a full Steinhaus set by induction on the equivalence classes of lattices. This leads us to the set theoretic part of the proof. To motivate these considerations, suppose first that the continuum hypothesis, CH, holds. The straightforward approach of simply enumerating the equivalence classes $\mathcal{L}_\alpha$, $\alpha < \omega_1$, and attempting to extend the partial Steinhaus set at stage α to meet each lattice in $\mathcal{L}_\alpha$ quickly runs into problems. Let $S_{<\alpha}$ be the set so far constructed, and $L \in \mathcal{L}_\alpha$. If $S_{<\alpha} \cap L = \emptyset$, then we must add a point of L to $S_{<\alpha}$, but it might be the case that every point of L is at a lattice distance from $S_{<\alpha}$. To avoid this problem, it is important that at each stage we enumerate not just a single equivalence class, but rather a countable collection of equivalence classes which is closed under certain operations. The main closure property we need arises from lemma 1.9, and the corresponding argument is given in claim 1.11 below. Rather than specify at the outset exactly what closure properties we wish our sets at each stage to satisfy, it is more convenient to follow a standard set-theoretic practice. Namely, at stage α we will consider those equivalence classes of lattices which lie in an elementary substructure (or hull) M_α of V_λ for some large λ (though actually $\lambda = \omega + \omega$ will suffice). Here V_λ denotes the initial segment of the universe of sets of rank less than λ, and an elementary substructure denotes a subset closed under the functions $f \colon (V_\lambda)^n \to V_\lambda$, for some n, which are definable in V_λ (the so-called Skolem functions of V_λ). For the benefit of the reader unfamiliar with set-theoretic terminology, we note that this is merely a convenient shorthand for requiring that the sets we consider be sufficiently closed without having to specify in advance exactly which functions we want them to be closed under. Thus, in place of M_α the reader could substitute "a sufficiently closed collection of points and lattices" (the closure properties needed will be apparent from the following argument, and could be specified in advance).

In the general case (not assuming CH) the construction is essentially an iteration of this substructure construction. We will define $M_{\vec{\alpha}}$ for certain $\vec{\alpha} \in \mathrm{ON}^{<\omega}$ ($\mathrm{ON}^{<\omega}$ denotes the finite sequences of ordinals). Now the $M_{\vec{\alpha}}$ will not in general be closed under the Skolem functions, but only relatively closed inside of a larger set. That this is good enough for the argument is an important aspect of the proof. Finally we remark that we feel the general framework presented here is likely to useful in other problems. Although the set-theoretic terminology could be largely eliminated (see also the comments at the end of this paper), we believe doing so would lessen the generality of the method and hide our motivation (for example, it was by pursuing the "hull" strategy just outlined that we were led to consider lemma 1.9).

We now describe the particular enumeration or well-ordering of the equivalence classes which we will use.

First, some notation. If $L \subseteq \mathbb{R}^2$ is an isometric copy of $\mathbb{Z}^2$, let $[L]$ denote the equivalence class of L under the equivalence relation $\sim$ of definition 1.6. Let $\mathfrak{L}$ denote the family of all equivalence classes. Let $\mathcal{L} \to L(\mathcal{L})$ be a function which picks for each equivalence class $\mathcal{L}$ a member $L(\mathcal{L}) \in \mathcal{L}$.

The construction of the enumeration begins by letting $\{M_{\alpha_0} : \alpha_0 < 2^\omega\}$ be an increasing continuous (i.e., at limit stages we take unions) chain of elementary substructures of a large V_λ with $|M_{\alpha_0}| < 2^\omega$ for all $\alpha_0 < 2^\omega$ and such that each lattice and each equivalence class of lattices is in one of these substructures. Assume also that $M_0 = \emptyset$. (The starting set M_0 is an exception in that it is not an elementary substructure.) Let $N_{\alpha_0} = M_{\alpha_0+1} - M_{\alpha_0}$.

By simultaneous recursion we define a subtree T of $\mathrm{ON}^{<\omega}$ and an ordinal $\kappa(\vec{\alpha})$ and sets $M_{\vec{\alpha}}$, $N_{\vec{\alpha}}$ for $\vec{\alpha} \in T$. Set $\kappa(\emptyset) = 2^\omega$. In general, suppose that $M_{\vec{\alpha}}$ is defined for $\vec{\alpha}$ in a certain subtree of $\mathrm{ON}^{<\omega}$. If $M_{\alpha_0,\dots,\alpha_k}$ is defined, we assume also that $\kappa(\alpha_0,\dots,\alpha_{k-1})$ has been defined and is an infinite cardinal. Furthermore, we assume in this case that $M_{\alpha_0,\dots,\alpha_{k-1},\beta}$ is defined if and only if $\beta < \kappa(\alpha_0,\dots,\alpha_{k-1})$. We let $N_{\alpha_0,\dots,\alpha_k}$ denote $M_{\alpha_0,\dots,\alpha_k+1} - M_{\alpha_0,\dots,\alpha_k}$.

Suppose now that $M_{\alpha_0,\dots,\alpha_k}$ is defined. If $N_{\alpha_0,\dots,\alpha_k}$ contains only countably many equivalence classes of lattices, let $\{\mathcal{L}_{\alpha_0,\dots,\alpha_k;n}\}_{n\in s}$, where $s \leq \omega$, enumerate these equivalence classes. In this case, $(\alpha_0,\dots,\alpha_k)$ is a terminal node in the tree, T, of indices $\vec{\alpha}$ for which $M_{\vec{\alpha}}$ is defined. Otherwise, let $\kappa(\alpha_0,\dots,\alpha_k) = |N_{\alpha_0,\dots,\alpha_k}|$ and express

$$N_{\alpha_0,\dots,\alpha_k} = \bigcup_{\alpha_{k+1}<\kappa(\alpha_0,\dots,\alpha_k)} M_{\alpha_0,\dots,\alpha_k,\alpha_{k+1}},$$

as a continuous increasing union, where each $M_{\alpha_0,\dots,\alpha_k,\alpha_{k+1}}$ is the intersection of $N_{\alpha_0,\dots,\alpha_k}$ with an elementary substructure of V_λ, and each $M_{\alpha_0,\dots,\alpha_k,\alpha_{k+1}}$ has size $< \kappa(\alpha_0,\dots,\alpha_k)$. Assume also $M_{\alpha_0,\dots,\alpha_k,0} = \emptyset$. We note two facts. Easily, the tree of indices is well-founded (since the cardinals $\kappa_{\vec{\alpha}}$ are strictly decreasing along any branch). Also, if $a_1,\dots,a_m \in M_{\alpha_0,\dots,\alpha_k,\alpha_{k+1}}$ and f is a Skolem function of V_λ and $f(a_1,\dots,a_m) \in N_{\alpha_0,\dots,\alpha_k}$, then $f(a_1,\dots,a_m) \in M_{\alpha_0,\dots,\alpha_{k+1}}$.

Notice if $\vec{\alpha}$ is incompatible with $\vec{\beta}$, then $N_{\vec{\alpha}}$ and $N_{\vec{\beta}}$ have no equivalence class in common. Furthermore, every equivalence class occurs as some $\mathcal{L}_{\alpha_0,\dots,\alpha_k;n}$. Thus, the $\mathcal{L}_{\alpha_0,\dots,\alpha_k;n}$ precisely enumerate the equivalence classes of lattices. By an "ω-block" of lattices we mean the (equivalence classes of) lattices of the form $\mathcal{L}_{\alpha_0,\dots,\alpha_k;n}$ for some fixed terminal index $\alpha_0,\dots,\alpha_k$. We consider the indices to be (well) ordered lexicographically.

The following simple lemma will be used.

Lemma 1.8. *Suppose $\vec{\alpha}$ is an index for which $M_{\vec{\alpha}}$ is defined. Let $a_1,\dots,a_m \in M_{\vec{\alpha}}$ and suppose b is definable from $a_1,\dots,a_m$ in V_λ. Then $b \in \bigcup_{\vec{\beta}\leq\vec{\alpha}} M_{\vec{\beta}}$.*

Proof. Let $\vec{\alpha} = (\alpha_0,\dots,\alpha_k)$ and assume $b \notin \bigcup_{\vec{\beta}\leq\vec{\alpha}} M_{\vec{\beta}}$. Since $M_{\alpha_0,\dots,\alpha_k}$ is relatively closed under the Skolem functions of V_κ inside of $N_{\alpha_0,\dots,\alpha_{k-1}}$, it follows that $b \notin N_{\alpha_0,\dots,\alpha_{k-1}}$. Since $b \notin M_{\alpha_0,\dots,\alpha_{k-1}}$ by assumption, we thus have $b \notin M_{\alpha_0,\dots,\alpha_{k-1}+1}$. Continuing, we eventually have $b \notin M_{\alpha_0+1}$, a contradiction since M_{α_0+1} is an elementary substructure of V_λ containing the a_i. $\square$

The idea for constructing a Steinhaus set is by transfinite induction along the terminal nodes of the tree T. However, it turns out we need another lemma in pure plane geometry in order to make an inductive extension.

Lemma 1.9 (B). *Let c_1, c_2, c_3 be three distinct points in the plane, and let r_1, r_2, $r_3 > 0$ be real numbers. Let C_1 be the circle in the plane with center at c_1 and radius r_1, and likewise for C_2 and C_3. Let a, b, c be three distinct points in the plane. Then, except for the exceptional case described afterwards, there are only finitely many triples of points (p_1, p_2, p_3) in the plane such that*

1. *$p_1 \in C_1$, $p_2 \in C_2$, and $p_3 \in C_3$.*
2. *The triangle $p_1p_2p_3$ is isometric with the triangle abc (we allow the degenerate case where the points a, b, c are collinear).*

The exceptional case is when $r_1 = r_2 = r_3$ and the triangle abc is isometric with $c_1c_2c_3$.

This lemma seems to be known within engineering mathematics specifically regarding the geometry of mechanisms [10]. In fact, lemma B has a direct interpretation in the theory of mechanical linkages. A four-bar linkage may be described as two circles C_1, C_2, and a rigid "bar" connecting two points a, b constrained to lie on C_1, C_2 respectively. If we consider a third point c, and require that the triangle $\triangle abc$ be rigid, then the locus of points traced out by c is called a coupler curve for the linkage. We say the coupler point c is non-trivial if it is not one of the endpoints a, b. In this terminology lemma B is the statement that the curve traced out by a non-trivial coupler point of a four-bar linkage has, except in the degenerate case noted, a finite intersection with any circle. Although not explicitly stated, lemma B follows from the analysis of Gibson and Newstead in [8]. Newstead pointed out (private communication) that the "residual curve" R (a higher dimensional curve defined in [8]) and the coupler curve C, which is its projection, are both of degree 6. It follows that the irreducible components of C and R exactly correspond. The analysis of [8], however, describes exactly when and how R factors, and the result follows.

The Gibson-Newstead argument uses a significant amount of algebraic geometry. We also give two elementary proofs of lemma B in [9], an algebraic one using Gröbner bases and computer algebra, and the other a purely geometric proof. Actually, as we remark at the end of this paper, we could get by with a weaker version of the lemma, but the stated form arises naturally in the inductive proof and may also prove useful in studying other problems.

Fix now a terminal index $\vec{\alpha} = (\alpha_0, \dots, \alpha_k)$. Assume inductively we have defined for each terminal index $\vec{\beta} < \vec{\alpha}$ a set $S_{\vec{\beta}} \subseteq \mathbb{R}^2$ which satisfy the following:

1. If $\vec{\beta}_1 < \vec{\beta}_2 < \vec{\alpha}$, then $S_{\vec{\beta}_1} \subseteq S_{\vec{\beta}_2}$.
2. For every terminal index $\vec{\beta}$ less than $\vec{\alpha}$, $S_{\vec{\beta}}$ meets every lattice in every equivalence class $\mathcal{L}_{\vec{\beta};n}$.
3. Every point of $S_{\vec{\beta}} - \bigcup_{\vec{\gamma}<\vec{\beta}} S_{\vec{\gamma}}$ lies on some lattice of the form $\mathcal{L}_{\vec{\beta};n}$.
4. For all distinct $z_1, z_2 \in S_{\vec{\beta}}$, $\rho(z_1, z_2)^2 \notin \mathbb{Z}$.
5. Suppose $\vec{\beta}_1 < \vec{\beta}_2 < \vec{\alpha}$, $x \in S_{\vec{\beta}_1}$, and $y \in S_{\vec{\beta}_2} - \bigcup_{\vec{\gamma}<\vec{\beta}_2} S_{\vec{\gamma}}$. Then if $\rho(x, y)^2 \in \mathbb{Q}$ then x, y both have rational coordinates with respect to some lattice of the form $\mathcal{L}_{\vec{\beta}_2;n}$.

Let $S_{<\vec{\alpha}} = \bigcup_{\vec{\beta}<\vec{\alpha}} S_{\vec{\beta}}$. We show how to extend $S_{<\vec{\alpha}}$ to a set $S_{\vec{\alpha}}$ also satisfying 4, 5 and such that $S_{\vec{\alpha}}$ meets every lattice in each equivalence class $\mathcal{L}_{\vec{\alpha},n}$. This suffices to prove theorem 1.2.

To ease notation, let $\mathcal{L}_n = \mathcal{L}_{\vec{\alpha};n}$, and let $L_n = L(\mathcal{L}_n)$. From lemma 1.7, it suffices to maintain property 4, have property 5 when $\vec{\beta}_2 = \vec{\alpha}$, and have $S_{\vec{\alpha}}$ meet every rational translation of each L_n (recall a rational translation of L_n refers to a motion which is a translation in the coordinate system of L_n).

For integers n, d, i, j, let $L_n^{d,i,j}$ denote the translation of L_n by the amount $(\frac{i}{d}, \frac{j}{d})$ (in the coordinate system of L_n). We also need the following easy lemma.

Lemma 1.10. *Let L be a lattice and $z \in \mathbb{R}^2$. Suppose z has coordinates (x, y) with respect to the lattice L, where at least one of x, y is irrational. Then there is a line $l = l(z, L)$ such that if w has rational coordinates with respect to L and $w \notin l$, then $\rho(w, z)^2 \notin \mathbb{Q}$.*

These last two geometric lemmas allow us to demonstrate the following claim.

Claim 1.11. For each n and rationals $\frac{i}{d}, \frac{j}{d}$, there is a finite set of lines $G_n(\frac{i}{d}, \frac{j}{d})$ with the following property: if $c \in S_{<\vec{\alpha}}$ does not have rational coordinates with respect to L_n, if $z \in L_n^{d,i,j}$, and if $\rho(c, z)^2 \in \mathbb{Q}$, then $z \in \cup G_n(\frac{i}{d}, \frac{j}{d})$.

Proof. Suppose there is a $z_1 \in L_n^{d,i,j}$ and a $c_1 \in S_{<\vec{\alpha}}$ not rational with respect to L_n such that $\rho(z_1, c_1)^2 \in \mathbb{Q}$ (otherwise there is nothing to prove). Let $l_1 = l(c_1, L_n^{d,i,j})$ be the line (necessarily through z_1) given by lemma 1.10. Suppose there is a $z_2 \notin l_1$, $z_2 \in L_n^{d,i,j}$, and a $c_2 \in S_{<\vec{\alpha}}$ not rational with respect to L_n with $\rho(z_2, c_2)^2 \in \mathbb{Q}$ (necessarily $c_2 \neq c_1$). Let $l_2 = l(c_2, L_n^{d,i,j})$ be given by lemma 1.10. Continuing, construct $z_m \in L_n^{d,i,j}$, $c_m \in S_{<\vec{\alpha}}$ if possible so that $z_m \notin l_1 \cup \cdots \cup l_{m-1}$ and $\rho(z_m, c_m)^2 \in \mathbb{Q}$. If the construction fails at some point, then the claim is proved. Assume toward a contradiction that we continue to produce an infinite sequence $z_1, c_1, z_2, c_2, \ldots$. Note that the c_i are distinct. Let $\vec{\beta}^m = (\beta_0^m, \ldots, \beta_l^m)$ be the terminal index (where l depends on m) such that $c_m \in N_{\vec{\beta}^m}$. Thus, $\vec{\beta}^m < \vec{\alpha}$. Easily, there is a $k' \leq k$ (recall $\vec{\alpha} = (\alpha_0, \ldots, \alpha_k)$) such that for infinitely many m we have $\beta_0^m = \alpha_0, \ldots, \beta_{k'-1}^m = \alpha_{k'-1}$, and $\beta_{k'}^m < \alpha_{k'}$ (we allow $k' = 0$, in which case we have $\beta_0^m < \alpha_0$). Let $\vec{\gamma} = (\alpha_0, \ldots, \alpha_{k'})$. Thus $\vec{\gamma} \leq \vec{\alpha}$, and for these infinitely many m we have $c_m \in M_{\vec{\gamma}}$. Let m_1, m_2, m_3 be three such m. Let $r_1 = \rho(c_{m_1}, z_{m_1})$, and similarly for r_2, r_3. We apply lemma B to the circles with centers at c_{m_i} of radii r_i and the points z_{m_i}. Note that we are not in the exceptional case of lemma B, as otherwise we would have $\rho(z_{m_1}, z_{m_2}) = \rho(c_{m_1}, c_{m_2})$. This contradicts the fact that $\rho(c_{m_1}, c_{m_2})^2 \notin \mathbb{Z}$ as they lie in $S_{<\vec{\alpha}}$ (note that $\rho(z_{m_1}, z_{m_2})^2 \in \mathbb{Z}$ as z_{m_1}, z_{m_2} lie in $L_n^{d,i,j}$). From lemma B, the points z_{m_1}, z_{m_2}, z_{m_3} are definable from c_{m_1}, c_{m_2}, and c_{m_3}. Since L_n is definable from z_{m_1}, z_{m_2}, z_{m_3} (in fact, from any two of them), L_n is definable from c_{m_1}, c_{m_2}, and c_{m_3}. It follows from lemma 1.8 that L_n lies in some $M_{\vec{\beta}}$, for $\vec{\beta} \leq \vec{\alpha}$. This contradicts $L_n \in N_{\vec{\alpha}}$. $\square$

Fix a bijection $(n, m) \mapsto \langle n, m \rangle \in \omega$ between ω^2 and ω. We may assume this bijection is increasing in each coordinate. Let Q_n be those points having rational coordinates with respect to L_n. We construct partial Steinhaus sets T_m^n in stages, with $T_0^n \subseteq T_1^n \subseteq \cdots \subseteq Q_n$. We will arrange it so that if $T^n = \bigcup_m T_m^n$, then T^n meets every lattice in $\mathcal{L}_n$. At stage $i = \langle n, m \rangle$ we extend T_{m-1}^n to T_m^n (or define T_0^n if $m = 0$). We will follow lemma 1.4 in doing each extension.

Fix $i = \langle n, m\rangle$, and assume that for all $\langle p, q\rangle < i$ that T^p_q is defined. Assume inductively that the T^p_q so far defined satisfy the following (below, interpret T^p_{q-1} as $\emptyset$ if $q = 0$):

(a) T^p_q contains T^p_{q-1}, and each T^p_q is a partial Steinhaus set (that is, $\rho(z_1, z_2)^2 \notin \mathbb{Z}$ for all distinct $z_1, z_2 \in T^p_q$).

(b) T^p_q is the range of a good selector f^p_q on $R^p_{d_q}$. Here $R^p_{d_q}$ is the set of points having coordinates with respect to L_p which can be written with denominators d_q, and $d_0, d_1, \ldots$ is a sequence (depends on p) with $d_j | d_{j+1}$ for all j and such that every integer divides some d_j.

(c) Let $E^p = \bigcup_{i,j,d}(G_p(\frac{i}{d}, \frac{j}{d}) \cap L^{d,i,j}_p)$. Note that E^p is small relative to Q_p. Then $(T^p_q - (S_{<\vec{\alpha}} \cup \bigcup_{\langle a,b\rangle < \langle p,q\rangle} T^a_b)) \cap E^p = \emptyset$.

(d) Suppose $z \notin S_{<\vec{\alpha}}$, and $i_2 = \langle p_2, q_2\rangle$ is least so that $z \in T^{p_2}_{q_2}$. If $i_1 = \langle p_1, q_1\rangle < i_2$ and $p_1 \neq p_2$, then z does not have rational coordinates with respect to L_{p_1}.

(e) Suppose $z \notin S_{<\vec{\alpha}}$, and $i_2 = \langle p_2, q_2\rangle$ is least so that $z \in T^{p_2}_{q_2}$. If $i_1 = \langle p_1, q_1\rangle < i_2$ and $y \in T^{p_1}_{q_1}$ does not have rational coordinates with respect to L_{p_2}, then $z \notin l(y, L_{p_2})$, where $l(y, L_{p_2})$ is as in lemma 1.10.

Consider now $i = \langle n, m\rangle$, and we define T^n_m so as to also satisfy the above properties.

Consider first the case $m = 0$, that is, we are at the beginning stage in the construction of T^n. We claim that there is at most one point in $S_{<\vec{\alpha}} \cup \bigcup_{\langle p,q\rangle<i} T^p_q$ which is also in Q_n. For suppose y, z were two such points. Note that $\rho(y, z)^2 \in \mathbb{Q}$. Suppose first that $y, z \in S_{<\vec{\alpha}}$. Say $y \in S_{\vec{\beta}_1} - \bigcup_{\vec{\gamma}<\vec{\beta}_1} S_{\vec{\gamma}}$, $z \in S_{\vec{\beta}_2} - \bigcup_{\vec{\gamma}<\vec{\beta}_2} S_{\vec{\gamma}}$ where $\vec{\beta}_1 \leq \vec{\beta}_2$. If $\vec{\beta}_1 = \vec{\beta}_2$, then each of y, z lies on a lattice in $N_{\vec{\beta}_2}$. Since L_n is definable from y and z, L_n is definable from two lattices in $M_{\vec{\beta}}$ for some $\vec{\beta} \leq \vec{\alpha}$. From lemma 1.8 it follows that $L_n \in \bigcup_{\vec{\gamma}\leq\vec{\alpha}} M_{\vec{\gamma}}$, a contradiction. If $\vec{\beta}_1 < \vec{\beta}_2$ then from inductive property 5, y, z both have rational coordinates with respect to some lattice L in $N_{\vec{\beta}_2}$. This would again imply that $L_n \in \bigcup_{\vec{\gamma}\leq\vec{\alpha}} M_{\vec{\gamma}}$, a contradiction. Suppose next that $y \in S_{<\vec{\alpha}}$ and $z \notin S_{<\vec{\alpha}}$. Let $\langle p, q\rangle$ be least so that $z \in T^p_q$ (so $p < n$). Since by (c), $z \notin \cup G_p(\frac{i}{d}, \frac{j}{d})$ we must have that y is rational with respect to L_p (as otherwise $\rho(y, z)^2 \notin \mathbb{Q}$). Thus, both y and z have rational coordinates with respect to both L_n and L_p, a contradiction to the fact that there can be at most one point with rational coordinates with respect to both lattices. Suppose now $y, z \notin S_{<\vec{\alpha}}$. Let $y \in T^{p_1}_{q_1}$, $z \in T^{p_2}_{q_2}$, with $i_1 = \langle p_1, q_1\rangle$, $i_2 = \langle p_2, q_2\rangle$ chosen minimally, so $p_1, p_2 < n$. Assume without loss of generality that $i_1 \leq i_2$. If $i_1 = i_2$ then y, z are rational with respect to both L_{p_2} and L_n, a contradiction. If $i_1 < i_2$ then from (e), y is rational with respect to L_{p_2}. So y, z are both rational with respect to L_{p_2} and L_n, again a contradiction.

Let E^n_0 be the union of $\bigcup_{i,j,d}(G_n(\frac{i}{d}, \frac{j}{d}) \cap L^{d,i,j}_n)$ together with $\bigcup_z l(z, L_n)$ where z ranges over the points in $\bigcup_{\langle p,q\rangle<i} T^p_q$ not having rational coordinates with respect to L_n, together with the (finitely many) points of Q_n which are rational with respect to one of the lattices L_p with $p \neq n$, and $\langle p, q\rangle < i$ for some q (for $m = 0$ this is equivalent to $p < n$). Clearly E^n_0 is small with respect to Q_n. Let w_n be the unique point in $(S_{<\vec{\alpha}} \cup \bigcup_{\langle p,q\rangle<i} T^p_q) \cap Q_n$ if it exists, and otherwise let w_n be any point of $L_n - E^n_0$. Apply now lemma 1.4 to get T^n_0 avoiding $E^n_0 - \{w_n\}$ and with $w_n \in T^n_0$. From the definition of E^n_0 it is clear that (c)-(e) are still satisfied.

Consider next the case $m > 0$. Define E^n_m exactly as above and apply lemma 1.4 to get T^n_m, adding only points which avoid the set E^n_m. Again, (c)-(e) are satisfied.

Note for the arguments below that if $z \notin S_{<\vec{\alpha}}$ and $\langle n, m\rangle$ is least so that $z \in T^n_m$, then $z \neq w_n$.

This completes the definition of the T^n_m, and we have verified (a)-(e). Let $T = \bigcup_{n,m} T^n_m$. Let $S_{\vec{\alpha}} = S_{<\vec{\alpha}} \cup T$. We must show that inductive hypotheses (1)-(5) are satisfied. Properties (1)-(3) are immediate from the construction. To see (5), let $\vec{\beta} < \vec{\alpha}$ and $y \in S_{\vec{\beta}}$, $z \in S_{\vec{\alpha}} - S_{<\vec{\alpha}}$, and suppose $\rho(y,z)^2 \in \mathbb{Q}$. Let $\langle n, m\rangle$ be least so that $z \in T^n_m$. Since $z \notin \bigcup_{i,j,d} G_n(\frac{i}{d}, \frac{j}{d})$ by construction, we must have that y is rational with respect to L_n.

Finally we verify (4), that is, we show $S_{\vec{\alpha}}$ is a partial Steinhaus set. Let y, z be distinct points in $S_{\vec{\alpha}}$, and assume $\rho(y,z)^2 \in \mathbb{Z}$. We may assume $z \in T - S_{<\vec{\alpha}}$. Suppose first that $y \in S_{<\vec{\alpha}}$. Let $i = \langle n, m\rangle$ be least with $z \in T^n_m$. As in the previous paragraph we must have y rational with respect to L_n as otherwise $\rho(y,z)^2 \notin \mathbb{Q}$. Let $i' = \langle n, 0\rangle$, so $i' \leq i$. In defining T^n_0, y would then have been the point w_n. So, $y \in T^n_m$, and since this is a partial Steinhaus set, $\rho(y,z)^2 \notin \mathbb{Z}$, a contradiction.

Assume now that $y, z \notin S_{<\vec{\alpha}}$. Let $y \in T^{n_1}_{m_1}$, $z \in T^{n_2}_{m_2}$, with $i_1 = \langle n_1, m_1\rangle$, $i_2 = \langle n_2, m_2\rangle$ chosen minimally. Clearly $i_1 \neq i_2$, in fact $n_1 \neq n_2$. Assume without loss of generality that $i_1 < i_2$. We must have y rational with respect to L_{n_2} as otherwise from the definition of $E^{n_2}_{m_2}$ and $l(y, L_{n_2})$ we would have $\rho(y,z)^2 \notin \mathbb{Q}$. Let $i_3 = \langle n_2, 0\rangle$ (note that $i_3 \neq i_1$). If $i_1 < i_3$, then y is the point w_{n_2}, which lies in $T^{n_2}_{m_2}$, and so $\rho(y,z)^2 \notin \mathbb{Z}$. If however $i_1 > i_3$, then from the definition of $E^{n_1}_{m_1}$ we would have that y is not rational with respect to L_{n_2} (since into $E^{n_1}_{m_1}$ we added the (at most one) point of $Q_{n_1} \cap Q_{n_2}$). This again implies that $\rho(y,z)^2 \notin \mathbb{Q}$, a contradiction.

This completes the proof of theorem 1.2.

Finally, we would like to point out some simplifications that could be made to the proof. First, our construction of an increasing continuous chain of elementary substructures is more general than what is needed for the proof of the theorem. However, we feel that these sorts of induction arguments are used over and over and it is useful to lay out how to proceed in general. We indicate how one could make the inductive construction more specific for this problem. Fix a transcendence basis for $\mathbb{R}$ over $\mathbb{Q}$, and let $\prec$ be a well-ordering of this transcendence basis. Then the finite $\prec$-decreasing sequences s from the basis are well-ordered lexicographically. For each such s, there are only countably many lattices which are algebraic over $\mathbb{Q}$ and s. Do the transfinite construction in the order of these sequences s, at each stage handling those countably many lattices that are algebraic over $\mathbb{Q}$ and s but not algebraic over $\mathbb{Q}$ and s' for any $s' \prec s$. Then all of the arguments given for the proof of the main theorem go through as before, if one replaces "definable" with "algebraic." The key point for this argument is: if three of the sequences β_m all precede α lexicographically and first differ from α at the same position k, then the union of these three sequences, arranged in $\prec$-decreasing order, still precedes α lexicographically.

Secondly, as we mentioned before, the full strength of lemma 1.9 is not needed for the proof. Let again (using the notation of claim 1.11) E^p be those points $z \in Q_p$ (i.e., having rational coordinates with respect to L_p) such that $\rho^2(c,z) \in \mathbb{Q}$ for some $c \in S_{<\vec{\alpha}}$ where c does not have rational coordinates with respect to L_p. Then it suffices to show that E^p is *semi-small* with respect to Q_p, which means for each rational translation $L = L_p^{d,i,j}$ of L_p there is a finite set of lines F such that for any line $l \notin F$, $l \cap L \cap E^p$ is finite. This is because the proof of lemma 1.4 shows that we may actually avoid any Q_p semi-small set in constructing the T_q^p. To see E^p is semi-small, it suffices to show that there is a bound $s \in \omega$ such that if $c_1, \dots, c_s$ are points in the plane with $\rho(c_i, c_j)^2 \notin \mathbb{Z}$ for distinct c_i, c_j, and if $z_1, \dots, z_s$ are *collinear* points with $\rho(c_i, z_i)^2 \in \mathbb{Q}$ and $\rho(z_i, z_j)^2 \in \mathbb{Z}$, then the z_i are definable from $\{c_1, \dots, c_s\}$; in fact, for fixed $c_1, \dots, c_s$, distances $\rho(c_i, z_i)$ and $\rho(z_i, z_j)$, there are only finitely many such $\{z_1, \dots, z_s\}$. This fact follows from lemma 1.1 of [13].

We thank the referees for pointing out these possible simplifications as well as the simplified proof of lemma A which we presented here.

References

[1] Beck, J. (1991) *Studia Sci. Math. Hung.* **24**, 263-268.

[2] Croft, H. T. (1982) *Quart. J. Math. Oxford* **33**, 71-83.

[3] Croft, H. T., Falconer, K. J. and Guy, R. K. (1991) *Unsolved Problems in Geometry,* (Springer-Verlag, New York).

[4] Erdős, P., Gruber, P. M. and Hammer, J. (1989) *Lattice points* (Longman Sci. Tech., Harlow).

[5] P. Erdős, P. (1985) *New York Academy of Sciences* **44**, 1-10.

[6] Erdős, P., Jackson, S. and Mauldin, R. D. (1994) *Fund. Math.* **145**, 101-119.

[7] Erdős, P., Jackson, S. and Mauldin, R. D. (1997) *Fund. Math.* **152**, 75-95.

[8] Gibson, C. G., and Newstead, P. E. (1986) *Acta Applicandae Mathematicae* **7**, 113-135.

[9] Jackson, S. and Mauldin, R. D. (200?) *Journal American Math. Society* , to appear.

[10] K. H. Hunt, Kinematic Geometry of Mechanisms, Oxford Engineering Science Series, 7 Oxford Science Publications, The Clarendon Press, Oxford University press, New York, 1990

[11] Kolountzakis, M. N. (1996) in *Analytic number theory, Vol. 2 (Allerton Park, IL, 1995),* Progr. Math., **139**, (Birkhäuser Boston, Boston), pp. 559-565.

[12] Kolountzakis, M. N. and Wolff, T. (1999) *Mathematika* **46**, 253-280.

[13] Komjáth, P. (1992) *Quart. J. Math. Oxford* **43**, 235-241.

[14] Sierpiński, W. (1958) *Fund. Math.* **46** (1958), 191-194.

Department of Mathematics, University of North Texas, Box 311430, Denton, TX 76203, USA

E-mail address: **jackson@unt.edu**

Department of Mathematics, University of North Texas, Box 311430, Denton, TX 76203, USA

E-mail address: **mauldin@unt.edu**